Operators
and
Matrices
Volume 6, Number 3 (2012), 577–588 doi:10.7153/oam-06-39

THE RIEMANNIAN MEAN AND MATRIX INEQUALITIES RELATED TO THE ANDO–HIAI INEQUALITY AND CHAOTIC ORDER

TAKEAKI YAMAZAKI

(*Communicated by T. Furuta*)

Abstract. The Riemannian mean on the convex cone of positive definite matrices is a kind of geometric mean of n-matrices which is an extension of the geometric mean of two-matrices. In this paper, we derive the Ando-Hiai inequality for the Riemannian mean which is an extension of the well-known Ando-Hiai inequality of two-matrices. Moreover, we shall show an extension of a characterization of chaotic order. Lastly, we will give a negative answer for the problem whether the same results are satisfied or not for other geometric means of n-matrices.

1. Introduction

For positive invertible matrices A and B, their weighted geometric mean $A\sharp_\alpha B$ is well known as

$$A\sharp_\alpha B = A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^\alpha A^{\frac{1}{2}} \quad \text{for } \alpha \in [0,1]. \tag{1.1}$$

Especially, in the case $\alpha = \frac{1}{2}$, we say $A\sharp_{\frac{1}{2}}B$ geometric mean, and denote it by $A\sharp B$, simply. If A and B are non-invertible positive matrices, then their geometric mean is defined by

$$A\sharp_\alpha B = \lim_{\varepsilon\to+0}(A+\varepsilon I)\sharp_\alpha(B+\varepsilon I) \quad \text{for } \alpha \in [0,1].$$

The problem of extending two-variable geometric mean to multivariable was a long standing problem. Recently, a nice definition of geometric mean of n-matrices was given in [3]. Since then, many authors have studied geometric means of n-matrices. Now, we know three kind of definitions of geometric mean. The one is defined by Ando-Li-Mathias in [3], the second one is defined in [10, 7] which is a modification of the geometric mean by Ando-Li-Mathias. The third one is called the Riemannian mean or the least squares mean defined in [5, 11, 13]. These geometric means satisfy the same ten properties including monotonicity and arithmetic-geometric mean inequality (which will be introduced in the later).

Mathematics subject classification (2010): Primary 47A64; Secondary 47A63, 47L25.
Keywords and phrases: Positive definite matrices, the Riemannian manifold, the Riemannian mean, matrix inequality, operator inequality, arithmetic-geometric mean inequality, the Ando-Hiai inequality, chaotic order.

Paper OaM-06-39

On the other hands, there are many results on the geometric mean of two-matrices. Especially, the following result is well known as the Ando-Hiai inequality [2]: Let $\alpha \in [0,1]$. Then for positive matrices A and B,

$$A\sharp_\alpha B \leqslant I \quad \text{implies} \quad A^p\sharp_\alpha B^p \leqslant I \quad \text{for all } p \geqslant 1,$$

where the order is defined by positive semi-definiteness. (In the whole paper, we use this order.)

For positive invertible matrices A and B, the order $\log A \geqslant \log B$ is called chaotic order. It is a weaker order than the usual order $A \geqslant B$ since $\log t$ is an operator monotone function. As a characterization of chaotic order, it is well known that the following statements are mutually equivalent [1, 8, 9, 16]:

(1) $\log A \geqslant \log B$,

(2) $A^p \geqslant (A^{\frac{p}{2}}B^pA^{\frac{p}{2}})^{\frac{1}{2}}$ for all $p \geqslant 0$,

(3) $A^r \geqslant (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{r}{p+r}}$ for all $p,r \geqslant 0$.

The Ando-Hiai inequality and the above characterization of chaotic order play important roles in the theory of matrix (operator) inequalities.

In this paper, we shall show some matrix inequalities for the Riemannian mean. One of them is an extension of the Ando-Hiai inequality, and the other one is an extension of the above characterization of chaotic order. In Section 2, we shall introduce the definition of the Riemannian mean and its basic properties. In Section 3, we will derive matrix inequalities for the Riemannian mean which include extensions of the Ando-Hiai inequality and a characterization of chaotic order. In Section 4, we will discuss whether our results hold for other two geometric means of n-variable or not.

2. The Riemannian mean and its basic properties

In this section, we shall introduce the definition of the Riemannian mean and its basic properties. In what follows let $M_m(\mathbb{C})$ be the set of all $m\times m$ matrices on $\mathbb{C}$, and let $P_m(\mathbb{C})$ be the set of all $m\times m$ positive invertible matrices. For $A,B \in M_m(\mathbb{C})$, define an inner product $\langle A,B\rangle$ by $\langle A,B\rangle = \mathrm{tr}A^*B$. Then $M_m(\mathbb{C})$ is an inner product space equipped with the norm $\|A\|_2 = (\mathrm{tr}A^*A)^{\frac{1}{2}}$, moreover $P_m(\mathbb{C})$ is a differential manifold, and we can consider the geodesic $[A,B] \subset P_m(\mathbb{C})$ which includes $A,B \in P_m(\mathbb{C})$. It can be parameterized as follows:

THEOREM A. ([4, 5]) *Let $A,B \in P_m(\mathbb{C})$. Then there exists a unique geodesic $[A,B]$ joining A and B. It has a parametrization*

$$\gamma(t) = A\sharp_t B = A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^tA^{\frac{1}{2}}, \quad t \in [0,1].$$

Furthermore, we have a distance $\delta_2(A,B)$ between A and B along the geodesic $[A,B]$ as

$$\delta_2(A,B) = \|\log A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\|_2.$$

We call the metric $\delta_2(A,B)$ between A and B by the Riemannian metric.

A vector $\omega=(w_1,w_2,\ldots,w_n)$ is called a probability vector if and only if its components satisfy $\sum_i w_i=1$ and $w_i>0$ for $i=1,2,\ldots,n$. Then the weighted Riemannian mean is defined as follows:

DEFINITION 1. ([4, 5, 11, 13]) Let $A_1,\ldots,A_n\in P_m(\mathbb{C})$, and $\omega=(w_1,\ldots,w_n)$ be a probability vector. Then the weighted Riemannian mean $\mathfrak{G}_\delta(\omega;A_1,\ldots,A_n)$ is defined by

$$\mathfrak{G}_\delta(\omega;A_1,\ldots,A_n)=\operatorname{argmin}_{X\in P_m(\mathbb{C})}\sum_{i=1}^n w_i\delta_2^2(A_i,X),$$

where argmin $f(X)$ means the point X_0 which attains minimum value of the function $f(X)$.

It is easy to see that the weighted Riemannian mean of two-matrices just coincides with the weighted geometric mean in (1.1) by the following property of the Riemannian metric.

$$\delta_2(A,A\sharp_\alpha B)=\alpha\delta_2(A,B)\quad\text{for }\alpha\in[0,1].$$

This definition is firstly introduced in [5, 13] for the case of $\omega=(\frac{1}{n},\ldots,\frac{1}{n})$. In this case, we denote the weighted Riemannian mean by $\mathfrak{G}_\delta(A_1,\ldots,A_n)$, simply, and we call it the Riemannian mean. Existence and uniqueness of the Riemannian mean have been already shown in [5, 13]. Recently, Lawson and Lim defined the weighted Riemannian mean in [11], generally.

It is known that the weighted Riemannian mean satisfies the following ten properties: Let $A_i\in P_m(\mathbb{C})$, $i=1,2,\ldots,n$, and $\omega=(w_1,\ldots,w_n)$ be a probability vector. Then

(P1) If $A_1,\ldots,A_n$ commute with each other, then

$$\mathfrak{G}_\delta(\omega;A_1,\ldots,A_n)=A_1^{w_1}\cdots A_n^{w_n}.$$

(P2) Joint homogeneity.

$$\mathfrak{G}_\delta(\omega;a_1A_1,\ldots,a_nA_n)=a_1^{w_1}\cdots a_n^{w_n}\mathfrak{G}_\delta(\omega;A_1,\ldots,A_n)$$

for positive numbers $a_i>0$ $(i=1,\ldots,n)$.

(P3) Permutation invariance. For any permutation π on $\{1,2,\ldots,n\}$,

$$\mathfrak{G}_\delta(\omega;A_1,\ldots,A_n)=\mathfrak{G}_\delta(\pi(\omega);A_{\pi(1)},\ldots,A_{\pi(n)}),$$

where $\pi(\omega)=(w_{\pi(1)},\ldots,w_{\pi(n)})$.

(P4) Monotonicity. For each $i=1,2,\ldots,n$, if $B_i\leqslant A_i$, then

$$\mathfrak{G}_\delta(\omega;B_1,\ldots,B_n)\leqslant\mathfrak{G}_\delta(\omega;A_1,\ldots,A_n).$$

(P5) Continuity. For each $i=1,2,\dots,n$, let $\{A_i^{(k)}\}_{k=1}^{\infty} \subset P_m(\mathbb{C})$ be sequences such that $A_i^{(k)} \to A_i$ as $k\to\infty$. Then

$$\mathfrak{G}_{\delta}(\omega;A_1^{(k)},\dots,A_n^{(k)}) \to \mathfrak{G}_{\delta}(\omega;A_1,\dots,A_n) \quad \text{as } k\to\infty.$$

(P6) Congruence invariance. For any invertible matrix S,

$$\mathfrak{G}_{\delta}(\omega;S^*A_1S,\dots,S^*A_nS) = S^*\mathfrak{G}_{\delta}(\omega;A_1,\dots,A_n)S.$$

(P7) Joint concavity.

$$\begin{aligned}&\mathfrak{G}_{\delta}(\omega;\lambda A_1+(1-\lambda)A_1',\dots,\lambda A_n+(1-\lambda)A_n')\\ &\qquad \geqslant \lambda\mathfrak{G}_{\delta}(\omega;A_1,\dots,A_n)+(1-\lambda)\mathfrak{G}_{\delta}(\omega;A_1',\dots,A_n') \quad \text{for } 0\leqslant\lambda\leqslant 1.\end{aligned}$$

(P8) Self-duality.

$$\mathfrak{G}_{\delta}(\omega;A_1^{-1},\dots,A_n^{-1})^{-1} = \mathfrak{G}_{\delta}(\omega;A_1,\dots,A_n).$$

(P9) Determinantial identity.

$$\det\mathfrak{G}_{\delta}(\omega;A_1,\dots,A_n) = \prod_{i=1}^{n}(\det A_i)^{w_i}.$$

(P10) Arithmetic-geometric-harmonic mean inequalities.

$$\left(\sum_{i=1}^{n} w_iA_i^{-1}\right)^{-1} \leqslant \mathfrak{G}_{\delta}(\omega;A_1,\dots,A_n) \leqslant \sum_{i=1}^{n} w_iA_i.$$

Moreover, instead of continuity of the weighted Riemannian mean, non-expansive property is satisfied as follows:

$$\delta_2(\mathfrak{G}_{\delta}(\omega;A_1,\dots,A_n),\mathfrak{G}_{\delta}(\omega;B_1,\dots,B_n)) \leqslant \sum_{i=1}^{n} w_i\delta_2(A_i,B_i). \tag{P5'}$$

(P3), (P5), (P6) and (P8) follow from the definition of the weighted Riemannian mean and properties of the Riemannian metric [4, 5, 11]. (P1), (P2), (P9) and (P10) follow from the following characterization of the weighted Riemannian mean [11, 13, 17].

THEOREM B. ([11, 13]) *Let $A_1,\dots,A_n\in P_m(\mathbb{C})$, and $\omega=(w_1,\dots,w_n)$ be a probability vector. Then $X=\mathfrak{G}_{\delta}(\omega;A_1,\dots,A_n)$ is a unique positive solution of the following matrix equation:*

$$w_1\log X^{\frac{-1}{2}}A_1X^{\frac{-1}{2}}+\cdots+w_n\log X^{\frac{-1}{2}}A_nX^{\frac{-1}{2}}=0.$$

(P4) and (P7) are not easy consequences. But very recently, Lawson and Lim have given proofs of (P4) and (P7) in [11] by using Sturm's result [15], and then Lawson and Lim showed that the weighted Riemannian mean satisfied (P5') in [11]. Alternative proof of (P4) is obtained in [6]. Theorem B has been obtained by Moakher in [13] in the case of $\omega=(\frac{1}{n},\dots,\frac{1}{n})$, and then Lawson and Lim obtained Theorem B in [11], completely.

3. Main results

In this section, we shall show further properties of the weighted Riemannian mean. Almost these results are matrix inequalities, and some of them extends well-known matrix (operator) inequalities introduced in Section 1.

THEOREM 1. *Let $A_1,\dots,A_n \in P_m(\mathbb{C})$, and $\omega = (w_1,\dots,w_n)$ be a probability vector. Then $w_1 \log A_1 + \cdots + w_n \log A_n \leqslant 0$ implies $\mathfrak{G}_\delta(\omega; A_1,\dots,A_n) \leqslant I$.*

Proof. If $w_1 \log A_1 + \cdots + w_n \log A_n \leqslant 0$, then there exists a matrix $A \in P_m(\mathbb{C})$ such that $A \geqslant I$ and

$$\frac{w_1}{2}\log A_1 + \cdots + \frac{w_n}{2}\log A_n + \frac{1}{2}\log A = 0.$$

Then $\omega_1 = (\frac{w_1}{2},\dots,\frac{w_n}{2},\frac{1}{2})$ is a probability vector, and by Theorem B, we have

$$\mathfrak{G}_\delta(\omega_1; A_1,\dots,A_n,A) = I.$$

Define a sequence $\{G_n\}_{n=0}^\infty \subset P_m(\mathbb{C})$ by

$$G_{n+1} = \mathfrak{G}_\delta(\omega_1; A_1,\dots,A_n,G_n) \quad \text{and} \quad G_0 = \mathfrak{G}_\delta(\omega_1; A_1,\dots,A_n,I).$$

Then by $A \geqslant I$ and the monotonicity of the weighted Riemannian mean, we have

$$I = \mathfrak{G}_\delta(\omega_1; A_1,\dots,A_n,A) \geqslant \mathfrak{G}_\delta(\omega_1; A_1,\dots,A_n,I) = G_0,$$

and hence we obtain

$$I \geqslant G_0 \geqslant G_1 \geqslant \cdots \geqslant G_n \geqslant \cdots \geqslant 0.$$

Therefore the sequence $\{G_n\}_{n=0}^\infty$ converges to a positive semi-definite matrix.

Let $X = \mathfrak{G}_\delta(\omega; A_1,\dots,A_n)$. We shall show that G_n converges to X. Noting that by Theorem B, we have

$$0 = \sum_{i=1}^n w_i \log X^{\frac{-1}{2}} A_i X^{\frac{-1}{2}} = \sum_{i=1}^n \frac{w_i}{2} \log X^{\frac{-1}{2}} A_i X^{\frac{-1}{2}} + \frac{1}{2}\log X^{\frac{-1}{2}} X X^{\frac{-1}{2}},$$

and hence

$$\mathfrak{G}_\delta(\omega_1; A_1,\dots,A_n,X) = X.$$

Then by the non-expansive property of the weighted Riemannian mean, we have

$$\begin{aligned}
\delta_2(X,G_k) &= \delta_2(\mathfrak{G}_\delta(\omega_1; A_1,\dots,A_n,X), \mathfrak{G}_\delta(\omega_1; A_1,\dots,A_n,G_{k-1})) \\
&\leqslant \frac{1}{2}\delta_2(X,G_{k-1}) \\
&\leqslant \cdots \\
&\leqslant \left(\frac{1}{2}\right)^k \delta_2(X,G_0) \to 0 \quad \text{as } k \to +\infty,
\end{aligned}$$

and hence $G_k \to X$ as $k \to +\infty$. Since $\{G_k\}_{k=0}^{\infty}$ is a contractive and decreasing sequence, we have

$$\mathfrak{G}_{\delta}(\omega; A_1, \dots, A_n) = X \leqslant I. \qquad \square$$

COROLLARY 2. *For $A_1, \dots, A_n \in P_m(\mathbb{C})$ and a probability vector $\omega = (w_1, \dots, w_n)$, the weighted Riemannian mean $\mathfrak{G}_{\delta}(\omega; A_1, \dots, A_n)$ is characterized by*

$$\mathfrak{G}_{\delta}(\omega; A_1, \dots, A_n) = \min\{X \in P_m(\mathbb{C}); \sum_{i=1}^{n} w_i \log X^{\frac{-1}{2}} A_i X^{\frac{-1}{2}} \leqslant 0\}.$$

Proof. By Theorem 1,

$$w_1 \log X^{\frac{-1}{2}} A_1 X^{\frac{-1}{2}} + \cdots + w_n \log X^{\frac{-1}{2}} A_n X^{\frac{-1}{2}} \leqslant 0$$

ensures $\mathfrak{G}_{\delta}(\omega; X^{\frac{-1}{2}} A_1 X^{\frac{-1}{2}}, \dots, X^{\frac{-1}{2}} A_n X^{\frac{-1}{2}}) \leqslant I$. By the congruence invariance property of the weighted Riemannian mean, we have

$$\mathfrak{G}_{\delta}(\omega; A_1, \dots, A_n) \leqslant X.$$

Hence the proof is completed. $\square$

THEOREM 3. *Let $A_1, \dots, A_n \in P_m(\mathbb{C})$. For any probability vector ω, $\mathfrak{G}_{\delta}(\omega; A_1, \dots, A_n) \leqslant I$ implies $\mathfrak{G}_{\delta}(\omega; A_1^p, \dots, A_n^p) \leqslant I$ for all $p \geqslant 1$.*

Theorem 3 is an extension of the following Ando-Hiai inequality, because $\mathfrak{G}_{\delta}(1-\alpha, \alpha; A, B) = A \sharp_{\alpha} B$.

THEOREM C. (Ando-Hiai inequality [2]) *Let A and B be positive matrices. For any $\alpha \in [0,1]$, $A \sharp_{\alpha} B \leqslant I$ implies $A^p \sharp_{\alpha} B^p \leqslant I$ for all $p \geqslant 1$.*

Proof of Theorem 3. Let $\omega = (w_1, \dots, w_n)$ and $X = \mathfrak{G}_{\delta}(\omega; A_1, \dots, A_n) \leqslant I$. Then for $p \in [1,2]$, we have

$$\begin{aligned} 0 &= p(w_1 \log X^{\frac{1}{2}} A_1^{-1} X^{\frac{1}{2}} + \cdots + w_n \log X^{\frac{1}{2}} A_n^{-1} X^{\frac{1}{2}}) \quad \text{by Theorem B} \\ &= w_1 \log (X^{\frac{1}{2}} A_1^{-1} X^{\frac{1}{2}})^p + \cdots + w_n \log (X^{\frac{1}{2}} A_n^{-1} X^{\frac{1}{2}})^p \\ &\leqslant w_1 \log X^{\frac{1}{2}} A_1^{-p} X^{\frac{1}{2}} + \cdots + w_n \log X^{\frac{1}{2}} A_n^{-p} X^{\frac{1}{2}}, \end{aligned}$$

where the last inequality holds since $\log t$ is operator monotone, $X \leqslant I$ and Hansen's inequality for $p \in [1,2]$. It is equivalent to

$$w_1 \log X^{\frac{-1}{2}} A_1^p X^{\frac{-1}{2}} + \cdots + w_n \log X^{\frac{-1}{2}} A_n^p X^{\frac{-1}{2}} \leqslant 0,$$

and by Theorem 1, we have

$$\mathfrak{G}_{\delta}(\omega; X^{\frac{-1}{2}} A_1^p X^{\frac{-1}{2}}, \dots, X^{\frac{-1}{2}} A_n^p X^{\frac{-1}{2}}) \leqslant I.$$

Hence we have

$$\mathfrak{G}_\delta(\omega;A_1^p,\dots,A_n^p) \leqslant X = \mathfrak{G}_\delta(\omega;A_1,\dots,A_n) \leqslant I$$

for $p\in[1,2]$ by (P6). Repeating this procedure for $\mathfrak{G}_\delta(\omega;A_1^p,\dots,A_n^p)\leqslant I$, the proof is completed. $\square$

Let $p_1,\dots,p_n$ be positive numbers. For $i=1,2,\dots,n$, we denote $\prod_{j\neq i}p_j$ by $p_{\neq i}$.

THEOREM 4. *Let $A_1,\dots,A_n\in P_m(\mathbb{C})$. Then the following assertions are mutually equivalent;*

(1) $\log A_1+\cdots+\log A_n\leqslant 0$,

(2) $\mathfrak{G}_\delta(A_1^p,\dots,A_n^p)\leqslant I$ *for all* $p>0$,

(3) $\mathfrak{G}_\delta(\omega';A_1^{p_1},\dots,A_n^{p_n})\leqslant I$ *for all* $p_i>0$, $i=1,2,\dots,n$,

where ω' *is a probability vector defined by*

$$\omega' = \left(\frac{p_{\neq 1}}{\sum_i p_{\neq i}},\dots,\frac{p_{\neq n}}{\sum_i p_{\neq i}}\right).$$

Theorem 4 is an extension of the following characterization of chaotic order:

THEOREM D. (Characterization of chaotic order [1, 8, 9, 16]) *Let A and B be positive invertible matrices. Then the following assertions are mutually equivalent:*

(1) $\log A\geqslant \log B$,

(2) $A^p\geqslant (A^{\frac{p}{2}}B^pA^{\frac{p}{2}})^{\frac12}$ *for all* $p\geqslant 0$,

(3) $A^r\geqslant (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{r}{p+r}}$ *for all* $p,r\geqslant 0$.

In fact, Theorem D can be rewritten in the following form:

THEOREM D'. *Let A and B be positive invertible matrices. Then the following assertions are mutually equivalent:*

(1) $\log A+\log B\leqslant 0$,

(2) $A^p\sharp B^p\leqslant I$ *for all* $p\geqslant 0$,

(3) $A^r\sharp_{\frac{r}{p+r}}B^p\leqslant I$ *for all* $p,r\geqslant 0$.

To prove Theorem 4, we need the following result:

THEOREM E. ([14]) *Let $A_1,\dots,A_n\in P_m(\mathbb{C})$. Then*

$$\lim_{p\to+0}\left(\frac{A_1^p+\cdots+A_n^p}{n}\right)^{\frac1p} = \exp\left(\frac{\log A_1+\cdots+\log A_n}{n}\right),$$

uniformly.

Proof of Theorem 4. Proof of (1) $\to$ (3). If $\log A_1 + \cdots + \log A_n \leqslant 0$, then we have

$$\frac{\prod_i p_i}{\sum_i p_{\neq i}}(\log A_1 + \cdots + \log A_n) \leqslant 0,$$

i.e.,

$$\frac{p_{\neq 1}}{\sum_i p_{\neq i}} \log A_1^{p_1} + \cdots + \frac{p_{\neq n}}{\sum_i p_{\neq i}} \log A_n^{p_n} \leqslant 0.$$

Hence by Theorem 1, we have

$$\mathfrak{G}_\delta(\omega'; A_1^{p_1}, \ldots, A_n^{p_n}) \leqslant I$$

for all $p_i > 0$, $i = 1, 2, \ldots, n$.

Proof of (3) $\longrightarrow$ (2) is easy by putting $p_1 = \cdots = p_n = p$.

Proof of (2) $\longrightarrow$ (1). By the geometric-harmonic mean inequality, we have

$$I \geqslant \mathfrak{G}_\delta(A_1^p, \ldots, A_n^p) \geqslant \left(\frac{A_1^{-p} + \cdots + A_n^{-p}}{n}\right)^{-1}.$$

By Theorem E, we have

$$\begin{aligned} I &\geqslant \lim_{p \to +0} \left(\frac{A_1^{-p} + \cdots + A_n^{-p}}{n}\right)^{\frac{-1}{p}} \\ &= \left(\exp \frac{\log A_1^{-1} + \cdots + \log A_n^{-1}}{n}\right)^{-1} = \exp\left(\frac{\log A_1 + \cdots + \log A_n}{n}\right). \end{aligned}$$

Hence we have (1). $\square$

4. Other geometric means

In the previous section, we showed further properties of the weighted Riemannian mean. Here one might expect that other geometric means satisfy the same properties stated in the previous section. In this section, we will give a negative answer for the problem.

It is known that there are two types of geometric means of n-matrices except the weighted Riemannian mean which satisfy all properties of (P1)–(P10) stated in Section 2. The most famous geometric mean has been defined by Ando-Li-Mathias in [3]. In this paper, we call it ALM mean. The other one is defined by Bini-Meini-Poloni and Izumino-Nakamura, independently in [7, 10]. We call it BMP mean in this paper. Weighted BMP mean has been considered in [7, 10], and recently weighted interpolation mean between ALM and BMP means has been defined in [12].

THEOREM 5. *Let $A_1,\dots,A_n \in P_m(\mathbb{C})$, ω be a probability vector and $\mathfrak{G}(\omega;A_1,\dots,A_n)$ be a weighted geometric mean satisfying all properties of (P1)–(P10). If the weighted geometric mean satisfies Theorem 1, then the weighted geometric mean $\mathfrak{G}$ coincides with the weighted Riemannian mean.*

Proof. Let $\omega = (w_1,\dots,w_n)$. If the weighted geometric mean satisfies Theorem 1, we have

$$\begin{aligned}\sum_{i=1}^{n} w_i \log A_i \geqslant 0 \Longleftrightarrow \sum_{i=1}^{n} w_i \log A_i^{-1} \leqslant 0 \\ \Longrightarrow \mathfrak{G}(\omega;A_1^{-1},\dots,A_n^{-1}) \leqslant I \\ \Longleftrightarrow \mathfrak{G}(\omega;A_1,\dots,A_n) \geqslant I \quad \text{by (P8)}.\end{aligned}$$

Hence we obtain

$$w_1 \log A_1 + \cdots + w_n \log A_n = 0 \Longrightarrow \mathfrak{G}(\omega;A_1,\cdots,A_n) = I. \tag{4.1}$$

Let $X = \mathfrak{G}_\delta(\omega;A_1,\dots,A_n)$ be the weighted Riemannian mean. Then by Theorem B, we have

$$w_1 \log X^{\frac{-1}{2}} A_1 X^{\frac{-1}{2}} + \cdots + w_n \log X^{\frac{-1}{2}} A_n X^{\frac{-1}{2}} = 0,$$

and by (4.1) and (P6),

$$\begin{aligned}&\mathfrak{G}(\omega;X^{\frac{-1}{2}} A_1 X^{\frac{-1}{2}},\dots,X^{\frac{-1}{2}} A_n X^{\frac{-1}{2}}) = I \\ &\Longleftrightarrow \mathfrak{G}(\omega;A_1,\dots,A_n) = X = \mathfrak{G}_\delta(\omega;A_1,\dots,A_n).\end{aligned}$$

This completes the proof. □

COROLLARY 6. *Let $A_1,\dots,A_n \in P_m(\mathbb{C})$, ω be a probability vector and $\mathfrak{G}(\omega;A_1,\dots,A_n)$ be a weighted geometric mean satisfying all properties of (P1)–(P10). If the weighted geometric mean satisfies Theorem 3, then the weighted geometric mean $\mathfrak{G}$ coincides with the weighted Riemannian mean.*

Proof. Let $\omega = (w_1,\dots,w_n)$. If $w_1 \log A_1 + \cdots + w_n \log A_n \leqslant 0$ is satisfied, then by arithmetic-geometric mean inequality, we have

$$I \geqslant w_1(I + \frac{\log A_1}{k}) + \cdots + w_n(I + \frac{\log A_n}{k}) \geqslant \mathfrak{G}\left(\omega; I + \frac{\log A_1}{k},\dots,I + \frac{\log A_n}{k}\right)$$

hold for sufficiently large k. Since the weighted geometric mean $\mathfrak{G}$ satisfies Theorem 3, we have

$$\mathfrak{G}\left(\omega;(I + \frac{\log A_1}{k})^k,\dots,(I + \frac{\log A_n}{k})^k\right) \leqslant I.$$

By the well-known formula $\lim_{k\to+\infty}(I + \frac{\log A_i}{k})^k = A_i$ and (P5), we have

$$\mathfrak{G}(\omega;A_1,\dots,A_n) \leqslant I,$$

i.e., the weighted geometric mean $\mathfrak{G}$ satisfies Theorem 1. Hence, by Theorem 5, the weighted geometric mean coincides with the weighted Riemannian mean. $\square$

Generally, ALM, BMP and the Riemannian means are different from each other. Here we shall introduce a concrete example. Before introducing an example, we shall introduce the definitions of ALM and BMP means of three-variable, briefly.

Let $A,B,C \in P_m(\mathbb{C})$. Define three sequences $\{A_n\}_{n=0}^{\infty}$, $\{B_n\}_{n=0}^{\infty}$ and $\{C_n\}_{n=0}^{\infty}$ as follows: $A_0 = A$, $B_0 = B$, $C_0 = C$ and

$$A_{n+1} = B_n \sharp C_n, \quad B_{n+1} = C_n \sharp A_n, \quad C_{n+1} = A_n \sharp B_n.$$

Then $\{A_n\}_{n=0}^{\infty}$, $\{B_n\}_{n=0}^{\infty}$ and $\{C_n\}_{n=0}^{\infty}$ converge the same limit, and we define it as ALM mean [3] (denoted by $\mathfrak{G}_{alm}(A,B,C)$), i.e.,

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n = \lim_{n\to\infty} C_n = \mathfrak{G}_{alm}(A,B,C).$$

On the other hand, BMP mean is defined as follows: Define three sequences $\{A_n\}_{n=0}^{\infty}$, $\{B_n\}_{n=0}^{\infty}$ and $\{C_n\}_{n=0}^{\infty}$ by $A_0 = A$, $B_0 = B$, $C_0 = C$ and

$$A_{n+1} = (B_n \sharp C_n)\sharp_{\frac{1}{3}} A_n, \quad B_{n+1} = (C_n \sharp A_n)\sharp_{\frac{1}{3}} B_n, \quad C_{n+1} = (A_n \sharp B_n)\sharp_{\frac{1}{3}} C_n.$$

Then $\{A_n\}_{n=0}^{\infty}$, $\{B_n\}_{n=0}^{\infty}$ and $\{C_n\}_{n=0}^{\infty}$ converge the same limit, and we define it as BMP mean [7, 10] (denoted by $\mathfrak{G}_{bmp}(A,B,C)$), i.e.,

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n = \lim_{n\to\infty} C_n = \mathfrak{G}_{bmp}(A,B,C).$$

EXAMPLE. Let

$$A = \begin{pmatrix} 18 & 5 \\ 5 & 2 \end{pmatrix}, \; B = \begin{pmatrix} 1 & 0 \\ 0 & 200 \end{pmatrix}, \; C = \begin{pmatrix} 75 & 54 \\ 54 & 40 \end{pmatrix}.$$

Then

$$G_1 = \mathfrak{G}_{alm}(A,B,C) = \begin{pmatrix} 9.06732 & 4.86436 \\ 4.86436 & 8.89146 \end{pmatrix}$$

and

$$\begin{aligned} &\log G_1^{\frac{1}{2}} A^{-1} G_1^{\frac{1}{2}} + \log G_1^{\frac{1}{2}} B^{-1} G_1^{\frac{1}{2}} + \log G_1^{\frac{1}{2}} C^{-1} G_1^{\frac{1}{2}} \\ &\qquad = \begin{pmatrix} -0.263706 & -0.0340424 \\ -0.0340424 & 0.263706 \end{pmatrix} \neq O. \end{aligned}$$

Hence by Theorem B, $\mathfrak{G}_{\delta}(A,B,C) \neq G_1 = \mathfrak{G}_{alm}(A,B,C)$. On the other hand,

$$G_2 = \mathfrak{G}_{bmp}(A,B,C) = \begin{pmatrix} 9.39875 & 4.91569 \\ 4.91659 & 8.63133 \end{pmatrix}$$

and

$$\log G_2^{\frac{1}{2}} A^{-1} G_2^{\frac{1}{2}} + \log G_2^{\frac{1}{2}} B^{-1} G_2^{\frac{1}{2}} + \log G_2^{\frac{1}{2}} C^{-1} G_2^{\frac{1}{2}} = \begin{pmatrix} -0.101249 & -0.0568546 \\ -0.0568546 & 0.101249 \end{pmatrix} \neq O.$$

Hence by Theorem B, $\mathfrak{G}_\delta(A,B,C) \neq G_2 = \mathfrak{G}_{bmp}(A,B,C)$.

COROLLARY 7. *ALM and BMP means do not satisfy Theorems 1 and 3.*

Proof. ALM and BMP means do not coincide with the Riemannian mean. Hence by Theorem 5 and Corollary 6, the proof is completed. □

Acknowledgment

The author expresses his thanks to the referee for his/her helpful comments.

REFERENCES

[1] T. ANDO, *On some operator inequalities*, Math. Ann. **279** (1987), 157–159.

[2] T. ANDO AND F. HIAI, *Log majorization and complementary Golden-Thompson type inequalities*, Linear Algebra Appl. **197**, **198** (1994), 113–131.

[3] T. ANDO, C. K. LI AND R. MATHIAS, *Geometric means*, Linear Algebra Appl. **385** (2004), 305–334.

[4] R. BHATIA, *Positive definite matrices*, Princeton Series in Applied Mathematics, Princeton University Press, Princeton, NJ, 2007.

[5] R. BHATIA AND J. HOLBROOK, *Riemannian geometry and matrix geometric means*, Linear Algebra Appl. **413** (2006), 594–618.

[6] R. BHATIA AND R. L. KARANDIKAR, *Monotonicity of the matrix geometric mean*, to appear in Math. Ann.

[7] D. A. BINI, B. MEINI AND F. POLONI, *An effective matrix geometric mean satisfying the Ando-Li-Mathias properties*, Math. Comp. **79** (2010), 437–452.

[8] M. FUJII, T. FURUTA AND E. KAMEI, *Furuta's inequality and its application to Ando's theorem*, Linear Algebra Appl. **179** (1993), 161–169.

[9] T. FURUTA, *Applications of order preserving operator inequalities*, Oper. Theory Adv. Appl. **59** (1992), 180–190.

[10] S. IZUMINO AND N. NAKAMURA, *Geometric means of positive operators II*, Sci. Math. Jpn. **69** (2009), 35–44.

[11] J. D. LAWSON AND Y. LIM, *Monotonic properties of the least squares mean*, Math. Ann. **351** (2011), 267–269.

[12] H. LEE, Y. LIM AND T. YAMAZAKI, *Multi-variable weighted geometric means of positive definite matrices*, Linear Algebra Appl. **435** (2011), 307–322.

[13] M. MOAKHER, *A differential geometric approach to the geometric mean of symmetric positive-definite matrices*, SIAM J. Matrix Anal. Appl. **26** (2005), 735–747.

[14] R. D. NUSSBAUM AND J. E. COHEN, *The arithmetic-geometric mean and its generalizations for noncommuting linear operators*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **15** (1988), 239–308.

[15] K.-T. STURM, *Probability measures on metric spaces of nonpositive curvature*, Heat kernels and analysis on manifolds, graphs, and metric spaces, 357–390, Contemp. Math. **338**, Amer. Math. Soc., Providence, RI, 2003.
[16] M. UCHIYAMA, *Some exponential operator inequalities*, Math. Inequal. Appl. **2** (1999), 469–471.
[17] T. YAMAZAKI, *An elementary proof of arithmetic-geometric mean inequality of the weighted Riemannian mean of positive definite matrices*, preprint.

(Received February 16, 2011)

Takeaki Yamazaki
Department of Electrical
Electronic and Computer Engineering
Toyo University
Kawagoe-Shi, Saitama, 350-8585
Japan
e-mail: t-yamazaki@toyo.jp