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ALL–DERIVABLE POINTS OF NEST ALGEBRAS ON BANACH SPACES

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Abstract. Let $\mathscr{N}$ be a nest on a real or complex Banach space X and let $\mathrm{Alg}\mathscr{N}$ be the associated nest algebra. $\Omega \in \mathrm{Alg}\mathscr{N}$ is called an additively all-derivable point if for any additive map $\delta : \mathrm{Alg}\mathscr{N} \to \mathrm{Alg}\mathscr{N}$, $\delta(AB) = \delta(A)B + A\delta(B)$ holds for any $A, B \in \mathrm{Alg}\mathscr{N}$ with $AB = \Omega$ implies that δ is an additive derivation. Assume that P is an idempotent operator with range $\mathrm{ran}(P) = N_0$ for some nontrivial $N_0 \in \mathscr{N}$. Let $\Omega \in \mathrm{Alg}\mathscr{N}$ be any operator satisfying that $P\Omega P = \Omega$ (or $(I-P)\Omega(I-P) = \Omega$). We show that, if $\Omega|_{\mathrm{ran}(P)}$ (or $\Omega|_{\mathrm{ran}(I-P)}$) is injective or has dense range, then Ω is an additively all-derivable point. Moreover, if X is infinite dimensional, then every additive map derivable at such an Ω is an inner derivation.

1. Introduction

Let $\mathscr{A}$ be an (operator) algebra. Recall that a linear (or an additive) map δ from $\mathscr{A}$ into itself is called a derivation if $\delta(AB) = \delta(A)B + A\delta(B)$ for all $A, B \in \mathscr{A}$. The class of derivations is one of the most important kinds of linear (or additive) maps both in theory and applications, and this topic has been studied intensively (Ref. [2]). The question of under what conditions that a linear (or an additive) map becomes a derivation attracted much attention of mathematicians (for instance, see [1], [4], [5], [9], [10] and the references therein). We say that a map $\delta : \mathscr{A} \to \mathscr{A}$ is derivable at a point Ω if $\delta(A)B + A\delta(B) = \delta(\Omega)$ for any $A, B \in \mathscr{A}$ with $AB = \Omega$, and such Ω is called a derivable point of δ. It is obvious that a linear map is a derivation if and only if it is derivable at all point. It is natural and interesting to ask the question whether or not a linear (an additive) map is a derivation if it is derivable only at one given point (Ref. [8]). Such points, if exist, are called linearly (additively) all-derivable points. The topic of characterizing the all-derivable points for various algebras has been studied by many authors and some all-derivable points have been found. However the set of all-derivable points is still far from being determined completely for almost all algebras.

Let $\mathscr{N}$ be a complete nest on a complex separable Hilbert space H. Suppose that M belongs to $\mathscr{N}$ with $\{0\} \neq M \neq H$ and write $\widehat{M}$ for M or $M^{\perp}$ and $P(\widehat{M})$ be the orthogonal projection on $\widehat{M}$. Let $\mathscr{N}_{\widehat{M}} = \{N \cap \widehat{M} : N \in \mathscr{N}\}$, which is a nest on $\widehat{M}$. It was shown in [8] that, for any $\Omega \in \mathrm{Alg}\mathscr{N}$ with $\Omega = P(\widehat{M})\Omega P(\widehat{M})$, if $\Omega|_{\widehat{M}}$ is invertible

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in $\mathrm{Alg}\mathcal{N}_{\widehat{M}}$, then Ω is a linearly all-derivable point in $\mathrm{Alg}\mathcal{N}$ for the strong operator topology, that is, every strongly continuous linear map from $\mathrm{Alg}\mathcal{N}$ into itself derivable at Ω is a derivation.

The purpose of this paper is to discuss a similar question for nest algebras on Banach spaces. Let $\mathcal{N}$ be a nest on a real or complex Banach space X and let $\mathrm{Alg}\mathcal{N}$ be the associated nest algebra. Assume that $\dim X = \infty$ and $\mathcal{N}$ is a nest in X. For an $N_0 \in \mathcal{N}$ which is complemented and for any idempotent operator P with range $\mathrm{ran}(P) = N_0$, we show that, if $\Omega \in \mathrm{Alg}\mathcal{N}$ satisfies that $P\Omega P = \Omega$ with $\Omega|_{\mathrm{ran}(P)}$ injective or of dense range as an operator on $\mathrm{ran}(P)$, or $(I-P)\Omega(I-P) = \Omega$ with $\Omega|_{\ker P}$ injective or of dense range as an operator on $\ker P$, then Ω is an additively all-derivable point of $\mathrm{Alg}\mathcal{N}$. In fact, every additive map derivable at Ω is a linear derivation and hence, an inner derivation. Comparing our result with the main result obtained in [8], we remark that, (1) we do not assume that the Banach space is separable or complex, and our result holds for any real or complex infinite dimensional Banach spaces; (2) the assumption on the nest is quite weak, and all nests on Hilbert spaces satisfy the assumption since every subspace in a Hilbert space is complemented; (3) we do not assume that the map is continuous under any topology, and the continuity is included in the conclusion; (4) we do not assume that $\Omega|_{\mathrm{ran}(P)}$ ($\Omega|_{\ker P}$) is invertible in $\mathrm{Alg}\mathcal{N}_{\widehat{M}}$, while only assume that it is injective or with dense range; (5) we do not assume that the map is linear, while only the additivity is assumed. Thus, our result generalizes the result of [8] remarkably. Our result is also a generalization of the main result of [5], in which it was shown that, if $\mathcal{N}$ satisfies that each $N \in \mathcal{N}$ with $N_- = N$ is complemented, then the above idempotent P as well as injective operators and operators with dense range in $\mathrm{Alg}\mathcal{N}$ are linearly all-derivable points. Here $N_- = \vee\{L : L \in \mathcal{N} \text{ and } L \subset N\}$.

2. Main Results

The following is our main result in this paper, which gives some new kinds of all-derivable points of nest algebras on Banach spaces. Recall that a map $\phi : \mathrm{Alg}\mathcal{N} \to \mathrm{Alg}\mathcal{N}$ is called an inner derivation if there exists $T \in \mathrm{Alg}\mathcal{N}$ such that $\phi(A) = AT - TA$ for all $A \in \mathrm{Alg}\mathcal{N}$. Inner derivations are linear.

THEOREM 2.1. *Let $\mathcal{N}$ be a nest on a real or complex Banach space X and let $\mathrm{Alg}\mathcal{N}$ be the associated nest algebra. Assume that $N_0 \in \mathcal{N}$ is non-trivial and complemented in X and P is any bounded idempotent operator on X with range N_0. Let $\delta : \mathrm{Alg}\mathcal{N} \to \mathrm{Alg}\mathcal{N}$ be an additive map.*

(a) For any operator $\Omega \in \mathrm{Alg}\mathcal{N}$ with $P\Omega P = \Omega$, if $\Omega|_{\mathrm{ran}(P)}$ is injective or has dense range as an operator on $\mathrm{ran}(P)$, then $\delta(AB) = \delta(A)B + A\delta(B)$ holds for any $A, B \in \mathrm{Alg}\mathcal{N}$ with $AB = \Omega$ implies that δ is an additive derivation.

(b) For any operator $\Omega \in \mathrm{Alg}\mathcal{N}$ with $(I-P)\Omega(I-P) = \Omega$, if $\Omega|_{\ker P}$ is injective or has dense range as an operator on $\ker P$, then $\delta(AB) = \delta(A)B + A\delta(B)$ holds for any $A, B \in \mathrm{Alg}\mathcal{N}$ with $AB = \Omega$ implies that δ is an additive derivation.

Furthermore, if X is infinite-dimensional, then δ is an inner derivation.

The last assertion is not valid for finite dimensional case. In fact, every additive derivation on the algebra $\mathcal{T}_n(\mathbb{F})$ of upper triangular matrix has the form $A \mapsto AT - TA +$

$(f(a_{ij}))$ for any $A=(a_{ij})\in\mathscr{T}_n(\mathbb{F})$, where $T\in\mathscr{T}_n(\mathbb{F})$ and f is an additive derivation on the complex field $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$ (ref. [7]). By [6], there exist many nontrivial additive derivations on $\mathbb{F}$.

Note that, every subspace of a Hilbert space is complemented. Hence as a consequence of Theorem 2.1, the following corollary is obvious, which is a generalization of the main result in [8].

COROLLARY 2.2. *Let $\mathscr{N}$ be a nest on a real or complex Hilbert space H. For any nontrivial $M\in\mathscr{N}$, write $\widehat{M}$ for M or $M^{\perp}$ and let $P(\widehat{M})$ be the orthogonal projection on $\widehat{M}$. Let $\Omega\in\mathrm{Alg}\mathscr{N}$ be an operator such that $\Omega=P(\widehat{M})\Omega P(\widehat{M})$ and $\Omega|_{\widehat{M}}$ is injective or of dense range as an operator on $\widehat{M}$. Let $\delta:\mathrm{Alg}\mathscr{N}\to\mathrm{Alg}\mathscr{N}$ be an additive map. If δ is derivable at Ω, that is, $\delta(AB)=\delta(A)B+A\delta(B)$ holds for any A,B with $AB=\Omega$, then δ is an additive derivation. In addition, if H is infinite-dimensional, then δ is inner.*

3. Proof of the main result

Proof of Theorem 2.1. Because $N_0\in\mathscr{N}$ is non-trivial, it follows from $\mathrm{ran}(P)=N_0$ that P is nontrivial and $P\in\mathrm{Alg}\mathscr{N}$. Let $\mathscr{N}_1=\{N\mid N\in\mathscr{N},N\subseteq N_0\}$, $\mathscr{N}_2=\{N\cap(\ker P)\mid N\in\mathscr{N}\}$. Then

$$\mathrm{Alg}\mathscr{N}=\{\begin{pmatrix} C & W\\ 0 & D\end{pmatrix}:C\in\mathrm{Alg}\mathscr{N}_1,W\in\mathscr{B}(\ker P,N_0),D\in\mathrm{Alg}\mathscr{N}_2\}.$$

For any $C\in\mathscr{A}_{11}=\mathrm{Alg}\mathscr{N}_1$, $W\in\mathscr{A}_{12}=\mathscr{B}(\ker P,N_0)$, $D\in\mathscr{A}_{22}=\mathrm{Alg}\mathscr{N}_2$, if $\delta:\mathrm{Alg}\mathscr{N}\to\mathrm{Alg}\mathscr{N}$ is additive, then we can write

$$\delta\left(\begin{pmatrix} C & W\\ 0 & D\end{pmatrix}\right)=\begin{pmatrix}\delta_{11}(C)+\tau_{11}(W)+\varphi_{11}(D) & \delta_{12}(C)+\tau_{12}(W)+\varphi_{12}(D)\\ 0 & \delta_{22}(C)+\tau_{22}(W)+\varphi_{22}(D)\end{pmatrix},$$

where $\delta_{ij}:\mathscr{A}_{11}\to\mathscr{A}_{ij}$, $\tau_{ij}:\mathscr{A}_{12}\to\mathscr{A}_{ij}$, $\varphi_{ij}:\mathscr{A}_{22}\to\mathscr{A}_{ij}$ are additive maps, $i,j\in\{1,2\}$ with $i\leqslant j$. Denote by I_i the unit of $\mathscr{A}_{ii}$, $i=1,2$.

(a) Assume that $\Omega\in\mathrm{Alg}\mathscr{N}$ satisfies the conditions that $P\Omega P=\Omega$, $\Omega|_{\mathrm{ran}(P)}$ is injective or has dense range as an operator on $\mathrm{ran}(P)$. Then $\Omega=\begin{pmatrix}\Omega_1 & 0\\ 0 & 0\end{pmatrix}$ with Ω_1 is injective or of dense range. We shall show that Ω is an all-derivable point of the nest algebra.

In the sequel we assume that δ is an additive map that is derivable at Ω.

Case 1. Ω_1 is injective. We will show that δ is a derivation step by step.

Step 1.1. $\delta_{11}(I_1)=0$, $\delta_{22}(\Omega_1)=0$, and for any $C_1,C_2\in\mathscr{A}_{11}$ with $C_1C_2=I_1$, we have $C_1\delta_{12}(C_2)=\delta_{12}(I_1)$.

For any $C_1, C_2 \in \mathscr{A}_{11}$ with $C_1C_2 = I_1$, any $D_1, D_2 \in \mathscr{A}_{22}$ with $D_1D_2 = 0$, take $S = \begin{pmatrix} \Omega_1 C_1 & 0 \\ 0 & D_1 \end{pmatrix}$ and $T = \begin{pmatrix} C_2 & 0 \\ 0 & D_2 \end{pmatrix}$; then $ST = \Omega$. So we have

$$\begin{aligned} &\begin{pmatrix} \delta_{11}(\Omega_1) & \delta_{12}(\Omega_1) \\ 0 & \delta_{22}(\Omega_1) \end{pmatrix} = \delta(S)T + S\delta(T) \\ &= \begin{pmatrix} \delta_{11}(\Omega_1C_1) + \varphi_{11}(D_1) & \delta_{12}(\Omega_1C_1) + \varphi_{12}(D_1) \\ 0 & \delta_{22}(\Omega_1C_1) + \varphi_{22}(D_1) \end{pmatrix} \begin{pmatrix} C_2 & 0 \\ 0 & D_2 \end{pmatrix} \\ &\quad + \begin{pmatrix} \Omega_1 C_1 & 0 \\ 0 & D_1 \end{pmatrix} \begin{pmatrix} \delta_{11}(C_2) + \varphi_{11}(D_2) & \delta_{12}(C_2) + \varphi_{12}(D_2) \\ 0 & \delta_{22}(C_2) + \varphi_{22}(D_2) \end{pmatrix} \\ &= \begin{pmatrix} T_1 & \delta_{12}(\Omega_1C_1)D_2 + \varphi_{12}(D_1)D_2 + \Omega_1C_1\delta_{12}(C_2) + \Omega_1C_1\varphi_{12}(D_2) \\ 0 & \delta_{22}(\Omega_1C_1)D_2 + \varphi_{22}(D_1)D_2 + D_1\delta_{22}(C_2) + D_1\varphi_{22}(D_2) \end{pmatrix}, \end{aligned}$$

where $T_1 = \delta_{11}(\Omega_1C_1)C_2 + \varphi_{11}(D_1)C_2 + \Omega_1C_1\delta_{11}(C_2) + \Omega_1C_1\varphi_{11}(D_2)$. It follows that

$$\delta_{11}(\Omega_1) = \delta_{11}(\Omega_1C_1)C_2 + \varphi_{11}(D_1)C_2 + \Omega_1C_1\delta_{11}(C_2) + \Omega_1C_1\varphi_{11}(D_2), \tag{3.1}$$

$$\delta_{12}(\Omega_1) = \delta_{12}(\Omega_1C_1)D_2 + \varphi_{12}(D_1)D_2 + \Omega_1C_1\delta_{12}(C_2) + \Omega_1C_1\varphi_{12}(D_2), \tag{3.2}$$

and

$$\delta_{22}(\Omega_1) = \delta_{22}(\Omega_1C_1)D_2 + \varphi_{22}(D_1)D_2 + D_1\delta_{22}(C_2) + D_1\varphi_{22}(D_2). \tag{3.3}$$

Letting $D_1 = D_2 = 0$, by Eqs.(3.2) and (3.3), one gets $\delta_{12}(\Omega_1) = \Omega_1C_1\delta_{12}(C_2)$ and $\delta_{22}(\Omega_1) = 0$. Letting $C_1 = C_2 = I_1, D_1 = D_2 = 0$, by Eqs.(3.1) and (3.2), one gets $\Omega_1\delta_{11}(I_1) = 0$ and $\delta_{12}(\Omega_1) = \Omega_1\delta_{12}(I_1)$. It follows from the injectivity of Ω_1 that

$$\delta_{11}(I_1) = 0,\ C_1\delta_{12}(C_2) = \delta_{12}(I_1),\ \delta_{22}(\Omega_1) = 0. \tag{3.4}$$

Step 1.2. $\delta_{12}(C) = CB$ holds for all $C \in \mathscr{A}_{11}$, where $B = \delta_{12}(I_1)$.

Taking any $C_0 \in \mathscr{A}_{11}$ which is invertible as an element in $\mathscr{A}_{11}$ and letting $C_2 = C_0, C_1 = C_0^{-1}$, by Step 1.1, we get that $\delta_{12}(C_0) = C_0\delta_{12}(I_1)$ holds for all invertible $C_0 \in \mathscr{A}_{11}$. For any $C \in \mathscr{A}_{11}$, take $n \in \mathbb{N}$ so that $n > \|C\|$. Then $nI_1 - C$ is an invertible operator with its inverse still in $\mathscr{A}_{11}$. Thus we have $\delta_{12}(nI_1 - C) = (nI_1 - C)\delta_{12}(I_1)$. It follows that $n\delta_{12}(I_1) - \delta_{12}(C) = n\delta_{12}(I_1) - C\delta_{12}(I_1)$ as δ is additive. Hence

$$\delta_{12}(C) = CB \tag{3.5}$$

for all $C \in \mathscr{A}_{11}$.

Step 1.3. $\delta_{22} = 0$.

Taking any invertible operator $C_0 \in \mathscr{A}_{11}$ and letting $C_2 = C_0, C_1 = C_0^{-1}$ and $D_1 = I_2, D_2 = 0$ in Eqs.(3.3), and by (3.4), we get $\delta_{22}(C_2) = 0$. Particularly, when $C_1 = C_2 = I_1$, we get $\delta_{22}(I_1) = 0$. For any $C \in \mathscr{A}_{11}$, we can take $n \in \mathbb{N}$ so that $n > \|C\|$. Then $nI_1 - C$ is invertible with its inverse still in $\mathscr{A}_{11}$. Hence $\delta_{22}(nI_1 - C) = 0$. Since δ is additive, we have

$$\delta_{22}(C) = n\delta_{22}(I_1) = 0 \tag{3.6}$$

holds for all $C \in \mathscr{A}_{11}$. Hence $\delta_{22} = 0$, as desired.

Step 1.4. $\varphi_{11}=0$ and $\varphi_{12}(D)=-BD$ for all $D\in\mathscr{A}_{22}$.

Letting $C_1=C_2=I_1, D_1=0, D_2=I_2$ in Eq.(3.2) we get $\Omega_1\delta_{12}(I_1)+\Omega_1\varphi_{12}(I_2)=0$. Because Ω_1 is injective, it follows that $\varphi_{12}(I_2)=-\delta_{12}(I_1)=-B$. Letting $C_1=C_2=I_1, D_2=0$ in Eqs.(3.1) and (3.4), we get $\varphi_{11}(D_1)=0$. Thus we have

$$\varphi_{11}(D)=0 \tag{3.7}$$

for all $D\in\mathscr{A}_{22}$.

Letting $C_1=C_2=I_1, D_1=0$ in Eqs.(3.2) and (3.5) we have $\Omega_1BD_2=-\Omega_1\varphi_{12}(D_2)$. Then $\varphi_{12}(D_2)=-BD_2$ as Ω_1 is injective. Thus for any $D\in\mathscr{A}_{22}$, we have

$$\varphi_{12}(D)=-BD. \tag{3.8}$$

Step 1.5. $\tau_{11}=0$ and $\tau_{22}=0$.

For any $W\in\mathscr{A}_{12}$, take $S=\begin{pmatrix}\Omega_1 & W\\ 0 & 0\end{pmatrix}$ and $T=\begin{pmatrix}I_1 & 0\\ 0 & 0\end{pmatrix}$. Then $ST=\Omega$. So, by Step 1.3,

$$\begin{aligned}
&\begin{pmatrix}\delta_{11}(\Omega_1) & \delta_{12}(\Omega_1)\\ 0 & 0\end{pmatrix}=\delta(S)T+S\delta(T)\\
&=\begin{pmatrix}\delta_{11}(\Omega_1)+\tau_{11}(W) & \delta_{12}(\Omega_1)+\tau_{12}(W)\\ 0 & \tau_{22}(W)\end{pmatrix}\begin{pmatrix}I_1 & 0\\ 0 & 0\end{pmatrix}+\begin{pmatrix}\Omega_1 & W\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & \delta_{12}(I_1)\\ 0 & 0\end{pmatrix}\\
&=\begin{pmatrix}\delta_{11}(\Omega_1)+\tau_{11}(W) & \Omega_1\delta_{12}(I_1)\\ 0 & 0\end{pmatrix}.
\end{aligned}$$

It follows that $\delta_{11}(\Omega_1)=\delta_{11}(\Omega_1)+\tau_{11}(W)$ for all $W\in\mathscr{A}_{12}$. Hence

$$\tau_{11}(W)=0 \tag{3.9}$$

holds for all $W\in\mathscr{A}_{12}$, that is, $\tau_{11}=0$.

For any $W\in\mathscr{A}_{12}$, take $S=\begin{pmatrix}I_1 & -W\\ 0 & 0\end{pmatrix}$ and $T=\begin{pmatrix}\Omega_1 & W\\ 0 & I_2\end{pmatrix}$; then $ST=\Omega$. Thus by Steps 1.1-1.4 (Eqs.(3.4)-(3.8)) and Eq.(3.9), we obtain that

$$\begin{aligned}
&\begin{pmatrix}\delta_{11}(\Omega_1) & \delta_{12}(\Omega_1)\\ 0 & 0\end{pmatrix}=\delta(S)T+S\delta(T)\\
&=\begin{pmatrix}0 & \delta_{12}(I_1)-\tau_{12}(W)\\ 0 & -\tau_{22}(W)\end{pmatrix}\begin{pmatrix}\Omega_1 & W\\ 0 & I_2\end{pmatrix}\\
&\quad+\begin{pmatrix}I_1 & -W\\ 0 & 0\end{pmatrix}\begin{pmatrix}\delta_{11}(\Omega_1) & \delta_{12}(\Omega_1)+\tau_{12}(W)+\varphi_{12}(I_2)\\ 0 & \tau_{22}(W)+\varphi_{22}(I_2)\end{pmatrix}\\
&=\begin{pmatrix}\delta_{11}(\Omega_1) & \delta_{12}(I_1)+\delta_{12}(\Omega_1)+\varphi_{12}(I_2)-W\tau_{22}(W)-W\varphi_{22}(I_2)\\ 0 & -\tau_{22}(W)\end{pmatrix}.
\end{aligned}$$

This entails that

$$\tau_{22}(W)=0 \tag{3.10}$$

for all $W\in\mathscr{A}_{12}$, i.e., $\tau_{22}=0$.

Step 1.6. $\tau_{12}(WD) = \tau_{12}(W)D + W\varphi_{22}(D)$ holds for all $W \in \mathscr{A}_{12}$ and $D \in \mathscr{A}_{22}$.

To see this, take $S = \begin{pmatrix} I_1 & W \\ 0 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} \Omega_1 & -WD \\ 0 & D \end{pmatrix}$, where $W \in \mathscr{A}_{12}$ and $D \in \mathscr{A}_{22}$. As $ST = \Omega$, by what proved in steps 1-5, we have

$$\begin{aligned} &\begin{pmatrix} \delta_{11}(\Omega_1) & \delta_{12}(\Omega_1) \\ 0 & 0 \end{pmatrix} = \delta(S)T + S\delta(T) \\ = &\begin{pmatrix} 0 & \delta_{12}(I_1) + \tau_{12}(W) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \Omega_1 & -WD \\ 0 & D \end{pmatrix} \\ &+ \begin{pmatrix} I_1 & W \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \delta_{11}(\Omega_1) & \delta_{12}(\Omega_1) - \tau_{12}(WD) + \varphi_{12}(D) \\ 0 & \varphi_{22}(D) \end{pmatrix} \\ = &\begin{pmatrix} \delta_{11}(\Omega_1) & \delta_{12}(I_1)D + \tau_{12}(W)D + \delta_{12}(\Omega_1) - \tau_{12}(WD) + \varphi_{12}(D) + W\varphi_{22}(D) \\ 0 & -\tau_{22}(W) \end{pmatrix}. \end{aligned}$$

Therefore,

$$\delta_{12}(\Omega_1) = \delta_{12}(I_1)D + \tau_{12}(W)D + \delta_{12}(\Omega_1) - \tau_{12}(WD) + \varphi_{12}(D) + W\varphi_{22}(D). \quad (3.11)$$

By Eqs.(3.5), (3.8) and (3.11), we get $\tau_{12}(W)D - \tau_{12}(WD) + W\varphi_{22}(D) = 0$. Thus

$$\tau_{12}(WD) = \tau_{12}(W)D + W\varphi_{22}(D) \quad (3.12)$$

holds for any $W \in \mathscr{A}_{12}$ and $D \in \mathscr{A}_{22}$.

Step 1.7. φ_{22} is a derivation.

The assertion of Step 1.6 implies that, for any $W \in \mathscr{A}_{12}$ and $D_1, D_2 \in \mathscr{A}_{22}$, we have

$$\begin{aligned} &\tau_{12}(W)D_1D_2 + W\varphi_{22}(D_1D_2) \\ = &\tau_{12}(WD_1D_2) = \tau_{12}(WD_1)D_2 + WD_1\varphi_{22}(D_2) \\ = &\tau_{12}(W)D_1D_2 + W\varphi_{22}(D_1)D_2 + WD_1\varphi_{22}(D_2). \end{aligned}$$

Thus one gets

$$W(\varphi_{22}(D_1D_2) - \varphi_{22}(D_1)D_2 - D_1\varphi_{22}(D_2)) = 0.$$

Because W is arbitrary, we see that

$$\varphi_{22}(D_1D_2) = \varphi_{22}(D_1)D_2 + D_1\varphi_{22}(D_2) \quad (3.13)$$

holds for all $D_1, D_2 \in \mathscr{A}_{22}$, that is, φ_{22} is a derivation.

Step 1.8. $\tau_{12}(CW) = C\tau_{12}(W) + \delta_{11}(C)W$ holds for all $C \in \mathscr{A}_{11}$ and $W \in \mathscr{A}_{12}$.

For any $W \in \mathscr{A}_{12}$ and any invertible operator $C_1 \in \mathscr{A}_{11}$, let $S = \begin{pmatrix} C_1 & -C_1W \\ 0 & 0 \end{pmatrix}$

and $T = \begin{pmatrix} C_1^{-1}\Omega_1 & W \\ 0 & I_2 \end{pmatrix}$; then $ST = \Omega$. By Eqs. (3.4)-(3.13),

$$\begin{aligned}
&\begin{pmatrix} \delta_{11}(\Omega_1) & \delta_{12}(\Omega_1) \\ 0 & 0 \end{pmatrix} = \delta(S)T + S\delta(T) \\
= &\begin{pmatrix} \delta_{11}(C_1) & \delta_{12}(C_1) - \tau_{12}(C_1W) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} C_1^{-1}\Omega_1 & W \\ 0 & I_2 \end{pmatrix} \\
&+ \begin{pmatrix} C_1 & -C_1W \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \delta_{11}(C_1^{-1}\Omega_1) & \delta_{12}(C_1^{-1}\Omega_1) + \tau_{12}(W) + \varphi_{12}(I_2) \\ 0 & 0 \end{pmatrix} \\
= &\begin{pmatrix} \delta_{11}(C_1)C_1^{-1}\Omega_1 + C_1\delta_{11}(C_1^{-1}\Omega_1) & T_2 \\ 0 & 0 \end{pmatrix},
\end{aligned}$$

where

$$T_2 = \delta_{11}(C_1)W + \delta_{12}(C_1) - \tau_{12}(C_1W) + C_1\delta_{12}(C_1^{-1}\Omega_1) + C_1\tau_{12}(W) + C_1\varphi_{12}(I_2).$$

Hence we have

$$\delta_{12}(\Omega_1) = \delta_{11}(C_1)W + \delta_{12}(C_1) - \tau_{12}(C_1W) + C_1\delta_{12}(C_1^{-1}\Omega_1) + C_1\tau_{12}(W) + C_1\varphi_{12}(I_2).$$

By Eqs.(3.5) and (3.8), we see that

$$\tau_{12}(C_1W) = \delta_{11}(C_1)W + C_1\tau_{12}(W) \tag{3.14}$$

holds for all invertible $C_1 \in \mathscr{A}_{11}$ and all $W \in \mathscr{A}_{12}$. For any $C \in \mathscr{A}_{11}$, substitute $C_1 = nI_1 - C$ with $n \in \mathbb{N}$ and $n > \|C\|$ in Eq.(3.14). Then we get $\delta_{11}(nI_1 - C)W + (nI_1 - C)\tau_{12}(W) = \tau_{12}((nI_1 - C)W)$. Thus

$$n\delta_{11}(I_1)W - \delta_{11}(C)W + n\tau_{12}(W) - C\tau_{12}(W) = n\tau_{12}(W) - \tau_{12}(CW).$$

Comparing the two sides of the above equation and applying Eq.(3.4), one sees that

$$\tau_{12}(CW) = C\tau_{12}(W) + \delta_{11}(C)W \tag{3.15}$$

holds for all $C \in \mathscr{A}_{11}$ and $W \in \mathscr{A}_{12}$.

Step 1.9. δ_{11} is a derivation.
For any $W \in \mathscr{A}_{12}$ and $C_1, C_2 \in \mathscr{A}_{11}$, it follows from Eq.(3.15) that

$$\begin{aligned}
&C_1C_2\tau_{12}(W) + \delta_{11}(C_1C_2)W \\
= &\tau_{12}(C_1C_2W) = C_1\tau_{12}(C_2W) + \delta_{11}(C_1)C_2W \\
= &C_1C_2\tau_{12}(W) + C_1\delta_{11}(C_2)W + \delta_{11}(C_1)C_2W,
\end{aligned}$$

which implies that

$$\delta_{11}(C_1C_2) = C_1\delta_{11}(C_2) + \delta_{11}(C_1)C_2 \tag{3.16}$$

for all $C_1, C_2 \in \mathscr{A}_{11}$, that is, δ_{11} is a derivation.

Step 1.10. δ is a derivation.
Now we are in a position to check that δ is a derivation.

For any $\begin{pmatrix} C_1 & W_1 \\ 0 & D_1 \end{pmatrix}, \begin{pmatrix} C_2 & W_2 \\ 0 & D_2 \end{pmatrix} \in \mathrm{Alg}\mathscr{N}$, where $C_1, C_2 \in \mathscr{A}_{11}$, $W_1, W_2 \in \mathscr{A}_{12}$, $D_1, D_2 \in \mathscr{A}_{22}$, by steps 1.1-1.9, we have

$$\begin{aligned} &\delta\left(\begin{pmatrix} C_1 & W_1 \\ 0 & D_1 \end{pmatrix}\begin{pmatrix} C_2 & W_2 \\ 0 & D_2 \end{pmatrix}\right) = \delta\begin{pmatrix} C_1C_2 & C_1W_2 + W_1D_2 \\ 0 & D_1D_2 \end{pmatrix} \\ &= \begin{pmatrix} \delta_{11}(C_1)C_2 + C_1\delta_{11}(C_2) & C_1C_2B - BD_1D_2 + \tau_{12}(C_1W_2) + \tau_{12}(W_1D_2) \\ 0 & \varphi_{22}(D_1)D_2 + D_1\varphi_{22}(D_2) \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} &\delta\left(\begin{pmatrix} C_1 & W_1 \\ 0 & D_1 \end{pmatrix}\right)\begin{pmatrix} C_2 & W_2 \\ 0 & D_2 \end{pmatrix} + \begin{pmatrix} C_1 & W_1 \\ 0 & D_1 \end{pmatrix}\delta\left(\begin{pmatrix} C_2 & W_2 \\ 0 & D_2 \end{pmatrix}\right) \\ &= \begin{pmatrix} \delta_{11}(C_1) & C_1B - BD_1 + \tau_{12}(W_1) \\ 0 & \varphi_{22}(D_1) \end{pmatrix}\begin{pmatrix} C_2 & W_2 \\ 0 & D_2 \end{pmatrix} \\ &\quad + \begin{pmatrix} C_1 & W_1 \\ 0 & D_1 \end{pmatrix}\begin{pmatrix} \delta_{11}(C_2) & C_2B - BD_2 + \tau_{12}(W_2) \\ 0 & \varphi_{22}(D_2) \end{pmatrix} \\ &= \begin{pmatrix} \delta_{11}(C_1)C_2 + C_1\delta_{11}(C_2) & T_3 \\ 0 & \varphi_{22}(D_1)D_2 + D_1\varphi_{22}(D_2) \end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} T_3 = &\ \delta_{11}(C_1)W_2 + C_1BD_2 - BD_1D_2 \\ &+ \tau_{12}(W_1)D_2 + C_1C_2B - C_1BD_2 + C_1\tau_{12}(W_2) + W_1\varphi_{22}(D_2) \\ = &\ C_1C_2B - BD_1D_2 + \tau_{12}(C_1W_2) + \tau_{12}(W_1D_2). \end{aligned}$$

Thus we get

$$\begin{aligned} &\delta\left(\begin{pmatrix} C_1 & W_1 \\ 0 & D_1 \end{pmatrix}\begin{pmatrix} C_2 & W_2 \\ 0 & D_2 \end{pmatrix}\right) \\ &= \delta\left(\begin{pmatrix} C_1 & W_1 \\ 0 & D_1 \end{pmatrix}\right)\begin{pmatrix} C_2 & W_2 \\ 0 & D_2 \end{pmatrix} + \begin{pmatrix} C_1 & W_1 \\ 0 & D_1 \end{pmatrix}\delta\left(\begin{pmatrix} C_2 & W_2 \\ 0 & D_2 \end{pmatrix}\right), \end{aligned}$$

i.e., δ is a derivation if Ω_1 is injective, as desired.

Case 2. Ω_1 has dense range. Let us check that δ is still a derivation step by step.

Step 2.1. $\delta_{11}(I_1) = 0$, $\delta_{22}(\Omega_1) = 0$, and $\delta_{12}(\Omega_1) = C_1\delta_{12}(C_2\Omega_1)$ holds for any $C_1, C_2 \in \mathscr{A}_{11}$ with $C_1C_2 = I_1$.

For any $C_1, C_2 \in \mathscr{A}_{11}$ with $C_1C_2 = I_1$ and any $D_1, D_2 \in \mathscr{A}_{22}$ with $D_1D_2 = 0$, let $S = \begin{pmatrix} C_1 & 0 \\ 0 & D_1 \end{pmatrix}$ and $T = \begin{pmatrix} C_2\Omega_1 & 0 \\ 0 & D_2 \end{pmatrix}$; then $ST = \Omega$. As δ is derivable at Ω, we have

$$\begin{aligned} &\begin{pmatrix} \delta_{11}(\Omega_1) & \delta_{12}(\Omega_1) \\ 0 & \delta_{22}(\Omega_1) \end{pmatrix} = \delta(S)T + S\delta(T) \\ &= \begin{pmatrix} \delta_{11}(C_1) + \varphi_{11}(D_1) & \delta_{12}(C_1) + \varphi_{12}(D_1) \\ 0 & \delta_{22}(C_1) + \varphi_{22}(D_1) \end{pmatrix}\begin{pmatrix} C_2\Omega_1 & 0 \\ 0 & D_2 \end{pmatrix} \\ &\quad + \begin{pmatrix} C_1 & 0 \\ 0 & D_1 \end{pmatrix}\begin{pmatrix} \delta_{11}(C_2\Omega_1) + \varphi_{11}(D_2) & \delta_{12}(C_2\Omega_1) + \varphi_{12}(D_2) \\ 0 & \delta_{22}(C_2\Omega_1) + \varphi_{22}(D_2) \end{pmatrix} \\ &= \begin{pmatrix} T_4 & \delta_{12}(C_1)D_2 + \varphi_{12}(D_1)D_2 + C_1\delta_{12}(C_2\Omega_1) + C_1\varphi_{12}(D_2) \\ 0 & \delta_{22}(C_1)D_2 + \varphi_{22}(D_1)D_2 + D_1\delta_{22}(C_2\Omega_1) + D_1\varphi_{22}(D_2) \end{pmatrix}, \end{aligned}$$

where

$$T_4 = \delta_{11}(C_1)C_2\Omega_1 + \varphi_{11}(D_1)C_2\Omega_1 + C_1\delta_{11}(C_2\Omega_1) + C_1\varphi_{11}(D_2).$$

Therefore,

$$\delta_{11}(\Omega_1) = \delta_{11}(C_1)C_2\Omega_1 + \varphi_{11}(D_1)C_2\Omega_1 + C_1\delta_{11}(C_2\Omega_1) + C_1\varphi_{11}(D_2), \tag{3.17}$$

$$\delta_{12}(\Omega_1) = \delta_{12}(C_1)D_2 + \varphi_{12}(D_1)D_2 + C_1\delta_{12}(C_2\Omega_1) + C_1\varphi_{12}(D_2), \tag{3.18}$$

and

$$\delta_{22}(\Omega_1) = \delta_{22}(C_1)D_2 + \varphi_{22}(D_1)D_2 + D_1\delta_{22}(C_2\Omega_1) + D_1\varphi_{22}(D_2). \tag{3.19}$$

Letting $D_1 = D_2 = 0$, by Eqs.(3.18) and (3.19), we get $\delta_{12}(\Omega_1) = C_1\delta_{12}(C_2\Omega_1)$ and $\delta_{22}(\Omega_1) = 0$ respectively. When $C_1 = C_2 = I_1$, by Eq.(3.17), we get $\delta_{11}(I_1)\Omega_1 = 0$. Because operator Ω_1 has dense range, we must have $\delta_{11}(I_1) = 0$. Thus we have shown that

$$\delta_{11}(I_1) = 0,\ \delta_{12}(\Omega_1) = C_1\delta_{12}(C_2\Omega_1),\ \delta_{22}(\Omega_1) = 0 \tag{3.20}$$

holds for any $C_1, C_2 \in \mathscr{A}_{11}$ with $C_1C_2 = I_1$.

Step 2.2. $\delta_{22} = 0$ and $\delta_{12}(C) = CB'$ for all $C \in \mathscr{A}_{11}$, where $B' = \delta_{12}(I_1)$.

Letting $C_1 = C_2 = I_1, D_1 = 0, D_2 = I_2$, by Eqs.(3.18) and (3.20), we get $\delta_{12}(I_1) + \varphi_{12}(I_2) = 0$; by Eqs.(3.19) and (3.20), we get $\delta_{22}(I_1) = 0$. Thus $\varphi_{12}(I_2) = -B'$, where $B' = \delta_{12}(I_1)$. Hence

$$\delta_{12}(I_1) = B', \delta_{22}(I_1) = 0, \varphi_{12}(I_2) = -B'. \tag{3.21}$$

Letting $D_1 = 0, D_2 = I_2$, by Eqs.(3.18) and (3.20), we have $\delta_{12}(C_1) = -C_1\varphi_{12}(I_2)$; by Eqs.(3.19) and (3.20), we get $\delta_{22}(C_1) = \delta_{22}(\Omega_1) = 0$. For any $C \in \mathscr{A}_{11}$, we can take positive integer n such that $n > \|C\|$. Then $nI_1 - C$ is invertible with its inverse still in $\mathscr{A}_{11}$. Thus we get $\delta_{12}(nI_1 - C) = -(nI_1 - C)\varphi_{12}(I_2)$, and $\delta_{22}(nI_1 - C)$=0. Because δ is an additive map, $n\delta_{12}(I_1) - \delta_{12}(C) = -n\varphi_{12}(I_2) + C\varphi_{12}(I_2)$. By Eq.(3.21) we have $\delta_{12}(C) = CB'$ and $\delta_{22}(C) = n\delta_{22}(I_1) = 0$. Thus

$$\delta_{12}(C) = CB', \delta_{22}(C) = 0 \tag{3.22}$$

holds for all $C \in \mathscr{A}_{11}$.

Step 2.3. $\varphi_{11} = 0$, $\varphi_{12}(D) = -B'D$ for all $D \in \mathscr{A}_{22}$.

Letting $C_1 = C_2 = I_1$, $D_2 = 0$, by Eqs.(3.17) and (3.20), we get $\varphi_{11}(D_1)\Omega_1 = 0$. Letting $C_1 = C_2 = I_1$, $D_1 = 0$, by Eqs.(3.18) and (3.20), we get $\varphi_{12}(D_2) = -\delta_{12}(I_1)D_2 = -B'D_2$. Since the operator Ω_1 has dense range, it follows that

$$\varphi_{11}(D) = 0, \varphi_{12}(D) = -B'D \tag{3.23}$$

holds for any $D \in \mathscr{A}_{22}$.

Step 2.4. $\tau_{11} = 0$ and $\tau_{22} = 0$.

For any $W\in\mathscr{A}_{12}$, we take $S=\begin{pmatrix}\Omega_1 & W\\ 0 & 0\end{pmatrix}$ and $T=\begin{pmatrix}I_1 & 0\\ 0 & 0\end{pmatrix}$, then $ST=\Omega$. By Steps 2.1-2.3,

$$\begin{aligned}\begin{pmatrix}\delta_{11}(\Omega_1) & \delta_{12}(\Omega_1)\\ 0 & 0\end{pmatrix} &= \delta(S)T+S\delta(T)\\ &= \begin{pmatrix}\delta_{11}(\Omega_1)+\tau_{11}(W) & \delta_{12}(\Omega_1)+\tau_{12}(W)\\ 0 & \tau_{22}(W)\end{pmatrix}\begin{pmatrix}I_1 & 0\\ 0 & 0\end{pmatrix}+\begin{pmatrix}\Omega_1 & W\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & \delta_{12}(I_1)\\ 0 & 0\end{pmatrix}\\ &= \begin{pmatrix}\delta_{11}(\Omega_1)+\tau_{11}(W) & \Omega_1\delta_{12}(I_1)\\ 0 & 0\end{pmatrix}.\end{aligned}$$

Thus we get that $\delta_{11}(\Omega_1)=\delta_{11}(\Omega_1)+\tau_{11}(W)$. It follows that

$$\tau_{11}(W)=0 \tag{3.24}$$

holds for any $W\in\mathscr{A}_{12}$.

Taking $S=\begin{pmatrix}I_1 & -W\\ 0 & 0\end{pmatrix}$ and $T=\begin{pmatrix}\Omega_1 & W\\ 0 & I_2\end{pmatrix}$ yields

$$\tau_{22}(W)=0 \tag{3.25}$$

for all $W\in\mathscr{A}_{12}$.

Step 2.5. $\tau_{12}(WD)=\tau_{12}(W)D+W\varphi_{22}(D)$ holds for all $W\in\mathscr{A}_{12}$ and $D\in\mathscr{A}_{22}$.

Taking $S=\begin{pmatrix}I_1 & W\\ 0 & 0\end{pmatrix}$ and $T=\begin{pmatrix}\Omega_1 & -WD\\ 0 & D\end{pmatrix}$ leads to that

$$\delta_{12}(\Omega_1)=\delta_{12}(I_1)D+\tau_{12}(W)D+\delta_{12}(\Omega_1)-\tau_{12}(WD)+\varphi_{12}(D)+W\varphi_{22}(D) \tag{3.26}$$

are true for any $W\in\mathscr{A}_{12}$ and $D\in\mathscr{A}_{22}$. Then by Eqs.(3.22) and (3.23), we get that $\tau_{12}(W)D-\tau_{12}(WD)+W\varphi_{22}(D)=0$, that is,

$$\tau_{12}(WD)=\tau_{12}(W)D+W\varphi_{22}(D) \tag{3.27}$$

holds for any W and D.

Step 2.6. φ_{22} is a derivation.

By Eq. (3.27) and a similar argument to that in Step 1.7 of Case 1, one can show that

$$\varphi_{22}(D_1D_2)=\varphi_{22}(D_1)D_2+D_1\varphi_{22}(D_2) \tag{3.28}$$

holds for all $D_1,D_2\in\mathscr{A}_{22}$ and hence φ_{22} is derivation.

Step 2.7. $\tau_{12}(CW)=C\tau_{12}(W)+\delta_{11}(C)W$ holds for all $C\in\mathscr{A}_{11}$ and $W\in\mathscr{A}_{12}$, and δ_{11} is a derivation.

Taking $S=\begin{pmatrix}C_1 & -C_1W\\ 0 & 0\end{pmatrix}$ and $T=\begin{pmatrix}C_1^{-1}\Omega_1 & W\\ 0 & I_2\end{pmatrix}$ one can get that

$$\delta_{12}(\Omega_1)=\delta_{11}(C_1)W+\delta_{12}(C_1)-\tau_{12}(C_1W)+C_1\delta_{12}(C_1^{-1}\Omega_1)+C_1\tau_{12}(W)+C_1\varphi_{12}(I_2)$$

It follows from Eqs.(3.22) and (3.23) that

$$\tau_{12}(CW) = C\tau_{12}(W) + \delta_{11}(C)W \tag{3.29}$$

holds for any $C \in \mathscr{A}_{11}$ and $W \in \mathscr{A}_{12}$. Now, by use of Eq.(2.29) it is easily checked that δ_{11} is a derivation.

Step 2.8. δ is a derivation.

Now, by use of Step 2.1-2.7, a similar argument to that in Step 1.10 of Case 1, one shows that δ is a derivation.

If $\dim X = \infty$, by [3], every additive derivation of $\mathrm{Alg}\mathscr{N}$ is linear. Hence, if δ is additive and derivable at Ω, then δ is a linear derivation and thus an inner derivation, that is, there exists an operator $T \in \mathrm{Alg}\mathscr{N}$ such that $\delta(A) = AT - TA$ for every $A \in \mathrm{Alg}\mathscr{N}$. This completes the proof of (a).

(b) Assume that $(I-P)\Omega(I-P) = \Omega$.

In this case Ω has the form $\Omega = \begin{pmatrix} 0 & 0 \\ 0 & \Omega_2 \end{pmatrix}$, where Ω_2 is injective or has dense range as an operator acting on $\ker P$. By a similar approach as that in Case 1 and Case 2 of (a), one can show that, if δ is derivable at Ω, then δ is again a derivation. We omit its proof here. $\square$

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