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NULL–ORBIT REFLEXIVE OPERATORS

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Abstract. We introduce and study the notion of null-orbit reflexivity, which is a slight perturbation of the notion of orbit-reflexivity. Positive results for orbit reflexivity and the recent notion of $\mathbb{C}$-orbit reflexivity both extend to null-orbit reflexivity. Of the two known examples of operators that are not orbit-reflexive, one is null-orbit reflexive and the other is not. The class of null-orbit reflexive operators includes the classes of hyponormal, algebraic, compact, strictly block-upper (lower) triangular operators, and operators whose spectral radius is not 1. We also prove that every polynomially bounded operator on a Hilbert space is both orbit-reflexive and null-orbit reflexive.

1. Introduction

In a recent paper [3] the authors and M. McHugh introduced a new notion of reflexivity for operators, $\mathbb{C}$*-orbit reflexivity* as well as its linear-algebraic analogue. This notion is related to the notion of orbit reflexivity [5]. Examples of Hilbert space operators that are not orbit reflexive can be found in two very remarkable papers; the first example was given by S. Grivaux and M. Roginskaya [1], and the second, much simpler, example was given by V. Müller and J. Vršovský [11].

Although even in finite-dimensions there is an ample supply of operators that are not $\mathbb{C}$-orbit reflexive, it was easy to show that operators that are strictly block-upper (or lower)-triangular are $\mathbb{C}$-orbit reflexive. This fact combined with the example of a non-orbit-reflexive operator in [11], led us naturally to a new version of orbit reflexivity, null-orbit reflexivity, that includes all of the previously-proved orbit-reflexive operators but excludes the counterexample in [1].

Suppose T is a linear transformation on a vector space. We define the *null-orbit* of T as

$$\text{nullOrb}(T) = \left\{0, 1, T, T^2, \dots\right\}.$$

The *orbit* of T is $\text{Orb}(T) = \left\{1, T, T^2, \dots\right\}$. We define $\text{nullOrbRef}_0(T)$ to be the set of all linear transformations S such that for every vector x

$$Sx \in \text{null-Orb}(T)\,x$$

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and we say that T is *algebraically null-orbit reflexive* if

$$\text{nullOrbRef}_0(T) = \text{nullOrb}(T).$$

If T is a bounded operator on a Banach space, we define $\text{nullOrbRef}(T)$ to be the set of all operators S such that, for every vector x

$$Sx \in [\text{nullOrb}(T)x]^-,$$

and we say that T is *null-orbit reflexive* if $\text{nullOrbRef}(T)$ is the strong-operator closure of $\text{nullOrb}(T)$. Orbit reflexivity is defined as in the above definition replacing $\text{nullOrb}(T)$ with $\text{Orb}(T)$. The slight change in definitions causes drastic changes in the two notions.

In this paper we extend all of the positive known results for orbit reflexivity to null-orbit reflexivity, and we show that most of the positive results for $\mathbb{C}$-orbit reflexivity extend to null orbit reflexivity. Moreover, for the example in [11] of a Hilbert space operator T, that is not orbit reflexive, we show that T is null-orbit reflexive. In the example in [1] of a Hilbert space operator that is not orbit reflexive, the proof shows that the operator is also not null-orbit reflexive.

We first prove a number of results in the purely algebraic case, and we use these to prove several results for operators on a normed space or a Hilbert space. We next extend the results of [5] and [11] to the null-orbit reflexivity case. We finish with a new result that every polynomially bounded operator on a Hilbert space is both orbit-reflexive and null-orbit reflexive.

Suppose X is a normed space and $\mathscr{A}$ is an algebra of (bounded linear) operators on X. A (closed linear) subspace M of X is $\mathscr{A}$*-invariant* if $A(M) \subseteq M$ for every $A \in \mathscr{A}$. We let $\text{Lat}\mathscr{A}$ denote the set of all invariant subspaces for $\mathscr{A}$, and we let $\text{AlgLat}\mathscr{A}$ denote the algebra of all operators that leave invariant every $\mathscr{A}$-invariant subspace. The algebra $\mathscr{A}$ is *reflexive* if $\mathscr{A} = \text{AlgLat}\mathscr{A}$. If the algebra $\mathscr{A}$ contains the identity operator 1, then $S \in \text{AlgLat}\mathscr{A}$ if and only if, for every $x \in X$, Sx is in the closure of $\mathscr{A}x$. This characterization works equally well for a linear subspace $\mathscr{S}$ of $B(X)$ (the set of all operators on X), i.e., we define $\text{ref}\mathscr{S}$ to be the set of all operators A such that, for every $x \in X$, we have Ax is in the closure of $\mathscr{S}x$, and we say that $\mathscr{S}$ is reflexive if $\mathscr{S} = \text{ref}\mathscr{S}$. If we let T be a single operator and let $\mathscr{S} = \text{Orb}(T) = \{T^n : n \geqslant 0\}$, we apply the same process to obtain the notion of orbit reflexivity. (Note that in this case $\mathscr{S}$ is not a linear space.) We define $\text{OrbRef}(T)$ to be the set of all operators A such that, for every vector x, we have Ax is in the closure of $\text{Orb}(T,x) = \text{Orb}(T)x$. We say that T is *orbit reflexive* if $\text{OrbRef}(T)$ is the closure of $\text{Orb}(T)$ in the strong operator topology (SOT). In the same context, the operator T on X is called $\mathbb{C}$*-orbit reflexive* if $\mathbb{C}\text{-Orb}(T) = \mathbb{C}\text{-Orb}(T)^{-SOT}$, where

$$\mathbb{C}\text{-Orb}(T) = \{\alpha T^n \mid \alpha \in \mathbb{C}, n \geqslant 0\}$$

and

$$\mathbb{C}\text{-OrbRef}(T) = \{A \in \mathscr{B}(X) \mid \forall x \in X : Ax \in \{\alpha T_X^n \mid \alpha \in \mathbb{C}, n \geqslant 0\}^-\}$$

In case $\mathbb{F}$ is an arbitrary field and X is a vector space over $\mathbb{F}$, we define algebraically $\mathbb{F}$*-orbit reflexivity* in the obvious way, omitting the closures (see [3]).

2. Algebraic Results

Throughout this section $\mathbb{F}$ will denote an arbitrary field, X will denote a vector space over $\mathbb{F}$, and $\mathscr{L}(X)$ will denote the algebra of all linear transformations on X.

A transformation $T \in \mathscr{L}(X)$ is *locally nilpotent* if $X = \cup_{n \geqslant 1} \ker(T^n)$. More generally T is *locally algebraic* if, for each $x \in X$, there is a nonzero polynomial $p_x \in \mathbb{F}[t]$ such that $p_x(T)x = 0$. If $p_x(t)$ is chosen to be monic with minimal degree, we call p_x a *local polynomial* for T at x.

THEOREM 1. *Every locally nilpotent linear transformation on a vector space X over field $\mathbb{F}$ is algebraically null-orbit reflexive. Moreover, if $S \in \text{nullOrbRef}_0(T)$, $x \in X$, and $Sx = T^k x \neq 0$, then $S = T^k$.*

Proof. We know from [3, Theorem 1] that T is algebraically $\mathbb{F}$-orbit reflexive. Thus if $S \in \text{nullOrbRef}_0(T)$ and $S \neq 0$, then there is an $x \in X$ and an integer $n \geqslant 0$ such that $Sx = T^n x \neq 0$, and it follows from [3, Theorem 1] that $S = T^n$. $\square$

For infinite fields the next theorem reduces the problem of algebraic null-orbit reflexivity to the case of locally algebraic transformations. A key ingredient in the proof is an algebraic reflexivity result from [2] that says if $\mathbb{F}$ is infinite and $T \in \mathscr{L}(X)$ is not locally algebraic, then, whenever $S \in \mathscr{L}(X)$ and for every $x \in X$ there is a polynomial p_x such that $Sx = p_x(T)x$, we must have $S = p(T)$ for some polynomial p.

THEOREM 2. *Suppose X is a vector space over an infinite field $\mathbb{F}$, and suppose $T \in \mathscr{L}(X)$ is not locally algebraic. Then T is algebraically null-orbit reflexive.*

Proof. Suppose $S \in \text{nullOrbRef}_0(T)$. Then $Sx \in \text{nullOrb}(T)x$ for every $x \in X$. It follows from [2] that T is algebraically reflexive, so we know there is a polynomial $p \in \mathbb{F}[t]$ such that $S = p(T)$. Since T is not locally algebraic, there is a vector $e \in X$ such that for every nonzero polynomial $q \in \mathbb{F}[t]$, we have $q(T)e \neq 0$. Since $S \in \text{nullOrbRef}_0(T)$, we know that there is an $n \geqslant 0$ such that $Se = T^n e$. Hence $p(t) = t^n$, and thus $S \in \text{nullOrb}(T)$. $\square$

REMARK 3. If there is an $A \in \text{OrbRef}_0(T)$ such that $AT \neq TA$, then, since $\text{OrbRef}_0(T) \subseteq \text{nullOrbRef}_0(T)$, it follows that T is not algebraically null-orbit reflexive. Similarly, if T acts on a Banach space, and there is an $A \in \text{OrbRef}(T)$ such that $AT \neq TA$, then T is not null-orbit reflexive. Hence the Hilbert space operator constructed by S. Grivaux and M. Roginskaya [1] is not null-orbit reflexive.

The preceding remark naturally leads to a pair of questions.

QUESTION 1. If $S \in \text{nullOrbRef}_0(T)$ and $ST = TS$, must $S \in \text{nullOrb}(T)$?

QUESTION 2. If T acts on a Hilbert space, $S \in \text{nullOrbRef}(T)$ and $ST = TS$, must S be in the strong-operator closure of $\text{nullOrb}(T)$? What is the answer if we assume that S is in the double commutant of $\{T\}$?

Note that the example of V. Müller and J. Vršovský [11, Example 1], where $S = 0 \in \mathrm{OrbRef}(T) \setminus \mathrm{Orb}(T)^{-SOT}$ shows that the analog of Question 2 for orbit reflexivity has a negative answer. We will see later (Corollary 16) that their example is null-orbit reflexive, so it has no bearing on Question 2. In [11] an example is given of an operator on ℓ^1 that is reflexive but not orbit reflexive. In view of Theorem 2.8 and Proposition 3.1 in [4], it seems feasible that the operator T in Example 1 of [11, Example 1] is reflexive. We know that $\mathrm{AlgLat}\,T \subseteq \{T\}''$ and that if $S \in \mathrm{AlgLat}\,T$, then there is a sequence $\{a_n\}_{n\geqslant 0}$ such that, for every vector x, $Sx \sim \sum_{n=0}^{\infty} a_n T^n$ in the sense of [4].

QUESTION 3. Is the operator in Example 1 of [11] reflexive?

The proof of Theorem 2 shows that if T is algebraically $\mathbb{F}$-orbit reflexive (reflexive) and $\mathbb{F}$-$\mathrm{Orb}(T)$ ($\{p(T) : p \in \mathbb{F}[t]\}$) has a separating vector, then T is algebraically null-orbit reflexive. This immediately gives us the following (see [3, Theorem 3]).

THEOREM 4. *Suppose X is a finite-dimensional vector space over a field $\mathbb{F}$ not isomorphic to $\mathbb{Z}/p\mathbb{Z}$ for some prime p. Then every linear transformation on X whose minimal polynomial splits over $\mathbb{F}$ is algebraically null-orbit reflexive.*

COROLLARY 5. *If X is a finite-dimensional vector space over an algebraically closed field $\mathbb{F}$, then every linear transformation on X is algebraically null-orbit reflexive.*

Recall from ring theory that if $\mathscr{R}$ is a principal ideal domain, M is an $\mathscr{R}$-module, $0 \neq r \in \mathscr{R}$ and $rM = \{0\}$, then M is a direct sum of cyclic $\mathscr{R}$-modules; Applying this fact to $\mathscr{R} = \mathbb{F}[t]$, we get that any algebraic linear transformation on a vector space is a direct sum of transformations on finite-dimensional subspaces, and therefore has a Jordan form when the minimal polynomial splits over $\mathbb{F}$. (See [6] for details.) This gives us the following corollary.

COROLLARY 6. *Suppose X is a vector space over a field $\mathbb{F}$ not isomorphic to $\mathbb{Z}/p\mathbb{Z}$ for some prime p. Then every algebraic linear transformation on X whose minimal polynomial splits over $\mathbb{F}$ is algebraically null-orbit reflexive.*

3. Null-orbit reflexivity

The following result was proved in [5, Proposition 3].

LEMMA 7. *Suppose $\mathscr{N}$ is a commuting family of normal operators on a Hilbert space X and $A \in B(X)$ satisfies, for every $x \in X$, $Ax \in (\mathscr{N}x)^-$. Then A is in the SOT-closure of $\mathscr{N}$.*

If in the preceding lemma we let $\mathscr{N} = \{0, 1, T, T^2, \ldots\}$, we obtain the following.

PROPOSITION 8. *Every normal operator on a Hilbert space is null-orbit reflexive.*

The next two results are consequences of Theorem 1.

THEOREM 9. *Suppose T is a bounded linear operator on a real or complex normed space X such that $\cup_{n=1}^{\infty}\ker(T^n)$ is dense in X. Then T is null-orbit reflexive and* $\operatorname{nullOrb}(T)$ *is SOT-closed. Moreover, if $S\in\operatorname{nullOrbRef}(T)$, $x\in\cup_{n=1}^{\infty}\ker(T^n)$, $k\geqslant 0$, and $Sx=T^kx\neq 0$, then $S=T^k$.*

Proof. Suppose $S\in\operatorname{nullOrbRef}(T)$, and let $M=\cup_{n=1}^{\infty}\ker(T^n)$. It is clear that $S(M)\subseteq M$ and $T(M)\subseteq M$ and $S|_M\in\operatorname{nullOrbRef}_0(T|M)$. But $T|M$ is locally nilpotent, and if $x\in M$ and $T^nx=0$, then

$$\operatorname{nullOrb}(T)x=\{0\}\cup\{x,Tx,\ldots,T^{n-1}x\}$$

is norm closed. Hence, $\operatorname{nullOrbRef}(T|M)=\operatorname{nullOrbRef}_0(T|M)$, which, by Theorem 1 is $\operatorname{nullOrb}(T|M)$. Hence there is an $A\in\operatorname{nullOrb}(T)$ such that $S|M=A|M$. However, M is dense in X, so $S=A\in\operatorname{nullOrb}(T)$. □

The preceding theorem implies a stronger version of itself.

COROLLARY 10. *Suppose X is a real or complex normed space, and there is a decreasingly directed family $\{X_\lambda:\lambda\in\Lambda\}$ of T-invariant closed linear subspaces such that*

1. *for every $\lambda\in\Lambda$, $\cup_{n=0}^{\infty}(T^n)^{-1}(X_\lambda)$ is dense in X, and*
2. $\cap_{\lambda\in\Lambda}X_\lambda=\{0\}$.

Then T is null-orbit reflexive and $\operatorname{nullOrbRef}(T)=\operatorname{nullOrb}(T)$.

Proof. Suppose $S\in\operatorname{nullOrbRef}(T)$ and $S\neq 0$. Choose $e\in X$ such that $Se\neq 0$. It follows from (2) that both (1) and (2) remain true if we consider only those X_λ that contain neither e nor Se. Since $T(X_\lambda)\subseteq X_\lambda$, $\hat{T}_\lambda(x+X_\lambda)=Tx+X_\lambda$ defines a bounded linear operator $\hat{T}_\lambda$ on X/X_λ. Condition (1) implies that $\cup_{n=1}^{\infty}\ker\left(\hat{T}_\lambda^n\right)$ is dense in X/X_λ; whence, by Theorem 9, $\hat{T}_\lambda$ is null-orbit reflexive. However, $S\in\operatorname{nullOrbRef}(T)$ implies that $S(X_\lambda)\subseteq X_\lambda$, so $\hat{S}_\lambda(x+X_\lambda)=Sx+X_\lambda$ defines an operator on X/X_λ such that $\hat{S}_\lambda\in\operatorname{nullOrbRef}\left(\hat{T}_\lambda\right)$. Hence, by Theorem 9, there is a unique nonnegative integer n_λ such that $\hat{S}_\lambda=\hat{T}_\lambda^{n_\lambda}$. Suppose $\eta\in\Lambda$. Since the X_λ's are decreasingly directed, there is a $\sigma\in\Lambda$ such that $X_\sigma\subseteq X_\lambda\cap X_\eta$. Applying the same arguments we used on X_λ, there is a unique integer $n_\sigma\geqslant 0$ such that $\hat{S}_\sigma=\tilde{T}_\sigma^{n_\sigma}$. However, it follows from (1) that there is a vector $x\in\left[\cup_{n=0}^{\infty}(T^n)^{-1}(X_\sigma)\right]\backslash X_\lambda$. Then there is an n such that $T^nx\in X_\sigma\subseteq X_\lambda$ and thus $\hat{T}_\lambda^n(x+X_\lambda)=0$ but $x+X_\lambda\neq 0$. However, $Sx-T^{n_\sigma}x\in X_\sigma\subseteq X_\lambda$, so

$$\hat{S}_\lambda(x+X_\lambda)=\bar{T}_\lambda^{n_\sigma}(x+X_\lambda)=\bar{T}_\lambda^{n_\lambda}(x+X_\lambda),$$

which implies that $n_\sigma=n_\lambda$. Hence there is an integer $n\geqslant 0$ such that, for every $\lambda\in\Lambda$, $n_\lambda=n$. Hence, for every $x\in X$ and every $\lambda\in\Lambda$,

$$Sx-T^nx\in X_{\lambda},$$

which, by (2), implies $S = T^n$. $\square$

The following corollary applies to operators that have a strictly upper-triangular operator matrix with respect to some direct sum decomposition.

COROLLARY 11. *If a normed space X over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ is a direct sum of spaces $\{X_n : n \in \mathbb{N}\}$ such that $T(X_1) = \{0\}$, and for every $n > 1$,*

$$T(X_n) \subseteq \left(\sum\nolimits_{k<n}^{\oplus} X_k\right)^{-},$$

then T is null-orbit reflexive and $\mathrm{nullOrbRef}(T) = \mathrm{nullOrb}(T)$.

The preceding corollary has some familiar special cases.

COROLLARY 12. *If T is an operator-weighted (unilateral or bilateral) shift or if T is a direct sum of nilpotent operators on a real or complex normed space X, then T is null-orbit reflexive.*

THEOREM 13. *Suppose X is a normed space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, $T \in B(X)$ and $\cap_{n=1}^{\infty} T^n(X)^{-} = \{0\}$. Then T is null-orbit reflexive and* $\mathrm{nullOrbRef}(T) = \mathrm{nullOrb}(T)$. *Moreover, if $S \in \mathrm{nullOrbRef}(T)$, $x \in X$, and $0 \neq Sx = T^k x$, then $S = T^k$.*

Proof. We will first show that T is algebraically null-orbit reflexive. If M is a finite-dimensional invariant subspace for T and $T|M$ is not nilpotent, then there is a nonzero T-invariant subspace N of M such that $\ker(T|N) = 0$. Thus $T(N) = N \neq 0$, which violates $\cap_{n=1}^{\infty} T^n(X)^{-} = \{0\}$. Thus, either T is not locally algebraic or T is locally nilpotent. In these cases it follows either from Theorem 2 or Theorem 1 that T is indeed algebraically null-orbit reflexive. Furthermore, the hypothesis on T implies, for each $x \in X$, that

$$\cap_{N=1}^{\infty} \left\{T^k x : k \geqslant N\right\}^{-} = \{0\},$$

so $\mathrm{nullOrb}(T)x$ is closed in X. Thus $\mathrm{nullOrbRef}(T) = \mathrm{nullOrbRef}_0(T) = \mathrm{nullOrb}(T)$. For the last statement suppose $x \in X$, and $k, n \geqslant 0$ are integers, and

$$0 \neq Sx = T^n x = T^k x.$$

Suppose $k < n$. Then $M = sp\left\{x, Tx, \ldots, T^{n-1}x\right\}$ is a nonzero finite-dimensional invariant subspace for T with $\dim M \leqslant n$. Since $T^n x \neq 0$, we know $T|M$ is not nilpotent, which, as remarked earlier, contradicts $\cap_{n=1}^{\infty} T^n(X)^{-} = \{0\}$. $\square$

This theorem also implies a stronger version of itself.

COROLLARY 14. *Suppose X is a real or complex normed space, $T \in B(X)$, and there is an increasingly directed family $\{X_\lambda : \lambda \in \Lambda\}$ of T-invariant linear subspaces such that*

1. *for every $\lambda \in \Lambda$, $\cap_{n=1}^{\infty} \overline{T^n(X_\lambda)} = \{0\}$, and*

2. $\cup_{\lambda\in\Lambda} X_\lambda$ *is dense in* X.

Then T *is null-orbit reflexive, and* $\mathrm{nullOrbRef}(T) = \mathrm{nullOrb}(T)$. *Moreover, if* $S \in \mathrm{nullOrbRef}(T)$, $x \in X$, *and* $0 \neq Sx = T^k x$, *then* $S = T^k$.

Proof. Suppose $0 \neq S \in \mathrm{nullOrbRef}(T)$. It follows from (2) that there is a $\lambda_0 \in \Lambda$ and an $f \in X_{\lambda_0}$ such that $0 \neq Sf$. However, we must have $S\left(X_{\lambda_0}\right) \subseteq X_{\lambda_0}$, and $S|X_{\lambda_0} \in \mathrm{nullOrbRef}\left(T|X_{\lambda_0}\right) = \mathrm{nullOrb}(T|X_\lambda)$ (by (1) and the preceding theorem). Thus there is an integer $k \geqslant 0$ such that

$$S|X_{\lambda_0} = T^k|X_{\lambda_0}.$$

The same k must work for any X_λ that contains X_{λ_0}. It follows from the fact that the family is increasingly directed and (2) that $S = T^k$. $\square$

COROLLARY 15. *Every backwards operator-weighted shift operator is null-orbit reflexive.*

If T is the operator constructed in [11] that is not orbit reflexive, it is easy to show that $\cap_{n\geqslant 0} T^n(X)^- = 0$.

COROLLARY 16. *The non orbit reflexive operator constructed in Example 1 of* [11] *is null-orbit reflexive.*

Irving Kaplansky [6] (see also [7], [8] , [10]) proved that a (bounded linear) operator on a Banach space is locally algebraic if and only if it is algebraic. This immediately gives us the following result from Theorem 2.

PROPOSITION 17. *Suppose* X *is a real or complex Banach space and* $T \in B(X)$ *is not algebraic. Then* T *is algebraically null-orbit reflexive.*

The results in the paper of [11] also extend to the null-orbit case. If T is an operator on a Banach space, then $r(T)$ denotes the spectral radius of T, i.e.,

$$r(T) = \max\{|\lambda| : \lambda \in \sigma(T)\}.$$

LEMMA 18. *If* X *is a normed space,* $T \in B(X)$ *and*

$$E = \{x \in X : \mathrm{nullOrb}(T)x \text{ is norm closed}\}$$

is not contained in a countable union of nowhere dense subsets of X, *then* T *is null-orbit reflexive and* $\mathrm{nullOrbRef}(T) = \mathrm{nullOrb}(T)$. *(Note that* E *contains all* $x \in X$ *such that* $T^n x \to 0$ *weakly or* $\|T^n x\| \to \infty$.*)*

Proof. If $S \in \mathrm{nullOrbRef}(\mathrm{T})$, then $E \subseteq \cup_{A\in \mathrm{nullOrb}(T)} \ker(S-A)$, so there is an $A \in \mathrm{nullOrb}(T)$ such that $\ker(S-A)$ has nonempty interior, which means that $S = A$. $\square$

COROLLARY 19. *If X is a Banach space, $T \in B(X)$ and $r(T) < 1$, then T is null-orbit reflexive.*

Proof. It follows that $\|T^n\| \to 0$, and thus the set E in Lemma 18 is all of X. □

The proof of the following theorem is almost exactly the same as the proof of Theorem 7 in [11].

THEOREM 20. *If X is a Banach space and $T \in B(X)$ and $\sum_{n=1}^{\infty} \frac{1}{\|T^n\|} < \infty$, then T is null-orbit reflexive. If X is a Hilbert space and $\sum_{n=1}^{\infty} \frac{1}{\|T^n\|^2} < \infty$, then T is null-orbit reflexive. In particular, if $r(T) \neq 1$, then T is null-orbit reflexive.*

COROLLARY 21. *The set of null-orbit reflexive operators on a Banach space X is norm dense in $B(X)$.*

THEOREM 22. *If X is a Hilbert space and $T \in B(X)$ and is polynomially bounded, then T is null-orbit reflexive and orbit reflexive.*

Proof. We prove the null-orbit reflexivity; the orbit reflexivity is proved in a similar fashion. Suppose T is polynomially bounded. It was proved by W. Mlak [9] that T is similar to the direct sum of a unitary operator U and an operator A with a weakly continuous H^∞ functional calculus. In particular, $A^n \to 0$ in the weak operator topology. We can assume $T = U \oplus A$. We can also assume that the A summand is present; otherwise, T is null-orbit reflexive by Proposition 8. Since $A^n \to 0$ in WOT, it follows from Lemma 18 that $\text{nullOrbRef}(A) = \text{nullOrb}(A)$. Hence we can assume that the U summand is also present. Suppose $S \in \text{nullOrbRef}(T)$. Then we can write $S = B \oplus C$. Hence $C \in \text{nullOrb}(A)$. We also know that $B \in \text{nullOrbRef}(U)$.

Case 1. $C = 0$, and $B \neq 0$. For a fixed x_0 with $Bx_0 \neq 0$ and any y there is a sequence $\{n_k\}$ of nonnegative integers such that $\|T^{n_k}(x_0 \oplus y) - Bx_0 \oplus 0\| \to 0$. In particular, $\|A^{n_k} y\| \to 0$. However, $A^n \to 0$ WOT implies there is an $M > 0$ such that $\|A^n\| < M$ for all $n \geqslant 0$. We want to show $\|A^n y\| \to 0$. Suppose $\varepsilon > 0$. Then there is an n_k such that $\|A^{n_k} y\| < \varepsilon/M$. If $n \geqslant n_k$, then

$$\|A^n y\| \leqslant \left\|A^{n-n_k}\right\| \left\|A_{n_k} y\right\| < M(\varepsilon/M) = \varepsilon.$$

We now know that $A^n \to 0$ in the strong operator topology.

Now suppose $m \geqslant 0$ and $A^m \neq 0$. Choose y_0 such that $A^m y_0 \neq 0$. For any x, there is a sequence $\{n_k\}$ of integers such that $T^{n_k}(x \oplus y_0) \to S(x \oplus y_0)$, and it follows that eventually $n_k > m$. Thus, for every x we have $Bx \in \{U^n x : n > m\}$, so it follows from Lemma 18 that $B \in \{U^n : n > m\}^{-SOT}$. It now follows that there is a net $\{n_\lambda\}$ of positive integers such that $T^{n_\lambda} \to S$ in the strong operator topology.

Case 2. $C \neq 0$. Since $C \in \text{nullOrb}(A)$, there is an integer $s \geqslant 0$ such that $C = A^s \neq 0$. Since $A^n \to 0$ in the WOT, it follows that $Ker\left(A^k - 1\right) = 0$ for $k > 0$. Thus if $A^n y = A^m y$ with $n < m$, then $(A^{m-n} - 1)A^n y = 0$, which implies that $A^n x = 0$ and therefore $A^m x = 0$. Choose y_1 so that $A^s y_1 \neq 0$. It follows that if $\{n_k\}$ is a sequence of nonnegative integers and $A^{n_k} y_1 \to A^s y_1$, then n_k must eventually become s. By considering vectors of the form $x \oplus y_1$, we see that $B = U^s$, and therefore $S = T^s$.

Since the only remaining case is $S = 0 \in \mathrm{nullOrb}(T)$, the proof is complete. □

COROLLARY 23. *If T is a Hilbert space operator and $\|T\| \leqslant 1$, then T is null-orbit reflexive.*

COROLLARY 24. *If T is a Hilbert space operator with $\|T\| = r(T)$ (e.g., T is hyponormal), then T is null-orbit reflexive.*

The following lemma is a consequence of Theorem 20.

LEMMA 25. *Suppose X is a Hilbert space, $T \in B(X)$, $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. If* $\ker(T - \lambda) \neq \ker(T - \lambda)^2$, *then T is null orbit reflexive.*

Proof. Suppose $\|x\| = 1$ and $(T - \lambda)^2 x = 0$ and $(T - \lambda)x \neq 0$. It follows that

$$\|T^n x\| = \|[\lambda + (T - \lambda)]^n x\| = \|\lambda^n x + n(T - \lambda)x\| \geqslant n\|(T - \lambda)x\| - \|x\| \geqslant \varepsilon n$$

for some $\varepsilon > 0$ and for sufficiently large n. Thus $\sum 1/\|T^n\|^2 < \infty$, which, by Theorem 20, implies T is null-orbit reflexive. □

THEOREM 26. *Suppose X is a Hilbert space, $T \in B(X)$, $r(T) = 1$ and no point in $E = \sigma(T) \cap \{z \in \mathbb{C} : |z| = 1\}$ is a limit point of the spectrum of T. If the restriction of T to the spectral subspace M_E for the clopen subset E of $\sigma(T)$ is an algebraic operator, then T is null-orbit reflexive. In particular, every compact operator, or algebraic operator on a Hilbert space is null-orbit reflexive. Hence every operator on a finite-dimensional space is null-orbit reflexive.*

Proof. It follows from Lemma 25 that we need only consider the case when $\ker(T - \lambda) = \ker(T - \lambda)^2$ for every $\lambda \in E$. This implies that the restriction of T to M_E is similar to a unitary operator, and since the restriction of T to $M_{\sigma(T)\setminus E}$ has spectral radius less than 1, we see that T is similar to a contraction. Hence, by Theorem 22, T is null-orbit reflexive. If T is compact or algebraic and $r(T) = 1$, then the first part of this theorem applies. If $r(T) \neq 1$, then T is null-orbit reflexive by Theorem 20. □

We conclude with another question.

QUESTION 4. Is every power bounded Hilbert space operator orbit reflexive or null-orbit reflexive?

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