Operators and Matrices
Volume 6, Number 3 (2012), 543–565 doi:10.7153/oam-06-37

ELEMENTARY MODELS OF UNBOUNDED JACOBI MATRICES WITH A FEW BOUNDED GAPS IN THE ESSENTIAL SPECTRUM

ANNE BOUTET DE MONVEL, JAN JANAS AND SERGUEI NABOKO

(*Communicated by Fritz Gesztesy*)

Abstract. This work contains a constructive example of a class of Jacobi operators with an arbitrary finite number of gaps in its unbounded essential spectrum. The construction of this class is based on elementary ideas of gluing finite-dimensional Jacobi matrices whose sizes grow to infinity. The precise analysis of the finite-dimensional pieces leads to a new "finite essential spectrum" besides the natural essential spectrum of two explicit infinite Jacobi matrices, determined by the above finite dimensional ones. This new finite essential spectrum is calculated explicitly. A connection to the ideas of the recent paper [12] is also given.

1. Introduction

The behaviour of the essential spectrum of unbounded Jacobi operators in comparison to bounded ones is more delicate. This phenomenon was already observed in earlier works [3, 6, 9, 10, 12, 13]. In particular in [6, 9, 10, 13] the first *explicit* examples of unbounded Jacobi operators with *one gap* in the essential spectrum were shown. These examples will be recalled briefly below. In turn, in [3] a class of unbounded Jacobi matrices with a *bounded essential spectrum* E such that $\mathbb{R}\setminus E$ consists of an arbitrary finite number of intervals was constructed. However, no constructive examples of Jacobi operators are known with *unbounded essential spectra* having a finite (greater than one) number of gaps.

The main purpose of the present paper is the construction of Jacobi operators with this property of the essential spectrum.

For given real sequences $\{a_k\}_1^\infty$, $a_k \neq 0$ for all k and $\{b_k\}_1^\infty$ the Jacobi operator $\mathscr{J}_0$ acts in the space $\ell_0^2 = \ell_0^2(\mathbb{N})$ of sequences $f = \{f_k\}_{k=1}^\infty$ having only a finite number of nonzero terms by the formula

$$(\mathscr{J}_0 f)_k = a_{k-1}f_{k-1} + b_k f_k + a_k f_{k+1},$$

where $k = 1,2,\dots$ and $a_0 := f_0 := 0$. The a_k's are called the "weights" of the Jacobi operator.

One can extend $\mathscr{J}_0$ to a unique self-adjoint operator $\mathscr{J} = \mathscr{J}(\{a_k\},\{b_k\})$ acting in $\ell^2 = \ell^2(\mathbb{N})$ provided that $\sum_k \frac{1}{|a_k|} = +\infty$ [1]. Denote by $\sigma_{\text{ess}}(\mathscr{J})$ the essential

Mathematics subject classification (2010): Primary 47B36; Secondary 47A10, 47B25.
Keywords and phrases: Jacobi matrix, essential spectrum, gaps, monodromy matrix.

Paper OaM-06-37

spectrum of $\mathscr{J}$. Surely for any given $X = \overline{X} \subset \mathbb{R}$ one can trivially construct a class of Jacobi matrices $\widetilde{\mathscr{J}}$ such that $\sigma_{\mathrm{ess}}(\widetilde{\mathscr{J}}) = X$. Indeed, for any sequence $\{a_n\}_{n=1}^{\infty}$ of positive numbers such that $a_n \to 0$ take a dense sequence $\{b_n\}_{n=1}^{\infty}$ in X. This construction is not so interesting (even useless) from our point of view. In fact this choice of $\{b_n\}_{n=1}^{\infty}$ defines irregular behaviour and cannot be used for example to produce a second order phase transition, as it was done in [10] in the case $a_n = n^{\alpha} + c_n f(n^{\gamma})$, $0 < \gamma < \frac{1-\alpha}{2}$, $\alpha \in (0,1)$, $\{c_k\}_{k=1}^{\infty}$ is 2-periodic, f is continuous and periodic of period T, and $b_n \equiv 0$.

We should also recall a general construction of $\mathscr{J}$ with $\sigma_{\mathrm{ess}}(\mathscr{J}) = X$ due to Stone [1]. This construction gives no information on matrix entries of $\mathscr{J}$, however.

The idea of construction of our class of examples of Jacobi operators can be explained as follows. For two given sequences $\{\mathscr{J}_{1s}\}$, $\{\mathscr{J}_{2s}\}$ of finite-dimensional Jacobi matrices whose sizes tend to infinity and a sequence $\{d_k\}_{k=1}^{\infty}$ of different from zero, real numbers such that $d_s \to 0$; define the infinite Jacobi matrix

$$\mathscr{J} = \begin{pmatrix} \boxed{\mathscr{J}_{11}} & d_1 & & & & \\ d_1 & \boxed{\mathscr{J}_{21}} & d_2 & & & \\ & d_2 & \boxed{\mathscr{J}_{12}} & d_3 & & \\ & & d_3 & \boxed{\mathscr{J}_{22}} & d_4 & \\ & & & d_4 & \ddots & \end{pmatrix} \tag{1}$$

We treat such infinite matrix as $\mathscr{J} = \mathscr{J}(\{a_k\},\{b_k\})$ defined on page 1 with $\{a_k\},\{b_k\}$ being the weights and diagonals of the Jacobi matrix, respectively. Using Weyl theorem [11], one can describe $\sigma_{\mathrm{ess}}(\mathscr{J})$ in terms of the accumulation points of the union $\cup_{s\geqslant 1}\big(\sigma(\mathscr{J}_{1s}) \cup \sigma(\mathscr{J}_{2s})\big)$, where $\sigma(\mathscr{J}_{js})$ denotes the spectrum of $\mathscr{J}_{js}$, $(j = 1,2)$ and each point of the union is counted with its multiplicity.

This idea will be made concrete below by choosing suitable blocks $\mathscr{J}_{1s}$ and $\mathscr{J}_{2s}$ which allow to find $\sigma_{\mathrm{ess}}(J)$ having the desired property ($\sigma_{\mathrm{ess}}(\mathscr{J})$ is unbounded and $\mathbb{R} \setminus \sigma_{\mathrm{ess}}(\mathscr{J})$ is the union of a finite number of intervals). Our choice of $\{\mathscr{J}_{1s}\}$ and $\{\mathscr{J}_{2s}\}$ is strongly related to the repetition of the pieces of two *infinite* Jacobi operators $\mathscr{J}_c$ and $\mathscr{J}_z$ in the sense described in the next sentences.

More precisely, $\mathscr{J}_c$ is the Jacobi operator defined by $a_n := n^{\alpha} + c_n$, where $\{c_k\}_{k=1}^{\infty}$ is a 2-periodic sequence $(c_1, c_2, c_1, c_2, \ldots)$, $\alpha \in (0,1)$ and $b_n \equiv 0$. This is the well known Jacobi operator studied in [5, 6, 10, 9, 13]. In particular if $c_1 > c_2$ then $\sigma_{\mathrm{ac}}(\mathscr{J}_c) = (-\infty, -c] \cup [c, +\infty)$, $c := c_1 - c_2$, and the spectrum is purely absolutely continuous in this set. Moreover, in the interval $(-c, c)$ there is no spectrum provided that $c_2 \geqslant -(2 - 2^{\alpha})$, see [5, Theorem 3.3].

In turn, $\mathscr{J}_z$ is determined by an N-periodic positive sequence

$$(a_k)_1^{\infty} = (z_1, z_2, \ldots, z_N; z_1, z_2, \ldots, z_N; \ldots),$$

i.e., $a_k := z_r$, for $k = l \cdot N + r$, $1 \leqslant r \leqslant N$ where $l = 0,1,\dots$, and $b_k \equiv 0$. We also denote $z_0 := z_N$ and $z_{-1} := z_{N-1}$.

With the above notations we define our basic models as follows. Fix natural numbers $N > 1$ and $r \in [0, N-1]$. Consider a sequence of natural numbers $\{k_s\}$ such that

i) $k_{2s-1} = m_s \cdot N + r$, where natural numbers $m_s \to +\infty$, as $s \to +\infty$;

ii) k_{2s} are even and $k_{2s} \to +\infty$, as $s \to +\infty$.

Then $\mathscr{J}_{1s} := \mathscr{J}_z(k_{2s-1})$, where $\mathscr{J}_z(k_{2s-1})$ is the k_{2s-1}-dimensional Jacobi matrix given by the $k_{2s-1} - 1$ weights $(z_1, z_2, \dots, z_N; z_1, z_2, \dots, z_N; \dots; z_1, \dots, z_{r-1})$, and the main diagonal $b_k \equiv 0$.

Define $\mathscr{J}_{2s}$ as the k_{2s}-dimensional Jacobi matrix $\mathscr{J}_c(k_{2s})$ with

$$a_p = p^\alpha + c_p, \quad p = 1, \dots, k_{2s} - 1, \quad b_p \equiv 0,$$

where

(a) $\{c_n\}$ is 2-periodic, $c_1 > c_2$, and $2 - 2^\alpha + c_2 > 0$.

Observe that this condition on c_n forces: $p^\alpha + c_p > 0$ for all $p \in \mathbb{N}$.

Note that $\mathscr{J}$ is a compact perturbation of

$$\mathscr{J}_{sp} := \bigoplus_{s \geqslant 1} \mathscr{J}_s, \tag{2}$$

where $\mathscr{J}_s$ coincides either with $\mathscr{J}_z(k_s)$ or with $\mathscr{J}_c(k_s)$, depending on the parity of s.

Before we formulate the main result of this work, let us recall a few basic notions and results related to infinite periodic matrix $\mathscr{J}_z$ and finite-dimensional matrices $\mathscr{J}_z(k)$ defined above. Some of these results will be used later in the work. Recall that the infinite matrix $\mathscr{J}_z$ acts in the standard way in the space of all scalar sequences. If $\{w_n\}$ is an arbitrary scalar sequence then we shall denote by $\mathscr{J}_z w$ the sequence $\{a_{n-1} w_{n-1} + a_n w_n\}$. For a given $\lambda \in \mathbb{R}$ denote by $B_s = B_s(\lambda)$ the transfer matrix of $\mathscr{J}_z$

$$B_s = \begin{pmatrix} 0 & 1 \\ -\frac{z_{s-1}}{z_s} & \frac{\lambda}{z_s} \end{pmatrix}, s \geqslant 1.$$

Then for $\lambda \in \mathbb{R}$ the system of equations $(\mathscr{J}_z u)_n = \lambda u_n$, $n > 1$, can be written in the form which will be used below many times

$$\vec{u}_{n+1} = B_n \vec{u}_n,$$

where $\vec{u}_n := \binom{u_{n-1}}{u_n} =: (u_{n-1}, u_n)^t$.

Let

$$M(\lambda) := B_1 \cdot B_N \cdot B_{N-1} \cdots B_2$$

be the monodromy matrix of the periodic problem. Note that $\det M(\lambda) = 1$. It is well known that the absolutely continuous spectrum $\sigma_{ac}(\mathcal{J}_z)$ of $\mathcal{J}_z$ is given by $\{\lambda \in \mathbb{R} \mid |\mathrm{Tr} M(\lambda)| \leqslant 2\}$ and has only a finite point spectrum in $\mathbb{R} \setminus \sigma_{ac}(\mathcal{J}_z)$, (see [4]).

By a $\mathbb{C}^2$-generalized eigenvector $\vec{u}(\lambda) = \{\vec{u}_n(\lambda)\}_2^\infty$ of $\mathcal{J}_z$, corresponding to $\lambda \in \mathbb{R}$, we mean any nontrivial solution of the equations $\vec{u}_{n+1} = B_n \vec{u}_n$, $n = 2, 3, \dots$.

In the case $\lambda \in$ "Int"$[\sigma(\mathcal{J}_z)] := \{x \in \mathbb{R} \mid |\mathrm{Tr} M(x)| < 2\}$ this $\mathbb{C}^2$-generalized eigenvector $\vec{u}(\lambda)$ satisfies the estimates

$$0 < c < \|\vec{u}_n(\lambda)\| < C \quad \text{for any } n = 2, 3, \dots, \tag{3}$$

for some constants c, C depending on λ and the initial conditions only, ([15, Theorem 7.3]). Generically the set "Int"$[\sigma(\mathcal{J}_z)]$ coincides with the interior of $\sigma_{ac}(\mathcal{J}_z)$.

Now fix $\lambda \notin \sigma_{ac}(\mathcal{J}_z)$. It follows that $|\mathrm{Tr} M(\lambda)| > 2$ and $M(\lambda)$ has two real eigenvalues $\mu_\pm$ such that $0 < |\mu_-| < 1 < |\mu_+|$. Let $P_\pm$ be the Riesz projections onto the eigenspaces corresponding to $\mu_\pm$. We have

$$M(\lambda) = \mu_+ P_+ + \mu_- P_-. \tag{4}$$

Choose

$$\overrightarrow{p}(\lambda) := \begin{pmatrix} 1 \\ \frac{\lambda}{z_1} \end{pmatrix}.$$

Let the sequence $\{k_{2s-1}\}_1^\infty$ be determined by N and r (see the condition i)described in the definition of $\mathcal{J}_c(k_{2s})$). The set $T_{N,r}$ which appears for the first time in Theorem 1 below is defined as follows

$$T_{N,r} := \left\{\lambda \in \mathbb{R} \mid \lambda \notin \sigma(\mathcal{J}_z) \text{ and } \left(B_{r+1} \cdots B_2 P_+ \overrightarrow{p}(\lambda)\right)_1 = 0\right\}, \tag{5}$$

where for a given vector $v \in \mathbb{C}^2$ we denote by v_1 its first coordinate. In the above $B_s \cdots B_t := I$ when $t > s$. In particular for $r = 0$ the above means $\left(P_+ \overrightarrow{p}(\lambda)\right)_1 = 0$. Note that $T_{N,r}$ is finite being a subset of the zero set of a polynomial. Special role of the set $T_{N,r}$ will be discussed later. Recall that for real λ the sequence of the orthogonal polynomials $P_n(\lambda)$ of the first kind related to $\mathcal{J}_z$ is defined by the recurrence relation

$$a_{n-1} P_{n-1}(\lambda) + a_n P_{n+1}(\lambda) = \lambda P_n(\lambda), \quad n = 1, 2, \dots,$$

where $P_0(\lambda) \equiv 0$, $P_1(\lambda) \equiv 1$ and a_k is the k-th weight of $\mathcal{J}_z$.

REMARK 1. Observe that $\lambda \in \sigma_p(\mathcal{J}_z)$ (the point spectrum) if and only if $\lambda \notin \sigma_{ess}(\mathcal{J}_z)$ and $P_+(1, P_2(\lambda))^t = 0$. Indeed, assume that $\lambda \in \sigma_p(\mathcal{J}_z)$ and $\mathcal{J}_z u = \lambda u$ with $u_1 = 1$ (u_1 must be $\neq 0$). It follows that $u_n = P_n(\lambda)$ and $\{P_{lN+r}(\lambda)\}_{l=1}^\infty \in l^2$. Using (4) we see that $P_+(1, P_2(\lambda))^t = 0$.

On the other hand if $\lambda \notin \sigma_{ess}(\mathcal{J}_z)$ and $P_+(1, P_2(\lambda))^t = 0$ then again (4) implies that $\{P_{lN+r}(\lambda)\}_{l=1}^\infty \in l^2$. Thus the whole sequence $\{P_n(\lambda)\} \in l^2$ (remember that all matrices B_s are invertible with uniformly bounded norms of their inverses).

The main result of this work is given by

THEOREM 1. *Let N and $r \in [0, N-1]$ be natural numbers. For a given sequence $\{m_s\}$ of natural numbers which tend to infinity let $k_{2s-1} = m_s \cdot N + r$ and let $\{k_{2s}\}$ be a sequence of even numbers with $k_{2s} \to +\infty$, as $s \to +\infty$. Fix a sequence $\{d_s\}$ of real nonzero numbers with $\lim_{s\to\infty} d_s = 0$. Fix any $\alpha \in (0,1)$. Suppose we are given a sequence $\{c_n\}$ of real numbers such that*

(a) $\{c_n\}$ is 2-periodic, $c_1 > c_2$, and $2 - 2^\alpha + c_2 > 0$.

Let $\mathscr{J}_{2s}$ be the k_{2s}-dimensional Jacobi matrix $\mathscr{J}_c(k_{2s})$ with

$$a_p = p^\alpha + c_p, \quad p = 1, \dots, k_{2s} - 1, \quad b_p \equiv 0.$$

Denote by $\mathscr{J}_{1s} := \mathscr{J}_z(k_{2s-1})$, where $\mathscr{J}_z(k_{2s-1})$ is the k_{2s-1}-dimensional Jacobi matrix given by the $k_{2s-1} - 1$ positive weights $(z_1, z_2, \dots, z_N; z_1, z_2, \dots, z_N; \dots; z_1, \dots, z_{r-1})$, and the main diagonal is equal to zero. If

$$c_1 - c_2 > 2 \max_{1 \leqslant s \leqslant N} z_s,$$

then for the operator $\mathscr{J}$ defined in (1) we have

$$\sigma_{\mathrm{ess}}(\mathscr{J}) = \sigma(\mathscr{J}_c) \cup \sigma(\mathscr{J}_z) \cup T_{N,r}.$$

REMARK 2. The matrix $\mathscr{J}$ considered in this theorem is defined ad hoc, as a compact perturbation of the direct sum $\oplus \mathscr{J}_s$, of finite-dimensional Jacobi matrices. Using this representation of $\mathscr{J}$ and the Kato–Rosenblum theorem, we know that the absolutely continuous spectrum $\sigma_{\mathrm{ac}}(\mathscr{J})$ of $\mathscr{J}$ is empty. Indeed, due to our assumptions one can choose a sequence $l_n \to +\infty$ such that $\{d_{l_n}\}_1^\infty \in \ell^1$ and this defines the decomposition of $\mathscr{J}$ as the sum $\bigoplus_s L_s + K$, where L_s are finite dimensional blocks of varying dimensions and K is a trace class operator.

We plan to construct in a forthcoming paper, using a different strategy, unbounded Jacobi operators with a finite number of bounded gaps in the essential spectrum and non-trivial absolutely continuous spectrum. In particular this strategy is partially based on the paper of Last-Simon [12], and will be discussed in Section 4 of this work.

2. Preparatory results

In this section we formulate and prove some results concerning the above mentioned finite Jacobi matrices $\mathscr{J}_c(k)$ and $\mathscr{J}_z(k)$ for arbitrary size k. We start by proving uniform (in k) estimates from below of the operator modulus $|\mathscr{J}_c(k)|$ of $\mathscr{J}_c(k)$.

LEMMA 1. *If the sequence $\{c_n\}_1^\infty$ satisfies* (c) *and if $k \in \mathbb{N}$ is even then*

$$\|\mathscr{J}_c(k)u\| \geqslant (c_1 - c_2)\|u\|, \qquad u \in \mathbb{C}^k. \tag{6}$$

Proof. The proof is based on the trick of Dombrowski used in [5] to show essentially the same inequality for the infinite matrix $\mathcal{J}_c$. For $u \in \mathbb{C}^k$ denote by

$$u_{\text{even}} = \sum_{t=1}^{k/2} u_{2t} e_{2t},$$
$$u_{\text{odd}} = \sum_{t=1}^{k/2} u_{2t-1} e_{2t-1},$$

where $\{e_s\}_1^k$ is the canonical basis in $\mathbb{C}^k$. Denote $T := \mathcal{J}_c(k)$ which is hermitian. Let V be the diagonal matrix $\operatorname{diag}\{(-1)^{p+1}\}_1^k$ in the basis $\{e_p\}_1^k$. It is easy to check that $TV = -VT$. By straightforward observation we verify that T is invertible due to the fact that k is even. Let λ be the least positive eigenvalue of T. If $Tv = \lambda v$ then $(v, Vv) = 0$ which is equivalent to

$$\|v_{\text{even}}\|^2 = \|v_{\text{odd}}\|^2. \tag{7}$$

But Tv is also eigenvector of T for the eigenvalue λ, hence

$$\|(Tv)_{\text{even}}\|^2 = \|(Tv)_{\text{odd}}\|^2. \tag{8}$$

We claim that

$$\|(Tv)_{\text{even}}\|^2 \geqslant (c_1 - c_2)^2 \|v_{\text{odd}}\|^2. \tag{9}$$

We have

$$\begin{aligned}
\|(Tv)_{\text{even}}\|^2 &= \sum_{s=1}^{k/2} |(Tv)_{2s}|^2 \\
&= \sum_s |a_{2s-1} v_{2s-1} + a_{2s} v_{2s+1}|^2 \\
&= \sum_s \big|[(2s-1)^\alpha + c_2] v_{2s-1} + [(2s)^\alpha + c_2] v_{2s+1} + (c_1 - c_2) v_{2s-1}\big|^2 \\
&= \sum_s \big|[(2s-1)^\alpha + c_2] v_{2s-1} + [(2s)^\alpha + c_2] v_{2s+1}\big|^2 + (c_1 - c_2)^2 \sum_s |v_{2s-1}|^2 \\
&\quad + 2\operatorname{Re} \sum_s \big[\big((2s-1)^\alpha + c_2\big) v_{2s-1} + \big((2s)^\alpha + c_2\big) v_{2s+1}\big] \overline{v}_{2s-1} (c_1 - c_2),
\end{aligned}$$

where $v_{k+1} := 0$. Note that

$$2\operatorname{Re} \sum_s \big[\big((2s-1)^\alpha + c_2\big) v_{2s-1} + \big((2s)^\alpha + c_2\big) v_{2s+1}\big] \overline{v}_{2s-1} \geqslant 0.$$

Indeed, the last sum is bounded from below by

$$\begin{aligned}
&2 \sum_s \Big[\big((2s-1)^\alpha + c_2\big) |v_{2s-1}|^2 - \frac{1}{2} \big((2s)^\alpha + c_2\big) \big(|v_{2s+1}|^2 + |v_{2s-1}|^2\big)\Big] \\
&= \sum_s \big[2(2s-1)^\alpha + 2c_2 - \big((2s)^\alpha + c_2\big)\big] |v_{2s-1}|^2 - \sum_s \big((2s)^\alpha + c_2\big) |v_{2s+1}|^2 \\
&= \sum_s \big[2(2s-1)^\alpha - (2s)^\alpha - (2s-2)^\alpha\big] |v_{2s-1}|^2 + (2 - 2^\alpha + c_2) |v_1|^2 \geqslant 0,
\end{aligned}$$

due to condition (c) and to concavity of n^α $(\alpha < 1)$. Combining (7), (8) and (9) we get the desired inequality (6). $\square$

REMARK 3. Note that for odd k the matrix $\mathscr{J}_c(k)$ is never invertible. Actually, since $TV = -VT$ and $\det V = -1$ so $\det T = 0$. We also emphasize that the above proof does not follow from the infinite-dimensional one because the truncation anihilates large a_n.

Now we recall general result which will be used in the proof of the next Lemma and also in other proofs of the next section.

LEMMA 2. *If A is a selfadjoint operator in a Hilbert space, $\lambda \in \mathbb{R}$ and $\|(A-\lambda I)x\| < \varepsilon$ for some $\varepsilon > 0$ and x in the domain of A with $\|x\| = 1$, then there exists $\mu \in \sigma(A)$ such that $|\lambda - \mu| < \varepsilon$.*

LEMMA 3.

$$T_{N,r} \subset \sigma_{\mathrm{ess}}(\mathscr{J}).$$

Proof. Let $\lambda \in T_{N,r}$. By definition of the monodromy matrix $M(\lambda)$ we can write

$$\begin{aligned}\begin{pmatrix} P_{k_{2s-1}+1}(\lambda) \\ P_{k_{2s-1}+2}(\lambda)\end{pmatrix} &= (B_{r+1}B_r\cdots B_2)[M(\lambda)]^{m_s}\overrightarrow{u_2}(\lambda) \\ &= (B_{r+1}\cdots B_2)\big(\mu_+^{m_s}P_+\overrightarrow{u_2}(\lambda) + \mu_-^{m_s}P_-\overrightarrow{u_2}(\lambda)\big),\end{aligned} \tag{10}$$

where $k_{2s-1} = m_s \cdot N + r$. Since $\big(B_{r+1}\cdots B_2 P_+\overrightarrow{u_2}(\lambda)\big)_1 = 0$ (by definition of $T_{N,r}$), (10) implies that

$$P_{k_{2s-1}+1}(\lambda) = \mathrm{O}(\mu_-^{m_s}), \text{ as } m_s \to +\infty. \tag{11}$$

Consider the sequence of $\mathbb{C}^{k_{2s-1}}$ vectors

$$\overrightarrow{w_s} := \big(1, P_2(\lambda), \ldots, P_{k_{2s-1}}(\lambda)\big)^t.$$

Using (11), we obtain

$$\big\|[\mathscr{J}_z(k_{2s-1}) - \lambda]\overrightarrow{w_s}\big\| = \big\|\big(0,\ldots,0,-a_{k_{2s-1}}P_{k_{2s-1}+1}(\lambda)\big)^t\big\| = \mathrm{O}(\mu_-^{m_s}), \text{ as } m_s \to +\infty.$$

Hence

$$\frac{\|[\mathscr{J}_z(k_{2s-1}) - \lambda]\overrightarrow{w_s}\|}{\|\overrightarrow{w_s}\|} = \mathrm{O}(\mu_-^{m_s}), \text{ as } m_s \to +\infty,$$

where a_j is the $j-th$ weight of $\mathscr{J}_z$. Using Lemma 2 we deduce that there exists an eigenvalue $\rho_s \in \sigma\big(\mathscr{J}_z(k_{2s-1})\big)$ with an eigenvector $\vec{f}_{m_s}$ $(\|\vec{f}_{m_s}\| = 1)$ such that

$$|\rho_s - \lambda| = \mathrm{O}(\mu_-^{m_s}), \text{ as } m_s \to +\infty.$$

Define the sequence of ℓ^2-vectors of the norm one written in the block form corresponding to the decomposition (2)

$$F_{m_s} = (\vec{\mathbf{0}}, \ldots, \vec{\mathbf{0}}, \vec{f}_{m_s}, \vec{\mathbf{0}}, \vec{\mathbf{0}}, \ldots),$$

here the position of the non-zero vector term equals $2s-1$. By definition $F_{m_s} \rightharpoonup 0$ weakly as $m_s \to \infty$, and

$$\begin{aligned}\|(\mathscr{J}_{sp} - \lambda)F_{m_s}\| &= \|[\mathscr{J}_z(k_{2s-1}) - \lambda]\vec{f}_{m_s}\| \\ &\leqslant \|(\mathscr{J}_z(k_{2s-1}) - \rho_s)\vec{f}_{m_s}\| + |\rho_s - \lambda| \\ &= |\rho_s - \lambda| = \mathrm{O}(\mu_-^{m_s}),\ \text{as } m_s \to +\infty.\end{aligned}$$

Thus $\lambda \in \sigma_{\mathrm{ess}}(\mathscr{J}_{sp})$ and the proof is complete. □

It is convenient to reformulate the definition of $T_{N,r}$. Assume that $\lambda \notin \sigma_{\mathrm{ess}}(\mathscr{J}_z)$ then $\lambda \in T_{N,r}$ if and only if

(i) $P_+\overrightarrow{u_2}(\lambda) \neq 0$,

(ii) $B_{r+1}\cdots B_2P_+\overrightarrow{u_2}(\lambda) = \begin{pmatrix} 0 \\ w(\lambda)\end{pmatrix}$, for a certain $w(\lambda) \neq 0$.

In other words

$$P_+\overrightarrow{u_2}(\lambda) = w(\lambda)(B_{r+1}\cdots B_2)^{-1}\begin{pmatrix}0\\1\end{pmatrix} =: \begin{pmatrix}R_1(\lambda)\\R_2(\lambda)\end{pmatrix}.$$

Since $P_+\overrightarrow{u_2}(\lambda)$ should be a nonzero eigenvector of $M(\lambda) \equiv \big(M_{ij}(\lambda)\big)_{i,j=1}^2$ corresponding to μ_+ it follows that

$$M(\lambda)\begin{pmatrix}R_1(\lambda)\\R_2(\lambda)\end{pmatrix} = \mu_+\begin{pmatrix}R_1(\lambda)\\R_2(\lambda)\end{pmatrix},$$

which can be written in an equivalent form:

1. $R_2(\lambda)[M_{11}(\lambda)R_1(\lambda) + M_{12}(\lambda)R_2(\lambda)] = R_1(\lambda)[M_{21}(\lambda)R_1(\lambda) + M_{22}(\lambda)R_2(\lambda)]$,

2. $\left|\left\langle M(\lambda)\begin{pmatrix}R_1(\lambda)\\R_2(\lambda)\end{pmatrix}, \begin{pmatrix}R_1(\lambda)\\R_2(\lambda)\end{pmatrix}\right\rangle\right| > \left\|\begin{pmatrix}R_1(\lambda)\\R_2(\lambda)\end{pmatrix}\right\|^2$,

because $|\mu_+| > 1$.

Since we know the monodromy matrix, the above equation is easy to solve(as a quadratic equation in $R_1(\lambda)/R_2(\lambda)$). On the other hand to verify the condition $P_+\overrightarrow{u_2}(\lambda) \neq 0$ is not easy to check cause P_+ is not known. Note that condition (ii) leads to the third restriction

(iii)

$$\big((B_{r+1}\cdots B_2)(R_1(\lambda), R_2(\lambda))^t\big)_1 = 0.$$

In other words one can say that $\lambda \notin \sigma_{\mathrm{ess}}(\mathscr{J}_z)$ belongs to $T_{N,r}$ if and only if the conditions (i),(ii) and (iii) are satisfied.

We shall find more efficient form of these conditions for $r = 0,1,2$ in Section 5, see Remark 4.

3. Proof of the main result

Let the 2-periodic sequence $\{c_k\}$ satisfy condition (c) (see Section 1) and the sequence $\{k_s\}$ fulfills conditions of Theorem 1. The condition $\lim d_s = 0$ implies that $\sigma_{\text{ess}}(\mathscr{J}_{sp}) = \sigma_{\text{ess}}(\mathscr{J})$. The infinite block matrix $\mathscr{J}_{sp}$ is defined by (2). The proof of Theorem 1 will be complete if we check the following two inclusions:

(i) *First inclusion*:

$$\sigma(\mathscr{J}_c) \cup \sigma(\mathscr{J}_z) \cup T_{N,r} \subset \sigma_{\text{ess}}(\mathscr{J}_{sp}). \tag{12}$$

(ii) *Second inclusion*:

$$\sigma_{\text{ess}}(\mathscr{J}_{sp}) \subset \sigma(\mathscr{J}_c) \cup \sigma(\mathscr{J}_z) \cup T_{N,r}. \tag{13}$$

Proof of the first inclusion (12). Due to Lemma 3 it is enough to verify that

$$\sigma(\mathscr{J}_c) \cup \sigma(\mathscr{J}_z) \subset \sigma_{\text{ess}}(\mathscr{J}_{sp}). \tag{14}$$

Fix $\lambda \in (-\infty, -c) \cup (c, +\infty) \subset \sigma(\mathscr{J}_c)$, $c := c_1 - c_2$. We may assume without loss of generality that $\lambda \neq \pm c$. For this λ any nonzero generalized eigenvector $\{u_n\}_1^\infty$ of $\mathscr{J}_c$ satisfies the estimates

$$dn^{-\alpha/2} \leqslant \|\vec{u}_n\| \leqslant Dn^{-\alpha/2}, \tag{15}$$

for some positive constants d, D and $n = 2, 3, \ldots$. See for instance [9] or [10].

We define a new sequence of complex numbers $\{v_n\}_1^\infty$ with finite supports – whose location and lengths are determined by $\{k_s\}_1^\infty$ – as follows. Let

$$\Delta_s := \frac{k_{2s}}{2}.$$

Recalling the standard trick of "linear cut-off of Weyl sequences" given for example in [12], we define the sequence

$$v_s(n) = \begin{cases} u_n[1 + \frac{1}{\Delta_s}(n - \Delta_s)], & 0 < n \leqslant \Delta_s, \\ u_n[1 - \frac{1}{\Delta_s}(n - \Delta_s - 1)], & \Delta_s < n \leqslant 2\Delta_s \\ 0, & n > k_{2s}. \end{cases}$$

Since the sequence $v_s(\cdot)$ has at most $2\Delta_s$ values different from zero we can estimate the ℓ^2-norm of $v_s(\cdot)$ from below as follows (assuming that $s \gg 1$)

$$\|v_s\|^2 \geqslant \sum_{-\Delta_s/4 \leqslant k - \Delta_s \leqslant \Delta_s/4} \|\vec{u}_k\|^2 \tag{16}$$

$$\geqslant \frac{1}{2} \sum_{-\Delta_s/8 \leqslant k - \Delta_s \leqslant \Delta_s/8} \|\vec{u}_k\|^2 \geqslant \frac{1}{2} \min_{-\Delta_s/8 \leqslant k - \Delta_s \leqslant \Delta_s/8} \|\vec{u}_k\|^2 \Delta_s/4.$$

Combining the last inequality with (15) we obtain

$$\|v_s\|^2 \geqslant \frac{1}{16} d^2 {\Delta_s}^{1-\alpha} =: p{\Delta_s}^{1-\alpha}. \tag{17}$$

In the reasoning given in this part of the proof a_k denotes $k-th$ weight of $\mathscr{J}_c$. For the k_{2s}-dimensional vector sequence

$$\overrightarrow{g_s} := (v_s(1),\dots,v_s(2\Delta_s))^t$$

we compute

$$\|[\mathscr{J}_c(k_{2s})-\lambda]\,\overrightarrow{g_s}\|^2 = {\Delta_s}^{-2}\|(2a_1u_2-\lambda u_1,\cdots,\pm(a_ru_{r+1}-a_{r-1}u_{r-1}), \qquad (18)$$

$$\cdots a_{k_{2s}-2}u_{k_{2s}-2}-a_{k_{2s}-1}u_{k_{2s}},\quad a_{k_{2s}-1}u_{k_{2s-1}}-a_{k_{2s}}u_{k_{2s}+1})^t\|^2.$$

The sign $\pm$ means $+$ if $r \leqslant \Delta_s$ and $-$ for $r > \Delta_s$.

The above identity can be easily checked by definition of $v_s(\cdot)$ and the recurrence equations satisfied by the generalized eigenvector $\{u_n\}_1^\infty$. Indeed, the $r-th$ coordinate (for $r > \Delta_s$) of the vector $(\mathscr{J}_c(k_{2s})-\lambda)\overrightarrow{g_s}$ equals

$$-\frac{r-1-\Delta_s}{\Delta_s}(a_ru_{r+1}+a_{r-1}u_{r-1}-\lambda u_r)+\frac{(a_{r-1}u_{r-1}-a_ru_{r+1})}{\Delta_s}=\frac{(a_{r-1}u_{r-1}-a_ru_{r+1})}{\Delta_s}.$$

Now the norm $\|[\mathscr{J}_c(k_{2s})-\lambda]\,\overrightarrow{g_s}\|^2$ can be estimated from above by

$$\frac{1}{\Delta_s^2}\big(|2a_1u_2-\lambda u_1|^2+\sum_{t=2}^{k_{2s}}\big(|a_{t-1}u_{t-1}|+|a_tu_{t+1}|\big)^2\big).$$

The last estimate and (15), prove that $\|[\mathscr{J}_c(k_{2s})-\lambda]\,\overrightarrow{g_s}\|^2$ is less than $M{\Delta_s}^{\alpha-1}$, for a certain constant $M>0$. Combining this and (17) we obtain

$$\frac{\|[\mathscr{J}_c(k_{2s})-\lambda]\,\overrightarrow{g_s}\|^2}{\|\overrightarrow{g_s}\|^2}\leqslant K{\Delta_s}^{2\alpha-2},$$

where $K := M/p$.

Since $\alpha < 1$

$${\Delta_s}^{2\alpha-2}\to 0,\ \text{as } s\to+\infty$$

and the above $\frac{\|[\mathscr{J}_c(k_{2s})-\lambda]\,\overrightarrow{g_s}\|^2}{\|\overrightarrow{g_s}\|^2}$ tends to zero as $s\to+\infty$.

Now define

$$G_s := (\mathbf{0},\dots,\mathbf{0},\overrightarrow{g_s},\mathbf{0},\dots)\in\ell^2,$$

where $\mathbf{0}$ are zero vectors of dimensions corresponding to the decomposition (2), and $\overrightarrow{g_s}$ occupies $2s$ coordinate of G_s. Therefore

$$\frac{\|(\mathscr{J}_{sp}-\lambda)G_s\|^2}{\|G_s\|^2}=\frac{\|[\mathscr{J}_c(k_{2s})-\lambda]\,\overrightarrow{g_s}\|^2}{\|\overrightarrow{g_s}\|^2}\xrightarrow[s\to+\infty]{}0.$$

Surely $\frac{G_s}{\|G_s\|}$ tends weakly to zero, so $\lambda\in\sigma_{\text{ess}}(\mathscr{J}_{sp})=\sigma_{\text{ess}}(\mathscr{J})$.

Consider now

$$y\in\text{“Int“}[\sigma(\mathscr{J}_z)]=\{x\in\mathbb{R}\mid|\mathrm{Tr}M(x)|<2\}.$$

We claim that there exists $C = C(y) > 0$ and $\mu_s \in \sigma\big(\mathscr{J}_z(k_{2s-1})\big)$ such that

$$|y - \mu_s| \leqslant \frac{C}{\sqrt{k_{2s-1}}}. \tag{19}$$

Letting $s \to +\infty$ in (19) we see that y is an accumulation point of $\cup_s \sigma\big(\mathscr{J}_z(k_{2s-1})\big)$ counted with multiplicity, thus y belongs to $\sigma_{\text{ess}}(\mathscr{J}_{sp}) = \sigma_{\text{ess}}(\mathscr{J})$.

To prove the above claim, let us consider the vectors $\overrightarrow{x_s} := (1, P_2(y), \dots, P_{k_{2s-1}}(y))^t$. In this part of the proof a_k will denote the $k-th$ weight of $\mathscr{J}_z$. Using (3) we have

$$\begin{aligned}\big\|(\mathscr{J}_z(k_{2s-1}) - y)\,\overrightarrow{x_s}\big\| &= \big\|\big(0,\dots,0, a_{k_{2s-1}-1}P_{k_{2s-1}-1}(y) - yP_{k_{2s-1}}(y)\big)^t\big\| \\ &= \big\|\big(0,\dots,0,-z_rP_{k_{2s-1}+1}(y)\big)^t\big\| \leqslant D\end{aligned}$$

for some $D = D(y)$ and all $s = 1, 2, \dots$ The last equation is derived from the relation $a_{k_{2s-1}-1}P_{k_{2s-1}-1}(y) + a_{k_{2s-1}}P_{k_{2s-1}+1}(y) = yP_{k_{2s-1}}(y)$, and $k_{2s-1} = m_s \cdot N + r$. Evoking (3) one can check that

$$\|\overrightarrow{x_s}\| \geqslant q\sqrt{k_{2s-1}}, \text{ for some } q = q(y) > 0.$$

Thus

$$\frac{\big\|(\mathscr{J}_z(k_{2s-1}) - y)\,\overrightarrow{x_s}\big\|}{\|\overrightarrow{x_s}\|} \leqslant \frac{D}{q}\frac{1}{\sqrt{k_{2s-1}}} \to 0, \text{ as } s \to +\infty$$

and (19) holds with $C := \frac{D}{q}$. Applying again Lemma 2 we can find an eigenvalue $\mu_s \in \sigma\big(\mathscr{J}_z(k_{2s-1})\big)$ which satisfies (19). Therefore $y \in \sigma_{\text{ess}}(\mathscr{J}_{sp})$. Since $TrM(\lambda)$ is a polynomial "$Int\,\sigma_{ac}(\mathscr{J}_z)$" differs from $\sigma_{ac}(\mathscr{J}_z)$ by a finite number of real points we have obtained to the desired inclusion $\sigma_{ac}(\mathscr{J}_z) \subset \sigma_{\text{ess}}(\mathscr{J}_{sp})$.

Finally consider $\sigma_{\text{p}}(\mathscr{J}_z)$. Remind that $\sigma(\mathscr{J}_z) = \sigma_{\text{ac}}(\mathscr{J}_z) \cup \sigma_{\text{p}}(\mathscr{J}_z)$ and $\sigma_{\text{p}}(\mathscr{J}_z)$ is finite and lies outside of $\sigma_{\text{ac}}(\mathscr{J}_z)$, [4]. Let $w \in \sigma_{\text{p}}(\mathscr{J}_z)$. The eigenvector $\big(1, P_2(w), P_3(w), \dots\big)^t$ is in ℓ^2. Similarly as above for the sequence $\overrightarrow{x_s} := (1, P_2(w), \dots, P_{k_{2s-1}}(w))^t$ we have

$$\big\|[\mathscr{J}_z(k_{2s-1}) - w]\,\overrightarrow{x_s}\big\| = \big\|(0,\dots,0, z_rP_{k_{2s-1}+1}(w))^t\big\|.$$

Since $P_{k_{2s-1}}(w) \to 0$ as $s \to +\infty$ and $\|\overrightarrow{x_s}\| \geqslant 1$, so $w \in \sigma_{\text{ess}}(\mathscr{J}_{sp})$ by the same reasoning as above for $y \in$ "Int"$[\sigma_{\text{ac}}(\mathscr{J}_z)]$.

This completes the proof of (12). □

Proof of the second inclusion (13). To prove the opposite inclusion (13) suppose that there exists $\lambda_0 \in \sigma_{\text{ess}}(\mathscr{J}_{sp})$ which does not belong to $\sigma(\mathscr{J}_c) \cup \sigma(\mathscr{J}_z) \cup T_{N,r}$. Since $\sigma(\mathscr{J}_c)$, $\sigma(\mathscr{J}_z)$ are closed and $T_{N,r}$ is finite (by (5)) we can choose $\varepsilon > 0$ so small that the interval around λ_0 with radius ε is contained in the set

$$Z_\varepsilon := \big\{w \in \mathbb{R} \mid \text{dist}\big(w, \sigma(\mathscr{J}_c) \cup \sigma(\mathscr{J}_z) \cup T_{N,r}\big) > \varepsilon\big\}.$$

We claim that there exists a number $k(\varepsilon) \in \mathbb{N}$ such that for all $k_{2s-1} > k(\varepsilon)$,

$$Z_\varepsilon \cap \sigma\big(\mathscr{J}_z(k_{2s-1})\big) = \varnothing. \tag{20}$$

Assume for a while that (20) is satisfied for all $k_{2s-1} > k(\varepsilon)$. This leads to a contradiction because then

$$(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \cap \sigma\big(\mathscr{J}_z(k_{2s-1})\big) = \varnothing$$

for $k_{2s-1} > k(\varepsilon)$ but λ_0 as a point of $\sigma_{\mathrm{ess}}(\mathscr{J}_{sp})$ must be an accumulation point of $\cup_s \sigma\big(\mathscr{J}_z(k_{2s-1})\big)$ counted with multiplicity. Remember that $\sigma\big(\mathscr{J}_c(k_{2s})\big) \subset \mathbb{R} \setminus (c_2 - c_1, c_1 - c_2)$ (for all s) due to Lemma 1, and therefore does not contribute to the part of spectrum under consideration. Therefore it remains to prove (20). Suppose on the contrary that for a certain sequence $w_i \in Z_\varepsilon$ there exists a subsequence $\{k_{2s_i-1}\}$ growing to infinity such that $w_i \in \sigma\big(\mathscr{J}_z(k_{2s_i-1})\big)$, for $i = 1, 2, \ldots$, with $k_{2s_i-1} = m_{s_i} \cdot N + r$. It follows (see the reasoning in the proof of Lemma 3) that for $\overrightarrow{g_s}(w_i) := \big(1, P_2(w_i), \ldots, P_{k_{2s_i-1}}(w_i)\big)^t$,

$$\begin{aligned}
&\big[\mathscr{J}_z(k_{2s_i-1}) - w_i\big]\overrightarrow{g_s}(w_i) = 0,\\
&P_{k_{2s_i-1}+1}(w_i) = 0.
\end{aligned}$$

Using (10) we get

$$\begin{aligned}
\begin{pmatrix} P_{k_{2s_i-1}+1}(w_i) \\ P_{k_{2s_i-1}+2}(w_i) \end{pmatrix} &= \mu_+^{m_{s_i}} B_{r+1}(w_i) \cdots B_2(w_i) P_+ \overrightarrow{u_2}(w_i) \\
&\quad + \mu_-^{m_{s_i}} B_{r+1}(w_i) \ldots B_2(w_i) P_- \overrightarrow{u_2}(w_i)
\end{aligned}$$

Looking at the first components of the last expression we obtain

$$\begin{aligned}
0 &= P_{k_{2s_i-1}+1}(w_i) \\
&= \mu_+{}^{m_{s_i}} \big(B_{r+1}(w_i) \ldots B_2(w_i) P_+ \overrightarrow{u_2}(w_i)\big)_1 + \mu_-{}^{m_{s_i}} \big(B_{r+1}(w_i) \ldots B_2(w_i) P_- \overrightarrow{u_2}(w_i)\big)_1.
\end{aligned}$$

We can assume that $w_i \to w_0 \in \overline{Z_\varepsilon}$, as $i \to +\infty$. Note that μ_-, P_+ and B_k depend on w_i and converge to $\mu_-(w_0)$, $P_+(w_0)$ and $B_k(w_0)$, as $i \to +\infty$. Therefore we arrive at uniform in the index i estimate

$$\big(B_{r+1}(w_i) \ldots B_2(w_i) P_+ \overrightarrow{u_2}(w_i)\big)_1 = \mathrm{O}(\mu_-{}^{2m_{s_i}}), \text{ as } i \to +\infty.$$

Since $|\mu_-(w_0)| < 1$ and $\mu_+(w_0) = 1/\mu_-(w_0)$ the last relation implies that

$$\big(B_{r+1}(w_0) \ldots B_2(w_0) P_+ \overrightarrow{u_2}(w_0)\big)_1 = 0,$$

and by definition of $T_{N,r}$ we conclude that $w_0 \in T_{N,r}$. This contradicts the inclusion $w_0 \in \overline{Z_\varepsilon}$.

The proofs of inclusions (13) and (12) are complete. $\square$

4. The essential spectrum by Last–Simon approach: heuristics

In recent years have appeared works concerning band dominated operators and their essential spectra, [7], [14]. In particular, in [14] the notion of *limit operator* is used to describe essential spectra of various classes of bounded operators. However, in what follows we shall use the ideas of the paper by Last–Simon [12] only, where for a *bounded* Jacobi matrix K was defined a *right limit point* of K. By definition a *right limit point* of $K = K(\{a_n\},\{b_n\})$ is a double-sided Jacobi matrix $K^{(r)}$ acting in $\ell^2(\mathbb{Z})$ with the entries $\{a_n^{(r)}, b_n^{(r)}\}_{n=-\infty}^{\infty}$ such that there is a sequence of natural numbers $\{n_j\}$ with $n_j \to +\infty$ which for each fixed $l \in \mathbb{Z}$ satisfies the relation

$$\begin{aligned} a_{n_j+l} &\xrightarrow[j\to\infty]{} a_l^{(r)}, \\ b_{n_j+l} &\xrightarrow[j\to\infty]{} b_l^{(r)}. \end{aligned} \tag{21}$$

THEOREM 2. (Last–Simon [12, Theorem 1.7]) *If $\mathscr{R}$ denotes the set of right limit points, then*

$$\sigma_{\mathrm{ess}}(K) = \bigcup_{r\in\mathscr{R}} \sigma(K^{(r)}). \tag{22}$$

Unfortunately, this description cannot be extended to general unbounded Jacobi matrices. Nevertheless, it inspired us in searching for another representation of $\sigma_{\mathrm{ess}}(\mathscr{J})$ by using heuristic arguments in the style of Last–Simon paper.

By applying formally the definition of $\mathscr{J}^{(r)}$ for our unbounded $\mathscr{J}$ we arrive at the following three classes of $\mathscr{J}^{(r)}$, i.e., $\mathscr{R}$ is divided into three subsets. Each of them is determined by one Jacobi matrix (up to a translation of the entries which does not change its spectrum). One may prove that there are no more reasonable new right limit point matrices, besides the three ones considered below.

First class

Fix $s \in \mathbb{Z}$, and define the subsequence

$$n_j = k_1 + k_2 + \ldots + k_{2j} + N\tau_j + s, \text{ with } \tau_j := \left\lfloor \tfrac{k_{2j+1}}{2N} \right\rfloor,$$

where $\lfloor \cdot \rfloor$ denotes the integer part. This choice of n_j is motivated by our requirement to place these integers strictly inside the "z_s" part of our definition of the entries of $\mathscr{J}$. It allows to prevent an influence of the "c" parts on the definition of the right limit matrix. The right limit matrix $\widetilde{\mathscr{J}}$ is a periodic one and is defined by periodic blocks $[z_1,\ldots,z_N]$ and the main diagonal equal to 0 (up to a translation depending on s). It is well known that $\sigma(\widetilde{\mathscr{J}})$ is purely absolutely continuous and consists of a finite number of intervals, see [15]. In particular, the Last–Simon result suggests the inclusion

$$\sigma(\widetilde{\mathscr{J}}) \subset \sigma_{\mathrm{ess}}(\mathscr{J}). \tag{23}$$

Note that due to the Glazman spliting lemma, see [2], $\sigma(\widetilde{\mathscr{J}}) = \sigma_{\mathrm{ess}}(\widetilde{\mathscr{J}}) = \sigma_{\mathrm{ess}}(\mathscr{J}_z)$.

Second class

Fix $s \in \mathbb{Z}$, and define the subsequence

$$n_j = k_1 + k_2 + \ldots + k_{2j-1} + s.$$

Now the right limit matrix $\widetilde{\mathscr{J}}$ is given (again up to a translation) by the weight sequence

$$(\ldots, [z_1, \ldots, z_N], [z_1, \ldots, z_N], z_1, \ldots, z_{r-1}, 0, 1^\alpha + c_1, \ldots, n^\alpha + c_n, \ldots).$$

The dots at the left mean an infinite sequence of blocks $[z_1, \ldots, z_N]$ whereas at the right they mean the infinite sequence $(n+1)^\alpha + c_{n+1}, (n+2)^\alpha + c_{n+2}, \ldots$. Hence one can decompose

$$\widetilde{\mathscr{J}} = T_1 \oplus T_2, \tag{24}$$

where $T_2 := \mathscr{J}_c$ and Jacobi matrix T_1 is defined on $\ell^2(\mathbb{Z}_-)$ where $\mathbb{Z}_- = \{n \in \mathbb{Z} \mid n < 0\}$ with the weights given by

$$(\ldots, [z_1, \ldots, z_N], [z_1, \ldots, z_N], z_1, \ldots, z_{r-2}, z_{r-1}).$$

Observe that T_1 is unitary equivalent by the reflection map to the Jacobi matrix $\mathscr{J}'_1$ acting in $\ell^2(\mathbb{N})$ and defined by the sequence of weights

$$(z_{r-1}, z_{r-2}, \ldots, z_1, [z_N, \ldots, z_1], [z_N, \ldots, z_1], \ldots).$$

The dots at the right end mean infinitely many repeated blocks $[z_N, \ldots, z_1]$. Indeed, if $R : e_{-k} \to e_k, k = 1, 2, \ldots$ is the reflection map, then using definitions of T_1 and $\mathscr{J}'_1$ one can check that

$$RT_1 = \mathscr{J}'_1 R. \tag{25}$$

Again, the Last–Simon theorem "gives" (by using its easier inclusion) the relation

$$\sigma(\widetilde{\mathscr{J}}) = \sigma(\mathscr{J}_c) \cup \sigma(T_1) \subset \sigma_{\text{ess}}(\mathscr{J}).$$

Third class

Note that by choosing $j \to \infty$ in the definition

$$n_j := k_1 + \ldots + k_{2j} + s$$

we obtain formally (up to a translation) the Jacobi matrix generated by the weight sequence

$$(\ldots, \infty, \infty, 0, [z_1, \ldots, z_N], [z_1, \ldots, z_N], \ldots).$$

The appearance of 0 here (as well as above for the second class) is determined by $d_j \to 0$. Similarly as above, this suggests that

$$\sigma(\mathscr{J}_z) \subset \sigma_{\text{ess}}(\mathscr{J}).$$

In Theorem 1 we have proved

$$\sigma_{\mathrm{ess}}(\mathscr{J}) = \sigma(\mathscr{J}_c) \cup \sigma_{\mathrm{ess}}(\mathscr{J}_z) \cup \sigma_{\mathrm{p}}(\mathscr{J}_z) \cup T_{N,r}.$$

Therefore the first class produces $\sigma_{\mathrm{ess}}(\mathscr{J}_z)$. The second one — by Glazman splitting lemma — gives $\sigma(\mathscr{J}_c) \cup \sigma_{\mathrm{ess}}(\mathscr{J}_z) \cup \sigma_{\mathrm{p}}(T_1)$.

Note that $\mathscr{J}'_1$ can be written as the sum $F + \tilde{\mathscr{J}}_z$, where F is the finite rank Jacobi matrix defined by $z_{r-1}, \ldots, z_1$ and $\tilde{\mathscr{J}}_z$ is the Jacobi matrix given by the weights

$$(0, 0, \ldots, 0, [z_N, \ldots, z_1], [z_N, \ldots, z_1], \ldots),$$

and the main diagonal equal to zero. Due to Theorem 2 it is easy to check that $\sigma_{\mathrm{ess}}(\tilde{\mathscr{J}}_z) = \sigma_{\mathrm{ess}}(\mathscr{J}_z)$. Thus

$$\sigma_{\mathrm{ess}}(T_1) = \sigma_{\mathrm{ess}}(\mathscr{J}'_1) = \sigma_{\mathrm{ess}}(\tilde{\mathscr{J}}_z) = \sigma_{\mathrm{ess}}(\mathscr{J}_z). \tag{26}$$

Therefore the third class adds only $\sigma(\mathscr{J}_z)$ which contains as a new part $\sigma_{\mathrm{p}}(\mathscr{J}_z)$. Note that the reasoning given in Section 3 allows to prove (by constructing suitable Weyl sequences) the inclusion

$$\sigma(\mathscr{J}_c) \cup \sigma(\mathscr{J}_z) \cup \sigma_{\mathrm{p}}(\mathscr{J}'_1) \subset \sigma_{\mathrm{ess}}(\mathscr{J}). \tag{27}$$

This inclusion justifies the simpler inclusion of the “Last–Simon theorem” for our unbounded $\mathscr{J}$. Using formally the opposite (more difficult) inclusion of the Last–Simon formula, we should obtain

$$\sigma_{\mathrm{ess}}(\mathscr{J}) = \sigma(\mathscr{J}_c) \cup \sigma(\mathscr{J}_z) \cup \sigma_{\mathrm{p}}(\mathscr{J}'_1). \tag{28}$$

In turn (26) entails $\sigma_{\mathrm{p}}(\mathscr{J}'_1) \cap \sigma_{\mathrm{ess}}(\mathscr{J}_z) = \varnothing$. Combining Theorem 1 and (28) we should expect that $T_{N,r} = \sigma_{\mathrm{p}}(\mathscr{J}'_1) \setminus \sigma_{\mathrm{p}}(\mathscr{J}_z)$.

Consequently, to prove (28) rigorously we shall establish below the last relation between $\sigma_{\mathrm{p}}(\mathscr{J}'_1)$ and $T_{N,r}$.

THEOREM 3. $T_{N,r} = \sigma_{\mathrm{p}}(\mathscr{J}'_1) \setminus \sigma_{\mathrm{p}}(\mathscr{J}_z)$.

Proof. First note that for real $\lambda \notin \sigma(\mathscr{J}_z) = \sigma_{\mathrm{p}}(\mathscr{J}_z) \cup \sigma_{\mathrm{ess}}(\mathscr{J}_z)$, $P_+\binom{1}{\lambda/z_1} \neq \vec{0}$. Thus $B_{r+1} \ldots B_2 P_+\binom{1}{\lambda/z_1} \neq \vec{0}$. If moreover $\lambda \in T_{N,r}$ then by definition of $T_{N,r}$

$$B_{r+1} \ldots B_2 P_+ \begin{pmatrix} 1 \\ \lambda/z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ x \end{pmatrix}, \text{ with } x \neq 0. \tag{29}$$

Multiply (29) by $B_1 B_N \ldots B_{r+2}$, then using the definition of $M(\lambda)$ and $M(\lambda)P_+ = \mu_+ P_+$ we obtain

$$P_+ \begin{pmatrix} 1 \\ \lambda/z_1 \end{pmatrix} = B_1 B_N \ldots B_{r+2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot x/\mu_+. \tag{30}$$

Since $P_-P_+ = 0$ so (30) implies that

$$P_-B_1B_N\dots B_{r+2}\begin{pmatrix}0\\1\end{pmatrix} = \vec{0}. \tag{31}$$

Conversely, if (31) holds then $B_1B_N\dots B_{r+2}\binom{0}{1} \neq \vec{0}$ is in the range of P_+. But $\mathscr{R}(P_+) = \left\{c\cdot P_+\binom{1}{\lambda/z_1}, c\in\mathbb{C}\right\}$ for $\lambda \notin \sigma(\mathscr{J}_z)$ (because P_+ is of rank 1 and $P_+\binom{1}{\lambda/z_1} \neq \vec{0}$) we get

$$B_1B_N\dots B_{r+2}\begin{pmatrix}0\\1\end{pmatrix} = c_1P_+\begin{pmatrix}1\\ \lambda/z_1\end{pmatrix}, \quad c_1 \neq 0,$$

which is equivalent to $\lambda \in T_{N,r}$ (see (30)). In this way we have found another description of $T_{N,r}$:

$$T_{N,r} = \left\{\lambda\in\mathbb{R} \mid \lambda\notin\sigma(\mathscr{J}_z) \text{ and } P_-B_1B_N\dots B_{r+2}\begin{pmatrix}0\\1\end{pmatrix} = \vec{0}\right\}. \tag{32}$$

Using (32), we shall complete the proof below. Consider the extension of T_1 to the periodic Jacobi matrix $\mathscr{J}_{\text{ex}}$ on $\ell^2(\mathbb{Z})$ (with the standard ordered enumeration of z_k). One can check that $\lambda\in\sigma_{\text{p}}(T_1)$ if and only if the formal nontrivial solution u of the spectral equation

$$\mathscr{J}_{\text{ex}}u = \lambda u, \tag{33}$$

is square summable at $-\infty$, and $\lambda\notin\sigma_{\text{ac}}(\mathscr{J}_{\text{ex}})$. Note that the condition $u_1 = 0$ is necessary and sufficient for the possibility to extend the eigenvector of T_1 to a solution of (33) (remind that all $z_k \neq 0$). These requirements say that $\lambda\in\sigma_{\text{p}}(T_1)$ if and only if the nonzero solution u of the spectral equation (33) satisfies the following conditions:

i) $u_1 = 0$ (continuation condition).

ii) $\lambda\notin\sigma_{\text{ac}}(\mathscr{J}_{\text{ex}}) \equiv \sigma_{\text{ess}}(\mathscr{J}_z)$ (hyperbolic condition).

iii) $P_-\vec{u}_{N-r+2} = \vec{0}$ (condition of exponential decay of u at $-\infty$).

Concerning the condition of exponential decay at $-\infty$ note that, for any $\ell\in\mathbb{Z}$,

$$\vec{u}_{N-r+2+\ell\cdot N} = M^\ell\cdot\vec{u}_{N-r+2} = \mu_+^\ell P_+\vec{u}_{N-r+2} + \mu_-^\ell P_-\vec{u}_{N-r+2}.$$

Therefore $\vec{u}_k$ is exponentially decaying as $k\to-\infty$(or equivalently as $l\to-\infty$) if and only if $P_-\vec{u}_{N-r+2} = \vec{0}$. Remind that $|\mu_-| < 1$, $|\mu_+| > 1$ provided that condition ii) is satisfied. If $B_s(ex)$ denotes the transfer matrix of $\mathscr{J}_{\text{ex}}$ at the above λ then using definition of $\mathscr{J}_{ex}$ we can write

$$\vec{u}_{N-r+2} = B_{N-r+1}(ex)\dots B_2(ex)\vec{u}_2 = B_1B_N\dots B_{r+2}\vec{u}_2. \tag{34}$$

By our choice $\vec{u}_2 = \binom{0}{1}$, see condition i). Using (32) and (34), we have

$$T_{N,r} = \sigma_{\text{p}}(T_1)\setminus\sigma_{\text{p}}(\mathscr{J}_z) = \sigma_{\text{p}}(\mathscr{J}_1')\setminus\sigma_{\text{p}}(\mathscr{J}_z). \quad \square$$

Theorem 3 being proved rigorously leads to the useful formula

$$\sigma_{\text{ess}}(\mathscr{J}) = \sigma(\mathscr{J}_c) \cup \sigma(\mathscr{J}_z) \cup \sigma_{\text{p}}(\mathscr{J}'_1). \tag{35}$$

Observe that this formula follows “formally“ from Last–Simon theorem provided we enlisted all cases of right limits.

5. Final results and examples

In this section we calculate a few examples of the *new essential spectrum* $T_{N,r}$ and locate it as a subset of the spectrum of a certain natural $(N-1)\times(N-1)$ Jacobi matrix $\widetilde{\mathscr{J}}_r$. This Jacobi matrix is defined by

$$\widetilde{\mathscr{J}}_r = \begin{pmatrix} 0 & z_{r+2} & 0 & \dots & 0 & 0 & 0 \\ z_{r+2} & 0 & z_{r+3} & \dots & 0 & 0 & 0 \\ 0 & z_{r+3} & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & z_{N+r-2} & 0 \\ 0 & 0 & 0 & \dots & z_{N+r-2} & 0 & z_{N+r-1} \\ 0 & 0 & 0 & \dots & 0 & z_{N+r-1} & 0 \end{pmatrix} \tag{36}$$

Let $\lambda \in T_{N,r}$. Since the polynomials $\{P_k(\lambda)\}_1^\infty$ satisfy the recurrence relation

$$z_{k-1}P_{k-1}(\lambda) + z_k P_{k+1}(\lambda) = \lambda P_k(\lambda),$$

$k = 1,2,\dots$, by repeating the reasoning given in the proof of Lemma 3 we have

$$\begin{pmatrix} P_{k+1}(\lambda) \\ P_{k+2}(\lambda) \end{pmatrix} = \mu_+^l (B_{r+1}\dots B_2 P_+ \overrightarrow{u_2}(\lambda)) + \mu_-^l (B_{r+1}\dots B_2 P_- \overrightarrow{u_2}(\lambda)), \tag{37}$$

for $k = l\cdot N + r$, $l = 1,2,\dots$. Using (37) and the definition of $T_{N,r}$, (5), we have the estimate

$$P_{k+1}(\lambda) = \mathrm{O}(|\mu_-|^l), \text{ as } l \to +\infty, \text{ with } k = l\cdot N + r. \tag{38}$$

On the other hand $P_+\overrightarrow{u_2}(\lambda) \neq 0$, see Remark 1, and the matrix $B_{r+1}\dots B_2$ is invertible. So, applying (37) once more we find that

$$|P_{k+2}(\lambda)| \geqslant c(\lambda)|\mu_+|^l, \tag{39}$$

for some positive $c(\lambda)$ and $k = l\cdot N + r$. By shifting the index $l \to l+1$ we have

$$P_{k+N+1}(\lambda) = \mathrm{O}(|\mu_-|^l), \tag{40}$$

as $l \to +\infty$, $k = l\cdot N + r$. These estimates enable us to prove

LEMMA 4. *Let $\widetilde{\mathscr{J}}_r$ be defined by* (36). *Then $T_{N,r} \subset \sigma(\widetilde{\mathscr{J}}_r)$.*

Proof. Assume that $N > 2$ and fix $\lambda \in T_{N,r}$. Define the sequence of vectors

$$\overrightarrow{H_k}(\lambda) := \left(P_{k+2}(\lambda), P_{k+3}(\lambda), \ldots, P_{k+N}(\lambda)\right)^t,$$

where $k = l \cdot N + r$. Using the recurrence relations we obtain the equations

$$\begin{pmatrix} 0 & z_{k+2} & 0 & \ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ z_{k+2} & 0 & z_{k+3} & \ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ 0 & z_{k+3} & 0 & \ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \ldots & \ldots & \ldots & \ldots & 0 & z_{k+N-2} & 0 \\ \ldots & \ldots & \ldots & \ldots & z_{k+N-2} & 0 & z_{k+N-1} \\ \ldots & \ldots & \ldots & \ldots & 0 & z_{k+N-1} & 0 \end{pmatrix} \overrightarrow{H_k}(\lambda) - \lambda \overrightarrow{H_k}(\lambda)$$

$$= \begin{pmatrix} z_{k+2}P_{k+3}(\lambda) - \lambda P_{k+2}(\lambda) \\ 0 \\ \vdots \\ 0 \\ z_{k+N-1}P_{k+N-1}(\lambda) - \lambda P_{k+N}(\lambda) \end{pmatrix} = \begin{pmatrix} -z_{k+1}P_{k+1}(\lambda) \\ 0 \\ \vdots \\ 0 \\ -z_{k+N}P_{k+N+1}(\lambda) \end{pmatrix}.$$

Applying (38), (39), (40) and the last equation we can estimate $\|(\widetilde{\mathscr{J}}_r - \lambda I)\overrightarrow{H_k}\|$ from above and $\|\overrightarrow{H_k}\|$ from below. In other words one can find positive constants $\tilde{C}(\lambda)$ and $\tilde{c}(\lambda)$ such that

$$\frac{\|(\widetilde{\mathscr{J}}_r - \lambda I)\overrightarrow{H_k}\|}{\|\overrightarrow{H_k}\|} \leqslant \frac{\tilde{C}(\lambda)|\mu_-|^l}{\tilde{c}(\lambda)|\mu_+|^l} \xrightarrow[l\to\infty]{} 0,$$

with $k = l \cdot N + r$. It follows that $\operatorname{dist}(\lambda, \sigma(\widetilde{\mathscr{J}}_r)) = 0$, which completes the proof in the case $N > 2$.

If $N = 2$, then using again the estimates (38), (39), (40) and the recurrence relation $z_{k+1}P_{k+1}(\lambda) + z_{k+2}P_{k+3}(\lambda) = \lambda P_{k+2}(\lambda)$ we conclude that $\lambda = 0$. $\square$

At the end of this work we describe $\sigma_{\mathrm{p}}(\mathscr{J}_z)$ and $T_{N,r}$ in terms of the monodromy matrix $M(\cdot)$ and compute examples of $T_{N,r}$ for $N = 2, 3, 4$, to illustrate the situation. We start with a characterization of $T_{N,r}$ for $r = 0, 1, 2$, in terms of the entries of the monodromy matrix $M(\lambda)$.

THEOREM 4. *Let $M(\cdot)$ be the monodromy matrix, and $\lambda \in \mathbb{R}$. Then $\lambda \in T_{N,0}$ if and only if the following conditions are satisfied:*

(i) $M_{12}(\lambda) = 0$,

(ii) $|M_{11}(\lambda)| < |M_{22}(\lambda)|$,

(iii) $\frac{\lambda}{z_1}\left(M_{11}(\lambda) - M_{22}(\lambda)\right) \neq M_{21}(\lambda)$.

Proof. Using the definition of $T_{N,0}$ (see (5)), we know that $\lambda \in T_{N,0}$ if and only if $\lambda \notin \sigma_{\mathrm{ess}}(\mathscr{J}_z)$ and

- $\big(P_+\overrightarrow{u_2}(\lambda)\big)_1 = 0$,
- $\lambda \notin \sigma_{\mathrm{p}}(\mathscr{J}_z)$.

By Remark 1 this is equivalent to the condition

- $P_+\overrightarrow{u_2}(\lambda) = \left(\begin{smallmatrix} 0 \\ w(\lambda) \end{smallmatrix}\right)$, $w(\lambda) \neq 0$,

provided that $\lambda \notin \sigma_{\mathrm{ess}}(\mathscr{J}_z)$. Thus assuming that $P_+\overrightarrow{u_2}(\lambda) = \left(\begin{smallmatrix} 0 \\ w(\lambda) \end{smallmatrix}\right)$ and $\lambda \notin \sigma_{\mathrm{ess}}(\mathscr{J}_z)$ we have $w(\lambda) \neq 0$ if and only if $\overrightarrow{u_2}(\lambda)$ is not an eigenvector of $M(\lambda)$. In other words, if $\lambda \notin \sigma_{\mathrm{ess}}(\mathscr{J}_z)$, then $\lambda \in T_{N,0}$ if and only if

- $M_{12}(\lambda) = 0$, $M_{11}(\lambda) = \mu_-$, $M_{22}(\lambda) = \mu_+$,
- $\frac{\lambda}{z_1}\big(M_{11}(\lambda) + \frac{\lambda}{z_1}M_{12}(\lambda)\big) \neq M_{21}(\lambda) + \frac{\lambda}{z_1}M_{22}(\lambda)$.

This completes the proof. $\square$

REMARK 4. A similar reasoning shows that

a) $\lambda \in T_{N,1}$ if and only if

  (i) $|M_{11}(\lambda)| > |M_{22}(\lambda)|$,

  (ii) $M_{21}(\lambda) = 0$,

  (iii) $M_{11}(\lambda) + \frac{\lambda}{z_1}M_{12}(\lambda) \neq M_{22}(\lambda)$.

b) $\lambda \in T_{N,2}$ if and only if

  (i) $z_1\big(M_{11}(\lambda)\cdot\lambda + M_{12}(\lambda)z_1\big) = \lambda\big(M_{21}(\lambda)\cdot\lambda + M_{22}(\lambda)z_1\big)$,

  (ii) $\big|M_{21}(\lambda)\frac{\lambda}{z_1} + M_{22}(\lambda)\big| > 1$,

  (iii) $\frac{\lambda}{z_1}\big(M_{11}(\lambda) + \frac{\lambda}{z_1}M_{12}(\lambda)\big) \neq M_{21}(\lambda) + \frac{\lambda}{z_1}M_{22}(\lambda)$ or in the case $\frac{\lambda}{z_1}\big(M_{11}(\lambda) + \frac{\lambda}{z_1}M_{12}(\lambda)\big) = M_{21}(\lambda) + \frac{\lambda}{z_1}M_{22}(\lambda)$ then $\big|M_{12}(\lambda)\frac{\lambda}{z_1} + M_{11}(\lambda)\big| > 1$.

Applying Theorem 4 and Remark 4 we have calculated $T_{2,r}$, $T_{3,r}$ and $T_{4,r}$ for $r = 0,1,2$ (see below). We omit straightforward reasoning behind these calculations because they are boring and we also have an easier method of finding all these sets by using Theorem 3. This easier method will be presented at the end of the work in calculation of $T_{4,3}$.

$$T_{2,0} = \varnothing.$$

$$T_{2,1} = \begin{cases} \varnothing, & \text{if } z_1 \leqslant z_2 \\ \{0\}, & \text{if } z_1 > z_2. \end{cases}$$

$$T_{3,0} = \begin{cases} \varnothing, & \text{if } z_1 \geqslant z_3 \text{ or } z_1 = z_2, \\ \{\pm z_2\}, & \text{if } z_1 < z_3 \text{ and } z_1 \neq z_2. \end{cases}$$

$$T_{3,1} = \begin{cases} \varnothing, & \text{if } z_1 \leqslant z_2 \text{ or } z_1 = z_3, \\ \{\pm z_3\}, & \text{if } z_1 > z_2 \text{ and } z_1 \neq z_3. \end{cases}$$

$$T_{3,2} = \begin{cases} \{\pm z_1\}, & \text{if } z_2 > z_3, \\ \varnothing, & \text{if } z_2 \leqslant z_3. \end{cases}$$

Similarly, for $T_{4,r}$ and $r = 0,1,2$:

$$T_{4,0} = \begin{cases} \varnothing, \text{iff } z_1 = z_3 \text{ or } z_1 z_2 \geqslant z_3 z_4, \\ \{\pm\sqrt{z_2^2 + z_3^2}\}, \text{iff } z_1 z_2 < z_3 z_4 \text{ and } z_1 \neq z_3. \end{cases}$$

$$T_{4,1} = \begin{cases} \varnothing, & \text{iff } z_1 z_3 \leqslant z_2 z_4 \text{ and } (z_1 z_4 \leqslant z_2 z_3 \text{ or } z_1^2 + z_2^2 = z_3^2 + z_4^2,) \\ \{0, \pm\sqrt{z_3^2 + z_4^2}\}, & \text{iff } z_1 z_3 > z_2 z_4,\ z_1 z_4 > z_2 z_3, \text{ and } z_1^2 + z_2^2 \neq z_3^2 + z_4^2, \\ \{0\}, & \text{iff } z_1 z_3 > z_2 z_4, \text{ and } (z_1 z_4 \leqslant z_2 z_3 \text{ or } z_1^2 + z_2^2 = z_3^2 + z_4^2,) \\ \{\pm\sqrt{z_3^2 + z_4^2}\}, & \text{iff } z_2 z_4 \geqslant z_1 z_3,\ z_1 z_4 > z_2 z_3 \text{ and } z_1^2 + z_2^2 \neq z_3^2 + z_4^2. \end{cases}$$

$$T_{4,2} = \begin{cases} \{\pm\sqrt{z_1^2 + z_4^2}\}, & \text{if } z_1 z_2 > z_3 z_4 \text{ and } z_2 \neq z_4 \\ \varnothing, \text{ otherwise.} \end{cases}$$

The above formulas for $T_{4,r}$ are related to the following one

$$T_{4,r} \subset \left\{0, \pm\sqrt{z_{r+2}^2 + z_{r+3}^2}\right\} = \sigma(\widetilde{\mathscr{J}}_r).$$

which can checked by direct calculations, see Lemma 4 also.

Note that Lemma 4 is also useful tool in calculations of $T_{N,r}$ for N not too large.

REMARK 5. The above description of $T_{3,r}$ and $T_{4,r}$ shows that the inclusion $T_{N,r} \subset \sigma(\widetilde{\mathscr{J}}_r)$ can be strict. However, we may also have the equality $T_{N,r} = \sigma(\widetilde{\mathscr{J}}_r)$. This can be seen from the relation

$$T_{4,1} = \left\{0, \pm\sqrt{z_3^2 + z_4^2}\right\} = \sigma(\widetilde{\mathscr{J}}_1),$$

by a suitable choice of (z_1, z_2, z_3, z_4).

In order to apply Theorem 3 we need the following elementary characterization of $\sigma_{\mathrm{p}}(\mathscr{J}_z)$ for arbitrary N. The same characterization holds for arbitrary periodic Jacobi matrix with nontrivial main diagonal $\{b_n\}$. One should only replace in (41) and (42) λ by $\lambda - b_1$.

THEOREM 5. *$\lambda \in \sigma_{\mathrm{p}}(\mathscr{J}_z)$ if and only if the following two conditions are satisfied:*

$$\frac{\lambda}{z_1}\left[M_{11}(\lambda)+\frac{\lambda}{z_1}M_{12}(\lambda)\right]=M_{21}(\lambda)+\frac{\lambda}{z_1}M_{22}(\lambda), \tag{41}$$

and

$$\left|M_{11}(\lambda)+\frac{\lambda}{z_1}M_{12}(\lambda)\right|<1. \tag{42}$$

Proof. We have $\lambda \in \sigma_{\mathrm{p}}(\mathscr{J}_z)$ if and only if $P_+\overrightarrow{u_2}(\lambda)=0$. In other words $P_-\overrightarrow{u_2}(\lambda)=\overrightarrow{u_2}(\lambda)$ or $M(\lambda)\overrightarrow{u_2}(\lambda)=\mu_-\overrightarrow{u_2}(\lambda)$. The last equation implies that

$$\frac{M_{21}(\lambda)+\frac{\lambda}{z_1}M_{22}(\lambda)}{M_{11}(\lambda)+M_{12}(\lambda)\cdot\frac{\lambda}{z_1}}=\frac{\lambda}{z_1},$$

which verifies (41). On the other hand,

$$\begin{aligned}|\langle M(\lambda)\overrightarrow{u_2}(\lambda),\overrightarrow{u_2}(\lambda)\rangle| &= \left[1+\left(\frac{\lambda}{z_1}\right)^2\right]\left|M_{11}(\lambda)+\frac{\lambda}{z_1}M_{12}(\lambda)\right| \\ &= |\mu_-|\,\|\overrightarrow{u_2}(\lambda)\|^2 \\ &< \|\overrightarrow{u_2}(\lambda)\|^2 = 1+\left(\frac{\lambda}{z_1}\right)^2\end{aligned}$$

proves (42).

Conversely, (41) gives

$$M(\lambda)\overrightarrow{u_2}(\lambda)=\left[M_{11}(\lambda)+\frac{\lambda}{z_1}M_{12}(\lambda)\right]\overrightarrow{u_2}(\lambda)$$

which combined with (42) leads to $M_{11}(\lambda)+\frac{\lambda}{z_1}M_{12}(\lambda)=\mu_-$, i.e., $M(\lambda)\overrightarrow{u_2}(\lambda)=\mu_-\overrightarrow{u_2}(\lambda)$. Hence $(1,P_2(\lambda),P_3(\lambda),\dots)^t \in \ell^2$ because $|\mu_-|<1$ and therefore $\lambda \notin \sigma_{\mathrm{ess}}(\mathscr{J}_z)$. Indeed,

$$\begin{pmatrix}P_{lN+1}(\lambda)\\P_{lN+2}(\lambda)\end{pmatrix}=M(\lambda)^l\overrightarrow{u_2}(\lambda), l=1,2,\dots \quad \square$$

Using Theorem 5 for $N=3$ we have

$$\sigma_{\mathrm{p}}(\mathscr{J}_z)=\begin{cases}\varnothing, & \text{if } z_3\leqslant z_2,\\ \{\pm z_1\}, & \text{if } z_2<z_3,\end{cases}$$

Similarly for $N=4$ by straightforward calculation we get

$$\sigma_{\mathrm{p}}(\mathscr{J}_z)=\begin{cases}\{0,\pm\sqrt{z_1^2+z_2^2}\}, & \text{if } z_1z_3<z_2z_4 \text{ and } z_2z_3<z_1z_4,\\ \{\pm\sqrt{z_1^2+z_2^2}\}, & \text{if } z_1z_3\geqslant z_2z_4 \text{ and } z_2z_3<z_1z_4,\\ \{0\}, & \text{if } z_1z_3<z_2z_4 \text{ and } z_1z_4\leqslant z_2z_3,\\ \varnothing, & z_1z_3\geqslant z_2z_4 \text{ and } z_2z_3\geqslant z_1z_4.\end{cases}$$

Observe that $\sigma_{\rm p}(\mathscr{J}_z) \subset \sigma(\widetilde{\mathscr{J}}_{N-1})$ for any N. However, there is no equality between $\sigma_{\rm p}(\mathscr{J}_z)$ and $\sigma(\widetilde{\mathscr{J}}_r)$ in general.

The last example of this work concerning $T_{N,r}$ illustrates an application of Theorem 5 and Theorem 3 for simpler calculations of $T_{N,r}$. Let us demonstrate this by finding $T_{4,3} = \sigma_{\rm p}(\mathscr{J}'_1) \setminus \sigma_{\rm p}(\mathscr{J}_z)$. First calculate $\sigma_{\rm p}(\mathscr{J}_z)$. Explicit calculations give the following expressions of the entries $M_{ij}(\lambda)$ of the monodromy matrix:

$$\begin{aligned}
M_{11}(\lambda) &= \frac{z_1 z_3}{z_2 z_4} - \frac{\lambda^2 z_1}{z_2 z_3 z_4},\\
M_{12}(\lambda) &= \frac{\lambda^3}{z_2 z_3 z_4} - \lambda\left(\frac{z_3}{z_2 z_4} + \frac{z_2}{z_3 z_4}\right),\\
M_{21}(\lambda) &= -\frac{\lambda^3}{z_2 z_3 z_4} + \lambda\left(\frac{z_3}{z_2 z_4} + \frac{z_4}{z_2 z_3}\right),\\
M_{22}(\lambda) &= \frac{\lambda^4}{z_1 z_2 z_3 z_4} - \lambda^2 \frac{z_2^2 + z_3^2 + z_4^2}{z_1 z_2 z_3 z_4} + \frac{z_2 z_4}{z_1 z_3}.
\end{aligned}$$

A straightforward application of Theorem 5 allows us to calculate the point spectrum of $\mathscr{J}_z$: $\sigma_{\rm p}(\mathscr{J}_z) \subset \{0, \pm\sqrt{z_1^2 + z_2^2}\}$. Moreover, $0 \in \sigma_{\rm p}(\mathscr{J}_z)$ iff $z_1 z_3 < z_2 z_4$, and $\pm\sqrt{z_1^2 + z_2^2} \in \sigma_{\rm p}(\mathscr{J}_z)$ iff $z_2 z_3 < z_1 z_4$. The permutation $(z_1, z_2, z_3, z_4) \to (z_2, z_1, z_4, z_3)$ immediately gives the same inclusion $\sigma_{\rm p}(\mathscr{J}'_1) \subset \{0, \pm\sqrt{z_1^2 + z_2^2}\}$. Now $0 \in \sigma_{\rm p}(\mathscr{J}'_1)$ iff $z_1 z_3 > z_2 z_4$, and $\pm\sqrt{z_1^2 + z_2^2} \in \sigma_{\rm p}(\mathscr{J}'_1)$ iff $z_1 z_4 < z_2 z_3$. In turn applying Theorem 3 we see that $0 \in T_{4,3}$ iff $z_1 z_3 > z_2 z_4$ and $z_1 z_3 \geqslant z_2 z_4$, which reduces to the only condition $z_1 z_3 > z_2 z_4$. On the other hand, $\pm\sqrt{z_1^2 + z_2^2} \in T_{4,3}$ iff $z_1 z_4 < z_2 z_3$ and $z_2 z_3 \geqslant z_1 z_4$, which again reduces to the only condition $z_1 z_4 < z_2 z_3$. Combining all these results we obtain

$$T_{4,3} = \begin{cases}
\varnothing & \text{iff } z_1 z_3 \leqslant z_2 z_4 \text{ and } z_1 z_4 \geqslant z_2 z_3,\\
\{0\} & \text{iff } z_1 z_3 > z_2 z_4 \text{ and } z_1 z_4 \geqslant z_2 z_3,\\
\{\pm\sqrt{z_1^2 + z_2^2}\} & \text{iff } z_1 z_3 \leqslant z_2 z_4 \text{ and } z_1 z_4 < z_2 z_3,\\
\{0, \pm\sqrt{z_1^2 + z_2^2}\} & \text{iff } z_1 z_3 > z_2 z_4 \text{ and } z_1 z_4 < z_2 z_3.
\end{cases}$$

It is easy to see that all four options can be realized for suitable choice of the z's. Similar calculations of $T_{N,r}$ can be done for arbitrary values of N and r provided one can explicitly calculate $\sigma_{\rm p}(\mathscr{J}_z)$.

Acknowledgments

The two last authors were supported by MSHE through grant N N201 426533, and S. N. was also partially supported by RFBR grant 09-01-00515a.

We are grateful to anonymous referee for many useful and essential comments which helped us to improve substantially this work.

REFERENCES

[1] N. I. AKHIEZER, *The classical moment problem and some related questions in analysis*, Translated by N. Kemmer. Hafner Publishing Co., New York, 1965.

[2] M. S. BIRMAN AND M. Z. SOLOMJAK, *Spectral theory of selfadjoint operators in Hilbert space*, Mathematics and its Applications (Soviet Series), D. Reidel Publishing Co., Dordrecht, 1987.

[3] A. BOUTET DE MONVEL, J. JANAS, AND S. NABOKO, *Unbounded Jacobi matrices with a few gaps in the essential spectrum. Constructive examples*, Integral Equations Operator Theory, 69(2):151–170, 2011.

[4] P. A. COJUHARI, *Discrete spectrum in the gaps for perturbations of periodic Jacobi matrices*, Jour. Comput. Appl. Math., 225:374–386, 2009.

[5] J. DOMBROWSKI, *Eigenvalues and spectral gaps related to periodic perturbations of Jacobi matrices*, In Spectral methods for operators of mathematical physics, volume 154 of Oper. Theory Adv. Appl., pages 91–100. Birkhäuser, Basel, 2004.

[6] J. DOMBROWSKI, J. JANAS, M. MOSZYŃSKI, AND S. PEDERSEN, *Spectral gaps resulting from periodic perturbations of a class of Jacobi operators*, Constr. Approx., 20(4):585–601, 2004.

[7] V. GEORGESCU, *On spectral analysis of quantum field Hamiltonians*, J. Func. Anal., 245(1):89–143, 2007.

[8] I. M. GLAZMAN, *Direct methods of qualitative spectral analysis of singular differential operators*, Translated from the Russian by the IPST staff. Israel Program for Scientific Translations, Jerusalem, 1965, 1966.

[9] J. JANAS AND M. MOSZYŃSKI, *Spectral properties of Jacobi matrices by asymptotic analysis*, J. Approx. Theory, 120(2):309–336, 2003.

[10] J. JANAS, S. NABOKO, AND G. STOLZ, *Decay bounds on eigenfunctions and the singular spectrum of unbounded Jacobi matrices*, Int. Math. Res. Notices, 4:736–764, 2009.

[11] T. KATO, *Perturbation theory for linear operators*, Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition.

[12] Y. LAST AND B. SIMON, *The essential spectrum of Schrödinger, Jacobi, and CMV operators*, J. Anal. Math., 98:183–220, 2006.

[13] M. MOSZYŃSKI, *Spectral properties of some Jacobi matrices with double weights*, J. Math. Anal. Appl., 280(2):400–412, 2003.

[14] V. S. RABINOVICH, S. ROCH AND B. SILBERMANN, *Limit Operators and Their Applications in Operator Theory*, Birkhäuser, 2004.

[15] G. TESCHL, *Jacobi operators and completely integrable nonlinear lattices*, volume 72 of Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, 2000.

(Received January 10, 2011)

Anne Boutet de Monvel
Institut de Mathématiques de Jussieu, Université Paris Diderot
175 rue du Chevaleret
75013 Paris, France
e-mail: aboutet@math.jussieu.fr
http://www.math.jussieu.fr/~aboutet/

Jan Janas
Institute of Mathematics, Polish Academy of Sciences
ul. Św. Tomasza 30
31-027 Kraków, Poland
e-mail: najanas@cyf-kr.edu.pl

Serguei Naboko
Department of Mathematical Physics, Institute of Physics,
St. Petersburg University
Ulianovskaia 1
198904 St. Petergoff
St. Petersburg, Russia
e-mail: naboko@snoopy.phys.spbu.ru