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SIMILARITY TO AN ISOMETRY OF COMPOSITION OPERATORS ON THE HALF–PLANE

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Abstract. Necessary and sufficient conditions are already known in the Hardy spaces of both the disc and the half plane for a composition operator to be an isometry, by Nordgren in the disc [6] and by Chalendar and Partington in the half plane [2]. All the same, conditions for such an operator to be similar to an isometry have taken much longer to find. We present some necessary conditions for general weighted composition operators to be similar to an isometry, and use them to produce a complete characterisation of the rational composition operators on $H^p(\mathbb{C}^+)$ which have this property.

1. Introduction

Let $\mathbb{C}^+$ denote the upper half of the complex plane, that is

$$\mathbb{C}^+ = \{z \in \mathbb{C} : \Im(z) > 0\}.$$

We recall from, for example [5], that the Hardy space $H^p(\mathbb{C}^+)$ is the collection of analytic functions f on $\mathbb{C}^+$ for which the norm

$$\|f\|^p = \sup_{y>0} \int_{\mathbb{R}} |f(x+iy)|^p dx$$

is finite. In fact, by extending H^p functions non-tangentially to $\mathbb{R}$, we see that H^p is a subspace of $L^p(\mathbb{R})$, and so in particular H^2 is a Hilbert space, with inner product

$$\langle f, g\rangle = \int_{\mathbb{R}} f(x)\overline{g(x)}dx.$$

For an analytic map $\phi : \mathbb{C}^+ \to \mathbb{C}^+$, the composition operator with symbol ϕ is the map

$$C_\phi f = f \circ \phi,$$

which can be defined on any space of functions on $\mathbb{C}^+$, in particular the Hardy spaces. Not all such operators are bounded, or indeed map H^p spaces into themselves, for characterisations of such properties, see [4].

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Letting m denote normalised Lebesgue measure on the unit circle $\mathbb{T}$, we recall (see for example [3]), that for $1 \leqslant p < \infty$ the Hardy space $H^p(\mathbb{D})$ is the collection of analytic functions f on the disc for which

$$\|f\|^p := \sup_{0<r<1} \int_{\mathbb{T}} |f(rt)|^p dm(t) < \infty.$$

We can similarly define composition operators on the disc in the natural way.

It is well know that the disc and half plane are conformally equivalent via the Möbius transformation

$$J : \mathbb{D} \to \mathbb{C}^+ \qquad z \mapsto i\left(\frac{1-z}{1+z}\right).$$

This map induces (not directly, but with the addition of a weight) an invertible isometry between corresponding Hardy spaces of the disc and half-plane, which also gives us the following Lemma which is easily verified (see for example [2]).

LEMMA 1.1. *If $\phi : \mathbb{C}^+ \to \mathbb{C}^+$ is an analytic self-map of the upper half-plane, then the composition operator $C_\phi : H^p(\mathbb{C}^+) \to H^p(\mathbb{C}^+)$ is equivalent via an invertible isometry to the weighted composition operator $L_\Phi : H^p(\mathbb{D}) \to H^p(\mathbb{D})$, given by*

$$(L_\Phi f)(z) = \left(\frac{1+\Phi(z)}{1+z}\right)^{2/p} C_\Phi f(z),$$

where $\Phi = J^{-1} \circ \phi \circ J$.

DEFINITION 1.2. We say an operator T is *similar to an isometry* if there is a bounded invertible operator S with bounded inverse and an isometry U such that

$$T = S^{-1} U S.$$

In [1], Frédéric Bayart proved that a composition operator C_ϕ on $H^2(\mathbb{D})$ is similar to an isometry if and only if ϕ is inner and has a fixed point in the disc. This result correlates well with Nordgren's result [6] that such an operator is a genuine isometry if and only if ϕ is inner and fixes 0.

In the case of the half plane, the condition for being an isometry is somewhat less concrete, but nonetheless has echoes of the case of the disc. As mentioned above, the Hardy spaces of the disc and half-plane are equivalent as Banach spaces, but as can be seen in Lemma 1.1, composition operators act somewhat differently in the two cases. Results from [2] tell us the situation here. In this setting, yet again, in order for C_ϕ to be isometric it must be have properties like an inner function, which is to say it must map the boundary of the half plane to itself almost everywhere. The second condition, however, is that

$$\int_{\mathbb{T}} \left|\frac{1+\Phi(z)}{1+z}\right|^2 dm(z) = 1,$$

where $\Phi = J^{-1} \circ \phi \circ J$, as above. This condition is a fairly natural one to arise, as we note from Lemma 1.1 that C_ϕ on $H^2(\mathbb{C}^+)$ is equivalent to the weighted composition operator

$$\frac{1+\Phi(z)}{1+z} C_\Phi$$

on $H^2(\mathbb{D})$. The result is a strong one, since it applies to all composition operators on $H^2(\mathbb{C}^+)$, but is relatively complicated to check, and doesn't provide a huge amount of insight into the nature of isometric composition operators on the half-plane.

In this paper, we will prove a number of results about similarity to an isometry, both of weighted composition operators on the disc, and composition operators on the half plane, ending with a complete and simple characterisation for rationally induced composition operators on $H^p(\mathbb{C}^+)$.

2. Some necessary conditions

We begin with the following theorem of Béla Szőkefalvi-Nagy [9].

THEOREM 2.1. *An operator T on a Hilbert space $\mathscr{H}$ is similar to an isometry if and only if there is some constant $k > 0$ such that*

$$\frac{1}{k}\|x\| \leqslant \|T^n x\| \leqslant k\|x\|,$$

for all $x \in \mathscr{H}$ and all $n \in \mathbb{N}$.

The property is certainly also necessary for an operator on a general Banach space to be similar to an isometry, since if $T = S^{-1}US$ then for all vectors x and all $n \in \mathbb{N}$, we have

$$\|T^n x\| = \left\|S^{-1}U^n Sx\right\| \leqslant \underbrace{\|S^{-1}\| \cdot \overset{1}{\cancel{\|U^n\|}} \cdot \|S\|}_{=k} \cdot \|x\|.$$

Similarly, since S^{-1} is bounded, S must be bounded below, and so the lower bound follows in the same fashion.

It will be helpful to consider weighted composition operators on the disc instead of composition operators on the half plane. We recall that for $\psi : \mathbb{D} \to \mathbb{C}$, and $\phi : \mathbb{D} \to \mathbb{D}$, the weighted composition operator $W_{\phi,\psi}$ is given by

$$W_{\phi,\psi} f(z) = \psi(z)(f \circ \phi)(z),$$

which, depending on ϕ and ψ, may be allowed to act on a number of function spaces on the disc. We begin with a Lemma.

LEMMA 2.2. *Let ϕ be a rational self map of $\mathbb{D}$. Then if ϕ is not inner, we must have $|\phi| = 1$ in at most finitely many places on $\mathbb{T}$.*

Proof. By transporting to the half plane, we note that the equivalent formulation is: if ϕ is a rational self-map of the half-plane, then on the real line the real part of ϕ must either always be zero, or be zero in at most finitely many places. Since for real z, the real part of ϕ is just another rational function, this is certainly true. □

The following proposition follows a similar proof to that of Bayart in [1].

PROPOSITION 2.3. *Let $W_{\phi,\psi}$ be a weighted composition operator on the disc such that ϕ is rational. Then if $W_{\phi,\psi}$ is both bounded and bounded below on $H^p(\mathbb{D})$, ϕ must be inner.*

Proof. Let $W_{\phi,\psi}$ be such an operator, namely

$$W_{\phi,\psi}f(z) = \psi(z)f \circ \phi(z).$$

Since $W_{\phi,\psi}$ is bounded, by considering its action on the constant function $\mathbf{1}$ we see that ψ must be in $H^p(\mathbb{D})$. Now since $W_{\phi,\psi}$ is bounded below, there is some $k > 0$ such that for all $f \in H^p(\mathbb{D})$

$$\frac{1}{k}\|f\| \leqslant \|W_{\phi,\psi}f\|. \tag{2.1}$$

Suppose that ϕ is not inner. Then since ϕ is rational, $|\phi| = 1$ in at most finitely many places on $\mathbb{T}$ by Lemma 2.2, that is

$$m(\{z \in \mathbb{T} : |\phi(z)| < 1\}) = 1.$$

Set $B = \{z \in \mathbb{T} : |\phi(z)| < 1\}$, and $B' = \phi^{-1}(B) \cap \mathbb{T}$. We now define a function $f \in H^p(\mathbb{D})$ such that

$$|f| = \begin{cases} 1 & \text{on } B \\ 1/2 & \text{on } \mathbb{T} \setminus B \end{cases},$$

which must exist since $\log|f|$ is p-integrable, so we may construct f using the theory of outer functions. Then $\|f^j\|^p \longrightarrow m(B)$ as $j \to \infty$, and

$$\|W_{\phi,\psi}f^j\|^p = \int_{B'} |\psi(z)|^p |f^j \circ \phi(z)|^p dm(z) + \int_{\mathbb{T}\setminus B'} |\psi(z)|^p |f^j \circ \phi(z)|^p dm(z).$$

By applying the maximum modulus principle to f, the integrand in the second term on the right hand side is dominated by $|\psi(z)|^p$ and converges pointwise to 0 as $j \to \infty$. As such, by the Dominated Convergence Theorem, as $j \to \infty$, the second term on the right hand side of the equations tends to zero, and so,

$$\|W_{\phi,\psi}f^j\|^p \to \int_{B'} |\psi(z)|^p dm(z).$$

Now taking limits and p^{th} powers in (2.1), we get

$$\left(\frac{1}{k^p}\right) m(B) \leqslant \int_{B'} |\psi(z)|^p dm(z). \tag{2.2}$$

and hence, since $m(B) = 1$,

$$\int_{B'} |\psi(z)|^p dm \geqslant \frac{1}{k^p}.$$

In particular, therefore, $m(B') > 0$. But $B \cap B' = \emptyset$, since $\phi(B) \subset \mathbb{D}$, so

$$m(B \cup B') = m(B) + m(B') = 1 + m(B') > 1,$$

which is a contradiction, since the measure of the entire circle is 1. □

COROLLARY 2.4. *Let $W_{\phi,\psi}$ be a weighted composition operator on the disc such that ϕ is rational. Then if $W_{\phi,\psi}$ is similar to an isometry on $H^p(\mathbb{D})$, ϕ must be inner.*

Proof. An operator which is similar to an isometry is both bounded and bounded below (by Theorem 2.1), so it satisfies the conditions for Proposition 2.3 □

Since composition operators on the half plane are equivalent to weighted composition operators on the disc, it is clear that we can make the same claims about such operators, under similar assumptions. It is a special case of Corollary 2.4, for example, that all rationally induced composition operators on the half plane which are similar to an isometry must have a symbol which take real values to real values almost everywhere. With this in mind, we can now use our results to help characterise similarity to an isometry on the half-plane.

3. Results for composition operators on the Half plane

We will be making use of the following from Pólya and Szegő's book [7], page 79.

LEMMA 3.1. *The rational maps r of $\mathbb{R}$ for which*

$$\int_{\mathbb{R}} f(r(t))dm(t) = \int_{\mathbb{R}} f(t)dm(t)$$

for all f for which the right hand side exists are precisely those maps of the form

$$r(z) = \pm \left(z + \alpha + \sum_{i=1}^{n} \frac{\mu_i}{z - \gamma_i} \right),$$

where $\alpha \in \mathbb{R}$, the γ_i are distinct with $\gamma_i \in \mathbb{R}$ for each i, and $\mu_i < 0$ for each i.

We note that if $r(\mathbb{C}^+) \subseteq \mathbb{C}^+$, then certainly r being of the above form will imply C_r is an isometry on $L^p(\mathbb{R})$ and hence $H^p(\mathbb{C}^+)$ by applying Lemma 3.1 to $|f|^p$.

In addition, we have the following, coming from Lemma 9.3.3 (p319) of [8].

LEMMA 3.2. *Let r be a rational function with real co-efficients of degree n, which maps $\mathbb{C}^+$ into itself. Then r is of the form*

$$r(z) = az + \alpha + \sum_{i=1}^{m} \frac{\mu_i}{z - \gamma_i},$$

for some real a, α, μ_i and γ_i, where each $\mu_i < 0$ and $a \geqslant 0$. If $a = 0$ then $m = n$, otherwise $m = n - 1$

The following result from [4] will help us look at norms of iterates of C_ϕ.

LEMMA 3.3. *Let ϕ be an analytic self-map of $\mathbb{C}^+$. Then for any $1 < p < \infty$, C_ϕ is bounded on $H^p(\mathbb{C}^+)$ if and only ϕ has a finite angular derivative at ∞. In this case, $\|C_\phi\|$ is the p^{th} root of that derivative.*

We note that as defined in [4], the angular derivative of a rational map at ∞ is equivalent to

$$\lim_{|z| \to \infty} \frac{z}{r(z)}.$$

We are now in a position to prove our main theorem: a classification of the rationally induced composition operators on $H^p(\mathbb{C}^+)$ which are similar to an isometry.

THEOREM 3.4. *If r is a rational self map of $\mathbb{C}^+$, the following are equivalent:*

1. *the composition operator C_r is an isometry on some (or equivalently every) $H^p(\mathbb{C}^+)$.*
2. *the composition operator C_r is similar to an isometry on some (or equivalently every) $H^p(\mathbb{C}^+)$.*
3. *r takes real values almost everywhere on $\mathbb{R}$ and $\lim_{|z| \to \infty} r(z)/z = 1$.*

Proof. That $(1) \Rightarrow (2)$ is of course trivial, and we have already seen as a special case of Corollary 2.4 that (2) implies r preserves the real line almost everywhere. By Lemma 3.3, we see that the second condition in (3) is equivalent to the statement that C_r has norm 1. The angular derivative at ∞ of ϕ composed with itself n times is equal to the n^{th} power of the angular derivative of ϕ, so $\|C_r^n\| = \|C_r\|^n$ for composition operators on the half plane. As such, if $\|C_r\| \neq 1$, we would have

$$\|C_r^n\| \to \infty$$

or

$$\|C_r^n\| \to 0$$

as $n \to \infty$, which would contradict the Szőkefalvi-Nagy condition (Theorem 2.1). As such, $(2) \Rightarrow (3)$.

To get $(3) \Rightarrow (1)$ we need to show that all rational functions satisfying (3) are of the form given in Lemma 3.1.

Since r maps the real line to itself almost everywhere, it must be possible to write it with only real co-efficients. Recall from Lemma 3.2 that such an r must have the form

$$r(z) = az + \alpha + \sum_{i=1}^{m} \frac{\mu_i}{z - \gamma_i},$$

for some real a, α, μ_i and γ_i, where each $\mu_i < 0$ and $a \geqslant 0$. Since $r(z)/z \to 1$ as $|z| \to \infty$, we must have $a = 1$, so for r of the form (3) we must have

$$r(z) = z + \alpha + \sum_{i=1}^{n} \frac{\mu_i}{z - \gamma_i}$$

for some real α, μ_i and γ_i, where each $\mu_i < 0$. This is precisely the condition we require to satisfy Lemma 3.1, so C_r is an isometry and $(3) \Rightarrow (1)$. The fact that condition (3) is independent of p tells us that the result holds equivalently for each p. $\square$

We note that although Theorem 3.4 only covers rational maps, unlike the Chalendar-Partington Theorem, condition (3) is extremely easy to check, and can simply be read off from any rational function. Additionally, this result also describes those operators which are similar to an isometry, for which no condition was previously known. It is also worthy of note that as with many results on properties of composition operators, the condition we have found is independent of the choice of p.

The result showns another in a growing list of genuine differences between the disc and half-plane cases. On the disc, even for rational maps, not all composition operators which are similar to an isometry are necessarily themselves isometries. Let us consider the standard automorphism f_a of the disc given by

$$f_a(z) = \frac{z - a}{1 - \bar{a}z}.$$

If R is any non-trivial rotation of $\mathbb{D}$ and $a \neq 0$, then certainly $\tau = f_a^{-1} \circ R \circ f_a$ is an inner rational map which fixes a but not 0, and hence C_τ is similar to an isometry, but is not a genuine isometry.

Although Theorem 3.4 holds only for rational functions, it seems not unreasonable to conjecture that, much as in the disc, all composition operators similar to an isometry on $H^2(\mathbb{C}^+)$ ought to have a symbol which preserves the real line almost everywhere. Indeed, it seems likely that this condition will extend to all p in both the disc and half-plane cases. The equivalence of all three conditions for general maps might be a little too much to ask, but would certainly be an interesting result if it were true, and the equivalence of (2) and (3) is certainly a possibility.

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