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POSITIVE COMMUTATORS AND COLLECTIONS OF OPERATORS

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Abstract. Let A and B be completely decomposable nonnegative matrices such that the commutator $AB-BA$ is also a nonnegative matrix. We prove that the set $\{A,B\}$ is completely decomposable, i.e., there exists a permutation matrix P such that PAP^{-1} and PBP^{-1} are upper triangular matrices. We show similar results for collections of completely decomposable nonnegative matrices. We also find conditions on commutators under which a given operator on a Riesz space is necessarily scalar.

1. Introduction

A collection $\mathscr{C}$ of real (resp. complex) $n \times n$ matrices is *reducible* if there exists a common invariant subspace other than the trivial ones $\{0\}$ and $\mathbb{R}^n$ (resp. $\mathbb{C}^n$), or equivalently, there exists an invertible matrix S such that the collection $S\mathscr{C}S^{-1}$ has a block upper-triangular form; otherwise, the collection $\mathscr{C}$ is said to be *irreducible*. If the matrix S can be chosen to be a permutation matrix, then the collection $\mathscr{C}$ is said to be *decomposable*; otherwise, it is called *indecomposable* or *ideal-irreducible*.

If there is an invertible matrix S such that the collection $S\mathscr{C}S^{-1}$ even consists of upper triangular matrices, then the collection $\mathscr{C}$ is said to be *triangularizable*. If the matrix S can be chosen to be a permutation matrix, then the collection $\mathscr{C}$ is said to be *completely decomposable* or *ideal-triangularizable*.

In a (real) partially ordered vector space E, we say that a vector $x \in E$ is of *constant-sign* if either x or $-x$ is a nonnegative vector. In particular, a real matrix A is of constant-sign if either A or $-A$ is a nonnegative matrix. We now recall three results on nonnegative matrices (see [2, Theorem 2.1], [4, Lemma 5.1.5] and [4, Theorem 5.1.2]). We will use their trivial generalization to matrices of constant-sign.

THEOREM 1.1. *Let A and B be matrices of constant-sign such that the commutator $C = AB - BA$ is of constant-sign as well. Then, up to similarity with a permutation matrix, C is a strictly upper triangular matrix, and so it is nilpotent.*

LEMMA 1.2. *A (multiplicative) semigroup $\mathscr{S}$ of matrices of constant-sign is decomposable if some non-zero ideal of $\mathscr{S}$ is decomposable.*

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THEOREM 1.3. *A (multiplicative) semigroup of nilpotent matrices of constant-sign is completely decomposable.*

In this paper we find some conditions under which a collection of completely decomposable matrices of constant-sign is completely decomposable (Section 2), and some conditions implying that a given operator on a Riesz space is necessarily scalar (Section 3).

2. From local to global complete decomposability

It is known and easy to prove that every commutative collection of complex matrices is triangularizable (see [4, Theorem 1.1.5]). In this section we seek for order analogs of this fact. We begin with a pair of nonnegative matrices.

THEOREM 2.1. *Let A and B be completely decomposable $n \times n$ nonnegative matrices such that $AB \leqslant BA$. Then the set $\{A, B\}$ is also completely decomposable, or equivalently, the sum $A+B$ is completely decomposable.*

Proof. After a permutation similarity, the matrix $C = A+B$ can be decomposed into a block triangular form whose diagonal blocks are indecomposable matrices. Since $C \geqslant A$ and $C \geqslant B$, the matrices A and B have the same block triangular form, and each of their diagonal blocks is completely decomposable. The latter fact follows easily from the theorem asserting that a nonnegative matrix is completely decomposable if and only if it becomes nilpotent upon replacement of its diagonal entries by zeros (see [4, Theorem 5.1.7]).

We want to prove that all diagonal blocks of C are one-dimensional. Assume the contrary. With no loss of generality we may assume that C is an indecomposable matrix of size $n \geqslant 2$. Since an indecomposable matrix is not nilpotent, we may also assume that the spectral radius of C equals 1.

By Perron-Frobenius Theorem [4, Corollary 5.2.13], there are strictly positive vectors u and v, unique up to a scalar multiple, such that $Cu = u$ and $C^T v = v$. Since $BA \geqslant AB$, the vector $(BA - AB)u = (CA - AC)u = CAu - Au$ is nonnegative. However, $v^T(CAu - Au) = 0$, so that $CAu - Au = 0$, as the vector v is strictly positive. This means that Au is an eigenvector of C corresponding to 1, and so there exists $\lambda \geqslant 0$ such that $Au = \lambda u$. In fact, $\lambda > 0$, since $A \neq 0$ and the vector u is strictly positive. Similarly, the vector $(A^T C^T - C^T A^T)v = A^T v - C^T A^T v$ is nonnegative, and it follows from $u^T(A^T v - C^T A^T v) = 0$ that $C^T A^T v = A^T v$, so that there exists $\mu > 0$ such that $A^T v = \mu v$. Since

$$\lambda v^T u = v^T A u = (A^T v)^T u = \mu v^T u$$

and $v^T u > 0$, we have $\lambda = \mu$.

Now, we may assume that the matrix $A = (a_{i,j})_{i,j=1}^n$ is upper triangular. The equalities $Au = \lambda u$ and $A^T v = \lambda v$ give $2n$ scalar equalities:

$$
\begin{aligned}
a_{1,1}u_1 + a_{1,2}u_2 + \cdots + a_{1,n}u_n &= \lambda u_1 \\
a_{2,2}u_2 + \cdots + a_{2,n}u_n &= \lambda u_2 \\
\ddots \quad \vdots \qquad \vdots \quad &\;\; \vdots \\
a_{n,n}u_n &= \lambda u_n
\end{aligned}
$$

$$
\begin{aligned}
a_{1,1}v_1 \qquad\qquad\qquad\qquad &= \lambda v_1 \\
a_{1,2}v_1 + a_{2,2}v_2 \qquad\qquad &= \lambda v_2 \\
\vdots \qquad \ddots \qquad\qquad \vdots \quad &\;\; \vdots \\
a_{1,n}v_1 + a_{2,n}v_2 + \cdots + a_{n,n}v_n &= \lambda v_n
\end{aligned}
$$

Having in mind that both vectors u and v are strictly positive, we conclude from them successively $a_{n,n} = \lambda$, $a_{1,1} = \lambda$, $a_{1,2} = a_{1,3} = \ldots = a_{1,n} = 0$, $a_{1,n} = a_{2,n} = a_{n-1,n} = 0$, $a_{2,2} = \lambda$, $a_{n-1,n-1} = \lambda$, etc. Thus, $A = \lambda I$, so that $B = C - A = C - \lambda I$ is indecomposable. This contradiction completes the proof. □

This theorem can be stated for general real matrices as follows. Here the absolute value $|A|$ of a real matrix A is taken entry-wise.

COROLLARY 2.2. *Let A and B be completely decomposable $n \times n$ real matrices such that $|A||B| \leqslant |B||A|$. Then $|A| + |B|$ is completely decomposable. In particular, the set $\{A, B\}$ is completely decomposable.*

In the case of collection of matrices we first consider the commutative case.

THEOREM 2.3. *A commutative collection $\mathscr{C}$ of matrices of constant-sign is completely decomposable if and only if each member of $\mathscr{C}$ is completely decomposable.*

Proof. We must only show that the condition is sufficient. Clearly, $|A||B| = |B||A|$ for every $A, B \in \mathscr{C}$. Let $\{C_1, C_2, \ldots, C_m\} \subseteq \mathscr{C}$ be the basis of the linear span of $\mathscr{C}$. By Corollary 2.2 and an easy induction, the sum $|C_1| + |C_2| + \ldots + |C_m|$ is completely decomposable. This implies easily that the whole collection is completely decomposable. □

THEOREM 2.4. *Let $\mathscr{S}$ be a semigroup of completely decomposable $n \times n$ matrices of constant-sign such that for every $A, B \in \mathscr{S}$ the commutator $AB - BA$ is of constant-sign. Then the semigroup $\mathscr{S}$ is completely decomposable.*

Proof. It suffices to show that the semigroup $\mathscr{S}$ is decomposable, because we can then apply induction on n or the Ideal-triangularization Lemma (see [3]).

By Theorem 2.3, we may assume that $\mathscr{S}$ is not commutative. Let A and B be matrices in $\mathscr{S}$ with the property $AB \neq BA$ and $AB - BA \geqslant 0$. Since the commutator of any pair of matrices from $\mathscr{S}$ is nilpotent by Theorem 1.1, the semigroup $\mathscr{S}$ is triangularizable by [4, Theorem 4.4.12]. This trivially implies that the semigroup $\mathscr{S}_1$

generated by the semigroup $\mathscr{S}$ and the nonnegative nonzero matrix $AB-BA$ is also triangularizable. Let $\mathscr{J}$ be the semigroup ideal generated by the matrix $AB-BA$ in $\mathscr{S}_1$. Since the spectral radius is submultiplicative on triangularizable families of matrices, every matrix in $\mathscr{J}$ is nilpotent. By Theorem 1.3, the semigroup ideal $\mathscr{J}$ is completely decomposable, and so the semigroup $\mathscr{S}_1$ is decomposable by Lemma 1.2. We finish the proof by noticing $\mathscr{S} \subseteq \mathscr{S}_1$. $\square$

The following example shows that Theorem 2.4 (for $n \geqslant 3$) does not hold without the assumption that the collection is a semigroup.

EXAMPLE 2.5. Let e_1, e_2, $\ldots$, e_n be the standard basis vectors of $\mathbb{R}^n$, where $n \geqslant 3$. Define completely decomposable nilpotent matrices by $A_i = e_i e_{i+1}^T$ for $i = 1,2,\ldots,n-1$, and $A_n = e_n e_1^T$. Then the collection $\{A_1,A_2,\ldots,A_n\}$ has the property that either $A_iA_j \geqslant A_jA_i$ or $A_iA_j \leqslant A_jA_i$ for every $i,j \in \{1,2,\ldots,n\}$, since either $A_iA_j = 0$ or $A_jA_i = 0$. We now show that this collection is not completely decomposable. Assume the contrary. Then the sum $S = A_1 + A_2 + \ldots + A_n$ is completely decomposable. Since all the diagonal entries of S are zero, S must be nilpotent which contradicts the fact that $S^n = I$.

The following theorem and its corollaries could be seen as extensions of [4, Corollary 1.7.5] in the setting of matrices of constant-sign. A collection $\mathscr{C}$ of matrices is called a *Lie set* if it is closed under commutation, i.e., $AB-BA$ is in $\mathscr{C}$ whenever A and B are in $\mathscr{C}$.

THEOREM 2.6. *A collection $\mathscr{C}$ of completely decomposable $n \times n$ matrices of constant-sign is completely decomposable if the Lie set $\mathscr{L}$ generated by $\mathscr{C}$ consists of matrices of constant-sign.*

Proof. If the collection $\mathscr{C}$ is commutative, then $\mathscr{C}$ is completely decomposable by Theorem 2.3. Suppose now that $\mathscr{C}$ is not commutative. Then there exist matrices $A,B \in \mathscr{C}$ such that $AB \geqslant BA$ and $AB \neq BA$. By Theorem 1.1, every commutator of matrices from $\mathscr{L}$ is a nilpotent matrix. Therefore, the Lie set $\mathscr{L}$ is triangularizable by [4, Corollary 1.7.8]. Let $\mathscr{S}$ be the (multiplicative) semigroup generated by $\mathscr{L}$ and let $\mathscr{J}$ be the semigroup ideal in $\mathscr{S}$ generated by the nonzero nonnegative nilpotent matrix $AB-BA$. Since the spectral radius is submultiplicative on triangularizable families of matrices, the semigroup ideal $\mathscr{J}$ consists of nilpotent matrices of constant-sign, and so it follows from [4, Theorem 5.1.2] that it is completely decomposable. Applying Lemma 1.2, we see that the semigroup $\mathscr{S}$ is decomposable, and so is the collection $\mathscr{C}$ as $\mathscr{C} \subseteq \mathscr{S}$. To finish the proof we apply Ideal-triangularization Lemma (see [3]) or we use induction on n. $\square$

COROLLARY 2.7. *A collection $\mathscr{C}$ of completely decomposable matrices of constant-sign is completely decomposable if for every pair $\{A,B\} \subseteq \mathscr{C}$ at least one of the commutators $AB-BA$ and $BA-AB$ is contained in $\mathscr{C}$.*

In view of Theorem 1.3 we have the following.

COROLLARY 2.8. *A Lie set of nilpotent matrices of constant-sign is completely decomposable.*

Recall that a Lie algebra of matrices is a Lie set that is also a linear space. The following result can be considered as an order analog of Engel's theorem asserting that a Lie algebra of nilpotent matrices is triangularizable (see [4, Corollary 1.7.6]).

THEOREM 2.9. *If a Lie algebra of nilpotent matrices is generated by the set of nonnegative matrices, then it is completely decomposable.*

Proof. Let $\mathscr{A}$ be a Lie algebra of nilpotent matrices generated by the set $\mathscr{F}$ of nonnegative matrices. By Engel's theorem [4, Corollary 1.7.6], $\mathscr{A}$ is triangularizable. It follows that in an appropriate basis all the matrices in $\mathscr{F}$ are strictly upper triangular. Therefore, the same is true for the semigroup $\mathscr{S}$ generated by $\mathscr{F}$. Hence, all the matrices in $\mathscr{S}$ are nilpotent and nonnegative (w.r.t. the original basis), so that $\mathscr{S}$ is completely decomposable by Theorem 1.3. Therefore, the (associative) algebra $\mathscr{A}_1$ generated by $\mathscr{F}$ is completely decomposable, because $\mathscr{A}_1$ is the linear span of $\mathscr{S}$. Since $\mathscr{A} \subseteq \mathscr{A}_1$, the Lie algebra $\mathscr{A}$ is completely decomposable as well. □

3. Conditions implying that an operator is scalar

It is well-known that only scalar operators (= multiples of the identity operator) commute with all (linear) operators on a vector space. Moreover, they are the only operators commuting with all rank-one operators. In this section we consider order analogs of this fact.

Let E be a real Riesz space, and let E^+ denote the positive cone of E. Denote by $\widetilde{E}$ the order dual of E, that is the vector space generated by positive linear functionals on E.

For operators A and T on E we write $[A,T] = AT - TA$. If $T = x \otimes \varphi$ with $x \in E$ and $\varphi \in \widetilde{E}$, then $[A,T] = Ax \otimes \varphi - x \otimes A^*\varphi$.

LEMMA 3.1. *Let A be an operator on a Riesz space E, and $x \in E^+$ a nonzero vector. Let Φ be a Riesz subspace of $\widetilde{E}$ that separates the points of E. If for every functional $\varphi \in \Phi^+$ the vector*

$$[A, x \otimes \varphi]x = \varphi(x)Ax - \varphi(Ax)x$$

is of constant-sign, then x in an eigenvector of A.

Proof. Define the sets

$$P = \{\varphi \in \Phi^+ : \varphi(x)Ax \geqslant \varphi(Ax)x\},$$
$$N = \{\psi \in \Phi^+ : \psi(x)Ax \leqslant \psi(Ax)x\}.$$

For any φ_1, $\varphi_2 \in P$, we have

$$\varphi_1(x)\varphi_2(Ax) \geqslant \varphi_1(Ax)\varphi_2(x).$$

If we change the role of these two functionals, we get

$$\varphi_1(x)\varphi_2(Ax) = \varphi_1(Ax)\varphi_2(x) \tag{1}$$

for all φ_1, $\varphi_2 \in P$. The same holds for pairs of functionals in N.

Since $\Phi^+ = P \cup N$ and Φ separates the points of E, there exists $\varphi_0 \in \Phi^+$ such that $\varphi_0(x) > 0$. Suppose that $\varphi_0 \in P$ (the case $\varphi_0 \in N$ is similar). If $\lambda = \frac{\varphi_0(Ax)}{\varphi_0(x)}$, then the equality (1) implies that $\varphi(Ax) = \lambda\varphi(x)$ for all $\varphi \in P$. Let us prove that also $\psi(Ax) = \lambda\psi(x)$ for all $\psi \in N$. We must consider two cases.

- There exists $\psi_0 \in N$ with $\psi_0(x) > 0$: If we denote $\mu = \frac{\psi_0(Ax)}{\psi_0(x)}$, then the equality (1) for the set N gives that $\psi(Ax) = \mu\psi(x)$ for all $\psi \in N$. Suppose that the functional $\varphi_0 + \psi_0$ is an element of P (the case when it belongs to N can be treated analogously). Then we have

  $$(\varphi_0 + \psi_0)(Ax) = \lambda(\varphi_0 + \psi_0)(x).$$

  Since

  $$\varphi_0(Ax) + \psi_0(Ax) = \lambda\varphi_0(x) + \mu\psi_0(x),$$

  we obtain that $\lambda = \mu$, and so

  $$\psi(Ax) = \lambda\psi(x)$$

  for all $\psi \in N$.

- For every $\psi \in N$ it holds that $\psi(x) = 0$: Choose any $\psi \in N$. The positive functional $\varphi_0 + \psi$ cannot be an element of N, since $(\varphi_0 + \psi)(x) = \varphi_0(x) \neq 0$. Therefore, we have $\varphi_0 + \psi \in P$, and so

  $$(\varphi_0 + \psi)(Ax) = \lambda(\varphi_0 + \psi)(x) = \lambda\varphi_0(x) = \varphi_0(Ax).$$

  Hence

  $$\psi(Ax) = 0 = \lambda\psi(x).$$

We have proved that $\varphi(Ax) = \lambda\varphi(x)$ for all $\varphi \in \Phi^+$ which implies that $\varphi(Ax - \lambda x) = 0$ for all $\varphi \in \Phi$. Since Φ separates the points of E, we conclude that $Ax = \lambda x$. This completes the proof. □

THEOREM 3.2. *Let A be an operator on a Riesz space E. Let Φ be a Riesz subspace of $\widetilde{E}$ that separates the points of E. If the commutator $[A,T]$ is of constant-sign for every positive rank-one operator $T = x \otimes \varphi$ with $x \in E$ and $\varphi \in \Phi$, then A is a scalar operator.*

Proof. If $x \in E^+$ is a nonzero vector, then for every functional $\varphi \in \Phi^+$ the vector

$$[A, x \otimes \varphi]x = \varphi(x)Ax - \varphi(Ax)x$$

is of constant-sign. Therefore, x in an eigenvector of A by Lemma 3.1. It is not difficult to show that A is necessarily a scalar operator. □

COROLLARY 3.3. *Let A be an operator on a normed Riesz space E. If the commutator $[A,T]$ is of constant-sign for every continuous positive rank-one operator T on E, then A is a scalar operator.*

Proof. Since the topological dual E' is an ideal of $\widetilde{E}$ (see [1, Theorem 3.49]) and it separates the points of E (see [1, Theorem 3.7]), we can apply Theorem 3.2. □

To define higher commutators, we introduce the notation $C_T(A) = [A,T] = AT - TA$, where A and T are operators on E. We now define inductively

$$C_T^1(A) = C_T(A),\ C_T^{n+1}(A) = C_T(C_T^n(A)) = [C_T^n(A), T] \text{ for } n \in \mathbb{N}.$$

If $T = x \otimes \varphi$ where $x \in E$ and $\varphi \in \widetilde{E}$, then it can be easily proved by induction that

$$C_{x\otimes\varphi}^n(A)x = \varphi(x)^{n-1}\left(\varphi(x)Ax - \varphi(Ax)x\right).$$

We conclude the paper with slight extensions of Lemma 3.1 and Theorem 3.2.

LEMMA 3.4. *Let A be an operator on a Riesz space E, and $x \in E^+$ a nonzero vector. Let Φ be a Riesz subspace of $\widetilde{E}$ that separates the points of E. If for every functional $\varphi \in \Phi^+$ there exists $n \in \mathbb{N}$ such that $C_{x\otimes\varphi}^n(A)x$ is of constant-sign, then x in an eigenvector of A.*

Proof. The assumption on commutators implies that if $\varphi(x) > 0$ then the vector

$$\varphi(x)Ax - \varphi(Ax)x$$

is of constant-sign. Since this clearly holds also in the case when $\varphi(x) = 0$, x must be an eigenvector of A by Lemma 3.1. □

THEOREM 3.5. *Let A be an operator on a Riesz space E. Let Φ be a Riesz subspace of $\widetilde{E}$ that separates the points of E. Suppose that, for each positive rank-one operator $T = x \otimes \varphi$ with $x \in E$ and $\varphi \in \Phi$, there exists $n \in \mathbb{N}$ such that the operator $C_T^n(A)$ is of constant-sign. Then A is a scalar operator.*

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