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IDEALS OF COMPACT OPERATORS WITH NAKANO TYPE NORMS IN A HILBERT SPACE

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Abstract. Let H be a separable Hilbert space with a norm $\|.\|_H$. For a compact linear operator A acting in H, let $\lambda_k(A)$ be the eigenvalues, $s_k(A)$ $(k=1,2,\dots)$ singular values and $\|A\|_H = \sup_{x\in H}\|Ax\|_H/\|x\|_H$. Let $\pi = \{p_k\}_{k=1}^{\infty}$ be a nondecreasing sequence of numbers $p_k \geqslant 1$. Put

$$\gamma_\pi(A) := \sum_{j=1}^{\infty} \frac{s_j^{p_j}(A)}{p_j}.$$

We investigate the ideal X_π of operators satisfying $\gamma_\pi(tA) < \infty$ for all $t > 0$. In particular, it is proved that for any $A \in X_\pi$ we have

$$\sum_{k=1}^{\infty} \frac{|\lambda_k(A)|^{p_k}}{p_k \nu_A^{p_k}} \leqslant \gamma_\pi(A/\nu_A),$$

where $\nu_A = \|A\|_H$ if $\|A\|_H > 1$ and $\nu_A = 1$ if $\|A\|_H \leqslant 1$.

1. Introduction and preliminaries

Let H be a separable Hilbert space with a scalar product $(.,.)$, the identity operator I and norm $\|.\|_H = \sqrt{(.,.)}$. For a compact linear operator A acting in H, A^* is the adjoint, $\lambda_k(A)$ are the eigenvalues and $s_k(A) = \sqrt{\lambda_k(A^*A)}$ $(k=1,2,\dots)$ are the singular values taken with their multiplicities and ordered in the decreasing way: $|\lambda_k(A)| \geqslant |\lambda_{k+1}(A)|$, $s_k(A) \geqslant s_{k+1}(A)$. Let $\pi = \{p_k\}_{k=1}^{\infty}$ be a nondecreasing sequence of numbers $p_k \geqslant 1$. Put

$$\gamma_\pi(A) := \sum_{j=1}^{\infty} \frac{s_j^{p_j}(A)}{p_j}$$

assuming that the series converges. We take the positive roots only. Denote by X_π the set of compact operators in H, such that $\gamma_\pi(tA) < \infty$ for all $t > 0$.

Let SN_p $(1 < p < \infty)$ be the Schatten-von Neumann ideal of operators A with the finite norm $N_p(A) := [\mathrm{Trace}(A^*A)^{p/2}]^{1/p}$. It is well-known that for any $A \in SN_p$,

$$\sum_{k=1}^{\infty} |\lambda_k(A)|^p \leqslant N_p^p(A). \tag{1.1}$$

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We will say that a compact operator in H is of infinite order if it does not belong to any Schatten-von Neumann ideal. Such operators arise in various applications. Many fundamental results on infinite order compact linear operators can be found in the well-known book [12, Section 3.1]. The literature on the ideals of compact operators and their applications is very rich, cf. the very interesting recent papers [1, 3, 5, 14, 17] and references cited therein. Especially, the Schatten-von Neumann ideals were deeply investigated [4, 8, 9, 15, 18, 20, 22]. Applications of the theory of the Schatten-von Neumann can be found in the papers [2, 7, 13, 21, 23]. About the classical results see [6, 10, 11]. Certainly we could not survey the whole subject here and refer the reader to the above listed publications and references given therein.

At the same time to the best of our knowledge, bounds for the eigenvalues of infinite order operators were almost not investigated in the available literature. The motivation of this paper is to generalize inequality (1.1) to the operators from X_π.

LEMMA 1.1. *X_π is a linear space.*

Proof. Indeed, $\gamma_\pi(ctA) \leqslant \gamma_\pi(|c|tA) < \infty$ for all $A \in X_\pi$ and $c \in \mathbb{C}$. In addition, as it is well-known, $s_{2k-1}(A+B) \leqslant s_k(A) + s_k(B)$ $(B \in X_\pi)$, cf. [11]. So

$$\gamma_\pi((A+B)/2) = \sum_{j=1}^{\infty} \frac{s_j^{p_j}((A+B)/2)}{p_j} = \sum_{k=1}^{\infty} \frac{s_{2k-1}^{p_{2k-1}}((A+B)/2)}{p_{2k-1}} + \frac{s_{2k}^{p_{2k}}((A+B)/2)}{p_{2k}}$$
$$\leqslant \sum_{k=1}^{\infty} \frac{1}{p_{2k-1}2^{p_{2k-1}}}(s_k(A)+s_k(B))^{p_{2k-1}} + \frac{1}{p_{2k}2^{p_{2k}}}(s_k(A)+s_k(B))^{p_{2k}}.$$

Take into account that

$$(a+b)^p \leqslant 2^{p-1}(a^p+b^p) \ \ (p \geqslant 1; a,b > 0). \tag{1.2}$$

Then

$$\gamma_\pi((A+B)/2) \leqslant \frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{p_{2k-1}}(s_k^{p_{2k-1}}(A) + s_k^{p_{2k-1}}(B)) + \frac{1}{p_{2k}}(s_k^{p_{2k}}(A) + s_k^{p_{2k}}(B)).$$

But, for all sufficiently large k, we have $s_k(A) \leqslant 1$ and therefore $s_k^{p_{2k-1}}(A) \leqslant s_k^{p_k}(A)$. Thus the series in the right-hand part of the latter inequality converge, since $\gamma_\pi(A)$, $\gamma_\pi(B) < \infty$. So $\gamma_\pi((A+B)/2) < \infty$. Now replacing $(A+B)/2$ by $t(A+B)$ we have $\gamma_\pi(t(A+B)) < \infty$. This proves the result. $\square$

LEMMA 1.2. *For all $A \in X_\pi$ and $c \in \mathbb{C}$ we have $\gamma_\pi(cA) \leqslant |c|\gamma_\pi(A)$ if $|c| \leqslant 1$ and $\gamma_\pi(cA) \geqslant |c|\gamma_\pi(A)$ if $|c| \geqslant 1$.*

Proof. Indeed, for all $p \geqslant 1$ we have $s_k^p(cA) = |c|^p s_k^p(A) \leqslant |c| s_k^p(A)$ if $|c| \leqslant 1$ and $s_k^p(cA) \geqslant |c| s_k^p(A)$ if $|c| \geqslant 1$. This proves the lemma. $\square$

Let Y be an arbitrary vector space over $\mathbb{C}$. A functional $m : Y \to [0,\infty)$ is called a modular, if it satisfies the properties: a) $m(x) = 0$ iff $x = 0$, b) $m(\alpha x) = m(x)$ for

$\alpha \in \mathbb{C}$ with $|\alpha| = 1$, c) $m(\alpha x + \beta y) \leqslant m(x) + m(y)$ if $\alpha, \beta > 0$ with $\alpha + \beta = 1$ for all $x, y \in Y$, cf. [19].

Now let Y be a space of sequences $x = (x_k)_{k=1}^{\infty}$, and $m(x) = m(x_1, x_2, \ldots)$ a modular on Y. For example,

$$m(x) = \sum_{k=1}^{\infty} \frac{|x_k|^{p_k}}{p_k}$$

is a modular, cf. [19].

For a compact operator A in H put

$$\hat{\gamma}(A) := m(s_1(A), s_2(A), \ldots).$$

Then $\hat{\gamma}(A)$ will be called a modular of A. So $\gamma_\pi(A)$ is a modular of A.

For an $A \in X_\pi$ put

$$\|A\|_\pi = \inf\{\lambda > 0 : \gamma_\pi(A/\lambda) \leqslant 1\}.$$

LEMMA 1.3. *$\|A\|_\pi$ is a norm in X_π.*

Proof. In the Nakano space of sequences $\{x_k\}_{k=1}^{\infty}$ satisfying

$$\sum_{k=1}^{\infty} \frac{|t x_k|^{p_k}}{p_k} < \infty \tag{1.3}$$

for all $t > 0$, introduce the (Luxemburg) norm

$$\|\{x_k\}\|_{\pi,L} = \inf\left\{\lambda > 0 : \sum_{k=1}^{\infty} \frac{|x_k/\lambda|^{p_k}}{p_k} \leqslant 1\right\}$$

cf. [19, Theorems 44.8 and 43.6]. We have

$$\|A\|_\pi = \|\{s_k(A)\}\|_{\pi,L}. \tag{1.4}$$

This proves the result. $\square$

Let us check that $\gamma_\pi(tA)$ is continuous in $t > 0$ for any $A \in X_\pi$. Indeed, for an integer p, $t > 0$ and $0 < \Delta < t$, we have

$$\frac{1}{t^p}[t^p - (t-\Delta)^p] \leqslant 1.$$

Hence,

$$\gamma_\pi(tA) - \gamma_\pi((t-\Delta)A) = \sum_{j=1}^{\infty} \frac{s_j^{p_j}(At)[t^{p_j} - (t-\Delta)^{p_j}]}{t^{p_j} p_j} \leqslant \gamma_\pi(tA).$$

Hence by the Lebesgue theorem, $\gamma_\pi(tA) - \gamma_\pi((t-\Delta)A) \to 0$ as $\Delta \to 0$.

Since $\gamma_\pi(A/\lambda)$ is continuous and decreases in $\lambda > 0$, we have

$$\gamma_\pi(A/\|A\|_\pi) = 1. \tag{1.5}$$

For a bounded linear operator T acting in H put $\|T\|_H := \sup_{x \in H} \|Tx\|_H / \|x\|_H$.

LEMMA 1.4. *The set X_π with the norm $\|A\|_\pi$ is a closed normed two-sided ideal in the algebra of all bounded linear operators on H. That is, if $A \in X_\pi$ and T is a bounded linear operator, then*

$$\|AT\|_\pi \leqslant \|A\|_\pi \|T\|_H, \ \|TA\|_\pi \leqslant \|T\|_H \|A\|_\pi.$$

Proof. It is well known that $s_j(AT) \leqslant s_j(A)\|T\|_H$ for all j (see e.g. [11, Chapter II, Section 2]). Assume that $\|A\|_\pi > 0$ and $\|T\|_H > 0$ (otherwise the proof is obvious). Then the definition of the norm $\|\cdot\|_\pi$ it follows that

$$\gamma_\pi(AT/\|A\|_\pi\|T\|_H) = \sum_{j=1}^{\infty} \frac{s_j^{p_k}(AT)}{p_k\|A\|_\pi^{p_k}\|T\|_H^{p_k}} \leqslant \sum_{j=1}^{\infty} \frac{s_j^{p_k}(A)}{p_k\|A\|_\pi^{p_k}} \leqslant 1.$$

But by (1.5) $\gamma_\pi(AT/\|AT\|_\pi) = 1$. Thus $\|AT\|_\pi \leqslant \|A\|_\pi\|T\|_H$. The second inequality is similarly proved. □

LEMMA 1.5. *The inequalities $\|A\|_\pi \leqslant 1$ and $\gamma_\pi(A) \leqslant \|A\|_\pi$ are fulfilled iff $\gamma_\pi(A) \leqslant 1$. In addition, we have $\|A\|_\pi \geqslant 1$ and $\gamma_\pi(A) \geqslant \|A\|_\pi$ iff $\gamma_\pi(A) \geqslant 1$.*

Proof. Clearly, $\gamma_\pi(A) \geqslant 1$, iff $\|A\|_\pi \geqslant 1$, since $\gamma_\pi(A/\|A\|_\pi) = 1$. Hence by Lemma 1.2 $\|A\|_\pi^{-1}\gamma_\pi(A) \geqslant 1$, as claimed. The rest of the proof is left to the reader. □

2. The main result

Put

$$\nu_A = \begin{cases} 1 & \text{if } \|A\|_H \leqslant 1, \\ \|A\|_H & \text{if } \|A\|_H > 1 \end{cases}.$$

THEOREM 2.1. *Let A be compact. Then*

$$\sum_{j=1}^{k} \frac{|\lambda_j(A)|^{p_j}}{\nu_A^{p_j} p_j} \leqslant \sum_{j=1}^{k} \frac{s_j^{p_j}(A)}{\nu_A^{p_j} p_j} \ \ (k = 1, 2, \ldots).$$

In particular, if $A \in X_\pi$, then

$$\sum_{j=1}^{\infty} \frac{|\lambda_j(A)|^{p_j}}{\nu_A^{p_j} p_j} \leqslant \gamma_\pi(A/\nu_A).$$

To prove this theorem, introduce the set Ω_π of operators $A \in X_\pi$ satisfying $s_1(A) = \|A\|_H \leqslant 1$.

LEMMA 2.2. *Let $A \in \Omega_\pi$. Then*

$$\sum_{j=1}^{k} \frac{|\lambda_j(A)|^{p_j}}{p_j} \leqslant \sum_{j=1}^{k} \frac{s_j^{p_j}(A)}{p_j} \quad (k = 1, 2, \ldots)$$

and therefore,

$$\sum_{j=1}^{\infty} \frac{|\lambda_j(A)|^{p_j}}{p_j} \leqslant \gamma_\pi(A).$$

Proof. Introduce the function

$$F(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} \frac{x_j^{p_j}}{p_j} \tag{2.1}$$

for $1 \geqslant x_1 \geqslant x_2 \geqslant \ldots \geqslant x_n \geqslant 0$. Then

$$x_k \frac{\partial F}{\partial x_k} = x_k^{p_k} \geqslant x_{k+1}^{p_{k+1}} = x_{k+1} \frac{\partial F}{\partial x_{k+1}}.$$

Now Theorem II.3.2 [11] implies the required result. □

Replacing A by $A/\|A\|_H$, by the previous lemma, we get the following result.

COROLLARY 2.3. *Let A be compact. Then*

$$\sum_{j=1}^{k} \frac{|\lambda_j(A)|^{p_j}}{\|A\|_H^{p_j} p_j} \leqslant \sum_{j=1}^{k} \frac{s_j^{p_j}(A)}{\|A\|_H^{p_j} p_j} \quad (k = 1, 2, \ldots).$$

In particular, if $A \in X_\pi$, then

$$\sum_{j=1}^{\infty} \frac{|\lambda_j(A)|^{p_j}}{\|A\|_H^{p_j} p_j} \leqslant \gamma_\pi(A/\|A\|_H).$$

Proof of Theorem 2.1. If $\|A\|_H \leqslant 1$, then Theorem 2.1 is valid due to Lemma 2.2. If $\|A\|_H \geqslant 1$, then required result is valid due to the previous corollary. □

Let us show that Theorem 2.1 really generalizes inequality (1.1). Indeed, let $p_k \equiv p \geqslant 1$. Then $\gamma(A) = N_p^p(A)/p$ and by Theorem 2.1,

$$\frac{1}{\nu_A^p p} \sum_{k=1}^{\infty} |\lambda_k(A)|^p \leqslant \frac{1}{p} N_p^p(A/\nu_A) = \frac{1}{\nu_A^p p} N_p^p(A).$$

So we have obtained (1.1).

Thanks to Theorem 2.1 and Lemma 1.5 we get.

COROLLARY 2.4. *Let* $\|A\|_\pi \leqslant 1$. *Then*

$$\sum_{j=1}^{\infty} \frac{|\lambda_j(A)|^{p_j}}{p_j} \leqslant \|A\|_\pi.$$

The following result shows that Ω_π is a convex set.

LEMMA 2.5. *Let* $A, B \in \Omega_\pi$. *Then*

$$\sum_{j=1}^{k} \frac{s_j^{p_j}((A+B)/2)}{p_j} \leqslant \sum_{j=1}^{k} \frac{s_j^{p_j}(A)+s_j^{p_j}(B)}{2p_j} \quad (k=1,2,\ldots)$$

and therefore

$$\gamma_\pi((A+B)/2) \leqslant \frac{1}{2}(\gamma_\pi(A)+\gamma_\pi(B)).$$

Proof. Using the function defined by (2.1) we have

$$\frac{\partial F}{\partial x_k} = x_k^{p_k-1} \geqslant x_k^{p_k}$$

since $x_k \leqslant 1$. Now Lemma II.3.5 and Remark II.3.3 from [11], imply

$$\sum_{j=1}^{k} \frac{s_j^{p_j}((A+B)/2)}{p_j} \leqslant \sum_{j=1}^{k} \frac{(s_j(A)+s_j(B))^{p_j}}{2^{p_j} p_j}$$

in view of the inequality

$$\sum_{j=1}^{k} s_j(A+B) \leqslant \sum_{j=1}^{k} s_j(A)+s_j(B). \tag{2.2}$$

Hence, applying inequality (1.2), we get the result. □

3. Dual ideals

In this section again $\pi = \{p_k\}_{k=1}^\infty$ is nondecreasing, but it is assumed that $p_k > 1$ $(k=1,2,\ldots)$. A non-increasing sequence $\pi^* = \{q_k\}_{k=1}^\infty$ of positive numbers $q_k \geqslant 1$ satisfying $1/q_k + 1/p_k = 1$ will be called *the sequence dual to* π. For a compact operators B acting in H put

$$\gamma_{\pi^*}(B) := \sum_{j=1}^{\infty} \frac{s_j^{q_j}(B)}{q_j},$$

provided the series converges. Denote by X_{π^*} the set of compact operators B in H, such that $\gamma_{\pi^*}(tB) < \infty$ for all $t > 0$. It is not hard to check that X_{π^*} is a linear space. We will call space X_{π^*} dual to X_π.

Furthermore, by the Young inequality [16], we arrive at the inequality

$$\sum_{k=1}^{\infty} s_k(A)s_k(B) \leqslant \gamma_\pi(A) + \gamma_{\pi^*}(B) \ \ (A \in X_\pi, B \in X_{\pi^*}). \tag{3.1}$$

Introduce the quantity

$$N_\pi(A) := \sup_{B \in X_{\pi^*}, \gamma_{\pi^*}(B)=1} \sum_{k=1}^{\infty} s_k(A)s_k(B) = \sup_{B \in X_{\pi^*}} \frac{1}{\gamma_{\pi^*}(B)} \sum_{k=1}^{\infty} s_k(A)s_k(B).$$

Clearly, $N_\pi(A) = 0$ iff $A = 0$; $N_\pi(cA) = |c|N_\pi(A)$ for all $c \in \mathbb{C}$. In addition, since $s_k(B)$ $(k = 1, 2, \ldots)$ decrease, according to (2.2), we obtain

$$\sum_{k=1}^{j} s_k(A + A_1)s_k(B) \leqslant \sum_{k=1}^{j} (s_k(A) + s_k(A_1))s_k(B),$$

and therefore

$$\begin{aligned} N_\pi(A + A_1) &= \sup_{B \in X_{\pi^*}, \gamma_{\pi^*}(B)=1} \sum_{k=1}^{\infty} (s_k(A) + s_k(A_1))s_k(B) \\ &\leqslant \sup_{B \in X_{\pi^*}, \gamma_{\pi^*}(B)=1} \sum_{k=1}^{\infty} s_k(A)s_k(B) + \sup_{B_1 \in X_{\pi^*}, \gamma_{\pi^*}(B_1)=1} \sum_{k=1}^{\infty} s_k(A_1)s_k(B_1) \\ &= N_\pi(A) + N_\pi(A_1). \end{aligned}$$

So $N_\pi(.)$ is a norm. Similarly the norm $N_{\pi^*}(B)$ for a $B \in X_{\pi^*}$ is defined.

LEMMA 3.1. *Space X_π with norm $N_\pi(.)$ is a two-sided ideal in the space of linear bounded operators in H. Moreover, $N_\pi(TA) \leqslant \|T\|_H N_\pi(A)$ and $N_\pi(AT) \leqslant \|T\|_H N_\pi(A)$ for any linear bounded operator T in H and any $A \in X_\pi$.*

Proof. Indeed,

$$\begin{aligned} N_\pi(TA) &:= \sup_{B \in X_\pi^*, \gamma_{\pi^*}(B)=1} \sum_{k=1}^{\infty} s_k(TA)s_k(B) \\ &\leqslant \sup_{B \in X_{\pi^*}, \gamma_{\pi^*}(B)=1} \sum_{k=1}^{\infty} \|T\|_H s_k(TA)s_k(B) = \|T\|_H N_\pi(A). \end{aligned}$$

Similarly, the second inequality is checked. □

LEMMA 3.2. *The generalized Hőlder inequality*

$$\sum_{k=1}^{\infty} s_k(A)s_k(B) \leqslant N_\pi(A)\|B\|_{\pi^*} \ \ (A \in X_\pi, B \in X_{\pi^*}) \tag{3.2}$$

is true.

Proof. We have

$$\sum_{k=1}^{\infty} s_k(A)s_k(B) = \|B\|_{\pi^*} \sum_{k=1}^{\infty} s_k(A)s_k(B_1)$$

where $B_1 = B/\|B\|_{\pi^*}(B)$. So $\|B_1\|_{\pi^*} = 1$. Hence $\gamma_{\pi^*}(B_1) \leqslant 1$ and

$$\sum_{k=1}^{\infty} s_k(A)s_k(B) \leqslant \|B\|_{\pi^*} \sup_{B_1 \in X_\pi^*, \gamma_{\pi^*}(B_1) \leqslant 1} \sum_{k=1}^{\infty} s_k(A)s_k(B_1) = \|B\|_{\pi^*} N_\pi(A),$$

as claimed. □

Due to (3.1),

$$\sum_{k=1}^{j} t s_k(A) t^{-1} s_k(B) = t \sum_{k=1}^{j} s_k(At^{-1}) s_k(B) \leqslant t(\gamma_\pi(At^{-1}) + \gamma_{\pi^*}(B)).$$

We thus have

$$N_\pi(A) \leqslant t(\gamma_\pi(At^{-1}) + 1) \tag{3.3}$$

for an arbitrary $t > 0$.

THEOREM 3.3. *The inequalities* $\|A\|_\pi \leqslant N_\pi(A) \leqslant 2\|A\|_\pi$ *are true.*

Proof. Take in (3.3) $t = \|A\|_\pi$. Then by (1.5)

$$N_\pi(A) \leqslant \|A\|_\pi (\gamma_\pi(A\|A\|_\pi^{-1}) + 1) \leqslant 2\|A\|_\pi. \tag{3.4}$$

Furthermore, take an operator B with $s_k(B) = s_k^{p_k - 1}(A)$. Then

$$\gamma_{\pi^*}(B) = \sum_{k=1}^{\infty} s_k^{q_k(p_k-1)}(A) = \sum_{k=1}^{\infty} s_k^{p_k}(A) = \gamma_\pi(A)$$

and

$$\sum_{k=1}^{\infty} s_k(A)s_k(B) = \gamma_\pi(A).$$

Hence, by the previous lemma

$$\gamma_{\pi^*}(B) = \gamma_\pi(A) \leqslant \|B\|_{\pi^*} N_\pi(A). \tag{3.5}$$

Now take $A_1 = A/\|A\|_\pi$ and $s_k(B_1) = s_k^{p_k - 1}(A_1)$. Then according to (3.5),

$$\gamma_\pi(A_1) \leqslant \|B_1\|_{\pi^*} N_\pi(A_1).$$

But

$$1 = \|A_1\|_\pi = \gamma_\pi(A_1) = \gamma_{\pi^*}(B_1) = \|B_1\|_{\pi^*}.$$

So

$$1 \leqslant N_\pi(A_1) = N_\pi(A)/\|A\|_\pi.$$

This and (3.4) prove the theorem. □

The previous result and Corollary 2.4 imply

COROLLARY 3.4. *Let* $N_\pi(A) \leqslant 1$. *Then*

$$\sum_{j=1}^{\infty} \frac{|\lambda_j(A)|^{p_j}}{p_j} \leqslant N_\pi(A).$$

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