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ON JOINT SPECTRUM OF INFINITE DIRECT SUMS

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Abstract. For families of uniformly bounded n-tuples $T_k=(T_k^1,\cdots,T_k^n),k=1,2,\cdots$ of commuting operators on $\mathscr{H}$, the joint spectrum of $\oplus_{k=1}^{\infty}T_k$ is considered.

Let $\mathscr{H}$ be an infinite dimensional complex separable Hilbert space and $\mathscr{B}(\mathscr{H})$ denote the algebra of all bounded linear operators acting on $\mathscr{H}$. By $\mathrm{Sp}(T)$ we denote the joint Taylor [5][6] spectrum of $T=(T_1,\cdots,T_n)$, an n-tuple of commuting operators on $\mathscr{H}$. Recall that $\mathrm{Sp}(T)$ consists of all points $\lambda=(\lambda_1,\cdots,\lambda_n)$ in $\mathbb{C}^n$ such that the Koszul complex $\mathrm{K}_*(T-\lambda,\mathscr{H})$ of the operators $(T_1-\lambda_1,\cdots,T_n-\lambda_n)$ is not exact. Let $\mathrm{Spp}(T)$ denote the joint point spectrum of $T=(T_1,\cdots,T_n)$, i.e.,

$$\mathrm{Spp}(T)=\{\lambda=(\lambda_1,\cdots,\lambda_n);\text{there exists } x\in\mathscr{H},x\neq 0,$$

$$\text{such that } (\lambda_i I-T_i)x=0,i=1,2,\cdots,n\}.$$

J. Pushpa and S. M. Patel [4] showed for two n-tuples $A=(A_1,\cdots,A_n)$ and $B=(B_1,\cdots,B_n)$ of commuting bounded operators on $\mathscr{H}$, the joint spectrum of $A\oplus B=(A_1\oplus B_1,\cdots,A_n\oplus B_n)$ equals to the union of the joint spectrum of A and B.

A natural question is: For families of uniformly bounded n-tuples $T_k=(T_k^1,\cdots,T_k^n)$ of commuting operators on $\mathscr{H}$, is the joint spectrum of $\oplus_{k=1}^{\infty}T_k$ the union of the joint spectrum of T_k?

Unfortunately, that is false.

EXAMPLE 1. Let $T_k=(T_k^1,T_k^2)$, and $T_k^1=T_k^2=\begin{bmatrix}0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & \\ & & & \end{bmatrix}_{n\times n}$.

Then we have:

$$\{(\lambda,\lambda),|\lambda|\leqslant 1\}=\mathrm{Sp}(\oplus_{k=1}^{\infty}T_k)\neq\bigcup_{k=1}^{\infty}\mathrm{Sp}(T_k)=\{(0,0)\}.$$

However, by considering the joint point spectrum, we obtain the following theorem:

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THEOREM 2. *Let $T_k = (T_k^1, \cdots, T_k^n), k = 1, 2, \cdots$ be families of uniformly bounded n-tuples of commuting operators on $\mathscr{H}$, then:*

$$\operatorname{Spp}(\oplus_{k=1}^{\infty} T_k) = \bigcup_{k=1}^{\infty} \operatorname{Spp}(T_k).$$

Proof. The fact $\oplus_{k=1}^{\infty} T_k^i$ is bounded on $\tilde{\mathscr{H}}$, for each $i = 1, \cdots, n$, follows from the fact $(T_k)_{k=1}^{\infty}$ are uniformly bounded operator tuples, i.e., there is $M \geqslant 0$, such that

$$||T_k^i|| \leqslant M, \;\; i = 1, 2, \cdots, n; \;\; k = 1, 2, \cdots,$$

where $\tilde{\mathscr{H}} = \mathscr{H} \oplus \mathscr{H} \oplus \mathscr{H} \oplus \cdots$.

Let $x = \oplus_{k=1}^{\infty} x_k \in \tilde{\mathscr{H}}$, $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{C}^n$, and assume that:

$$(\lambda - \oplus_{k=1}^{\infty} T_k)x = \oplus_{k=1}^{\infty}((\lambda_1 - T_k^1)x, \cdots, (\lambda_n - T_k^n)x) = 0.$$

Therefore, either $x = 0$ or $\lambda \in \operatorname{Spp}(\oplus_{k=1}^{\infty} T_k)$, hence

$$\operatorname{Spp}(\oplus_{k=1}^{\infty} T_k) = \bigcup_{k=1}^{\infty} \operatorname{Spp}(T_k). \quad \square$$

REMARK 3. By Theorem 2, the condition of $T_k = (T_k^1, \cdots, T_k^n), k = 1, 2, \cdots$ being n-tuples of commuting operators is not necessary. However we do not know much about the non-commutative operator tuples. The theorem can be seen as some work on non-commutative operator tuples.

To get the relation between $\operatorname{Sp}(\oplus_{k=1}^{\infty} T_k)$ and $\bigcup_{k=1}^{\infty} \operatorname{Sp}(T_k)$ in details, we need study the Koszul complex $\mathrm{K}_*(T, \mathscr{H})$.

Let n_k be a sequence of nonnegative numbers with $n_k = 0$, for $k < 0$, $\mathscr{H}_k = \mathscr{H} \otimes \mathbb{C}^{n_k}$ and $d_k \in \mathscr{B}(\mathscr{H}_k, \mathscr{H}_{k-1})$ such that for all k, $d_k \circ d_{k+1} = 0$. Then the complex is

$$\cdots \xrightarrow{d_{k+1}} \mathscr{H}_k \xrightarrow{d_k} \mathscr{H}_{k-1} \xrightarrow{d_{k-1}} \cdots \xrightarrow{d_2} \mathscr{H}_1 \xrightarrow{d_1} \mathscr{H}_0 \longrightarrow 0 .$$

If $T = (T_1, \cdots, T_n)$ is an n-tuple of commuting operators on $\mathscr{H}$, the Koszul complex $\mathrm{K}_*(T, \mathscr{H})$ is the one we get by taking $n_k = \begin{pmatrix} n \\ k \end{pmatrix}$ and

$$d_k(x \otimes e_{j_1} \wedge \cdots \wedge e_{j_k}) = \sum_{i=1}^{k} (-1)^{i+1} T_{j_i} x \otimes e_{j_1} \wedge \cdots \wedge \bar{e}_{j_i} \wedge \cdots \wedge e_{j_k}.$$

R. Curto [1] introduced an operator matrix corresponding to $T = (T_1, \cdots, T_n)$, defined as:

$$\hat{T} = \begin{pmatrix} d_1 & & \\ d_2^* & d_3 & \\ & \ddots & \ddots \end{pmatrix} \in \mathscr{B}(\mathscr{H} \otimes \mathbb{C}^{2^{n-1}}).$$

LEMMA 4. *Let* $T=(T_1,\cdots,T_n)$ *be an* n*-tuple of commuting operators on* $\mathscr{H}$*, then* $\lambda=(\lambda_1,\cdots,\lambda_n)\in \mathrm{Sp}(T)$ *if and only if* $(T-\lambda)\hat{}$ *is not invertible.*

LEMMA 5. *Let* $T_k=(T_k^1,\cdots,T_k^n), k=1,2,\cdots$ *be families of uniformly bounded* n*-tuples of commuting operators on* $\mathscr{H}$*, then* $(\oplus_{k=1}^{\infty}T_k)\hat{}$ *is unitarily equivalent to* $\oplus_{k=1}^{\infty}(T_k)\hat{}$ *.*

Proof. Since $(\oplus_{k=1}^{\infty}T_k)\hat{}$ is a bounded operator in $\mathscr{B}(\tilde{\mathscr{H}}\otimes\mathbb{C}^{2^{n-1}})$ and $\oplus_{k=1}^{\infty}(T_k)\hat{}$ is a bounded operator in $\mathscr{B}(\tilde{\mathscr{H}})$, where $\tilde{\mathscr{H}}=\mathscr{H}\oplus\mathscr{H}\oplus\mathscr{H}\oplus\cdots$, then let

$$U:\tilde{\mathscr{H}}\otimes\mathbb{C}^{2^{n-1}}\to\tilde{\mathscr{H}},$$

$U:(\xi_1^1,\xi_2^1,\cdots,\xi_1^2,\xi_2^2,\cdots,\xi_1^{2^{n-1}},\xi_2^{2^{n-1}},\cdots)\mapsto(\xi_1^1,\xi_1^2,\cdots,\xi_1^{2^{n-1}},\xi_2^1,\xi_2^2,\cdots,\xi_2^{2^{n-1}},\cdots)$, where $\xi_j^i\in\mathscr{H}, i=1,\cdots,2^{n-1}; j=1,2,\cdots,$ thus we have that $UU^*=I, U^*U=I$ and $U(\oplus_{k=1}^{\infty}T_k)\hat{}\,U^*=\oplus_{k=1}^{\infty}(T_k)\hat{}$, therefore $(\oplus_{k=1}^{\infty}T_k)\hat{}$ is unitarily equivalent to $\oplus_{k=1}^{\infty}(T_k)\hat{}$. $\square$

THEOREM 6. *Let* $T_k=(T_k^1,\cdots,T_k^n), k=1,2,\cdots$ *be families of uniformly bounded* n*-tuples of commuting operators on* $\mathscr{H}$*, then:*

$$\mathrm{Sp}(\oplus_{k=1}^{\infty}T_k)=\bigcup_{k=1}^{\infty}\mathrm{Sp}(T_k)\cup\sigma,$$

where $\sigma=\{\lambda\notin\mathrm{Sp}(T_k);$ *there exists* n_k*, such that* $||((\lambda-T_{n_k})^{-1})\hat{}||\to\infty\}$.

Proof. For $\lambda\in\mathrm{Sp}(T_k)$, it follows from Lemma 4 that $(\lambda-T_k)\hat{}$ is not invertible, thus $\oplus_{k=1}^{\infty}(\lambda-T_k)\hat{}$ is not invertible, then by Lemma 5, $(\oplus_{k=1}^{\infty}(\lambda-T_k))\hat{}$ is not invertible, that is $\lambda\in\mathrm{Sp}(\oplus_{k=1}^{\infty}T_k)$. Thus we get the inclusion

$$\bigcup_{k=1}^{\infty}\mathrm{Sp}(T_k)\subseteq\mathrm{Sp}(\oplus_{k=1}^{\infty}T_k).$$

If $\lambda\in\sigma$, then there is a sequence $\{x_{n_k}\}_{k=1}^{\infty}, x_{n_k}\in\mathscr{H}\otimes\mathbb{C}^{2^{n-1}}, ||x_{n_k}||=1$, such that

$$||(\lambda-T_{n_k})\hat{}\,x_{n_k}||\to 0,$$

thus if $u_k=\oplus_{i=1}^{n_k-1}0\oplus x_{n_k}\oplus_{i=n_k+1}^{\infty}0$, then $||u_k||=1, u_k\in\tilde{\mathscr{H}}$, and

$$||(\lambda-T_k)\hat{}\,u_k||\to 0,$$

that is $\oplus_{k=1}^{\infty}(\lambda-T_k)\hat{}$ is not invertible, by Lemma 5, $(\oplus_{k=1}^{\infty}(\lambda-T_k))\hat{}$ is not invertible. Hence

$$\mathrm{Sp}(\oplus_{k=1}^{\infty}T_k)\supseteq\bigcup_{k=1}^{\infty}\mathrm{Sp}(T_k)\cup\sigma.$$

For all $\lambda\notin\bigcup_{k=1}^{\infty}\mathrm{Sp}(T_k)\cup\sigma$, then there is $d>0$, such that for all k,

$$||((\lambda-T_k)^{-1})\hat{}||\leqslant d,$$

thus $\oplus_{k=1}^{\infty}(\lambda - T_k)\hat{}$ is invertible, it follows by Lemma 5 that $(\oplus_{k=1}^{\infty}(\lambda - T_k))\hat{}$ is invertible, therefore $\lambda \notin \mathrm{Sp}(\oplus_{k=1}^{\infty} T_k)$, hence

$$\mathrm{Sp}(\oplus_{k=1}^{\infty} T_k) = \bigcup_{k=1}^{\infty} \mathrm{Sp}(T_k) \cup \sigma. \quad \square$$

M. Chō and M. Takaguchi [3] showed that the joint spectrum of an n-tuple of commuting operators on finite Hilbert space is the joint point spectrum. The following corollary is a generalization of their result by Theorem 2.

COROLLARY 7. *Let $T_k = (T_k^1, \cdots, T_k^n), k = 1, 2, \cdots$ be families of uniformly bounded n-tuples of commuting operators on $\mathbb{C}^n$, then*

$$\mathrm{Spp}(\oplus_{k=1}^{\infty} T_k) = \bigcup_{k=1}^{\infty} \mathrm{Sp}(T_k).$$

The next corollary is a generalization of a special case of R. Curto and K. Yan [2].

COROLLARY 8. *Let $T_k = (T_k^1, \cdots, T_k^n), k = 1, 2, \cdots$ be families of uniformly bounded n-tuples of commuting operators on $\mathscr{H}$, if for all k, $\lambda \notin \mathrm{Sp}(T_k)$, where $k = 1, 2, \cdots$, there is $d > 0$, such that $||((\lambda - T_k)^{-1})\hat{}|| \leqslant d$, then:*

$$\mathrm{Sp}(\oplus_{k=1}^{\infty} T_k) = \bigcup_{k=1}^{\infty} \mathrm{Sp}(T_k).$$

COROLLARY 9. *Let $T_k = (T_k^1, \cdots, T_k^n), k = 1, 2, \cdots, m$ be families of uniformly bounded n-tuples of commuting operators on $\mathscr{H}$, then*

$$\mathrm{Sp}(\oplus_{k=1}^{m} T_k) = \bigcup_{k=1}^{m} \mathrm{Sp}(T_k).$$

It is noted that by using the Curto matrix, $\lambda \notin \mathrm{Sp}(\oplus_{k=1}^{m} T_k) \Leftrightarrow (\oplus_{k=1}^{m}(\lambda - T_k))\hat{}$ is invertible $\Leftrightarrow (\lambda - T_k)\hat{}$ is invertible for all $k = 1, \cdots, m \Leftrightarrow \lambda \notin \mathrm{Sp}(T_k)$ for all $k = 1, \cdots, m \Leftrightarrow \lambda \notin \bigcup_{k=1}^{m} \mathrm{Sp}(T_k)$.

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