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THE POSSIBLE SHAPES OF NUMERICAL RANGES

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Abstract. Which convex subsets of $\mathbb{C}$ are the numerical range $W(A)$ of some matrix A? This paper gives a precise characterization of these sets. In addition to this we show that for any A there exists a symmetric B of the same size such that $W(A) = W(B)$ thereby settling an open question from [2].

Consider $\mathbb{C}^d$, the standard complex inner product space. Let $\langle \cdot, \cdot \rangle$ denote its scalar product, and $\|\cdot\|$ the related norm. The *numerical range* $W(A)$ of a $d \times d$ matrix A is defined as

$$W(A) = \{\langle Ax, x\rangle : \ \|x\| = 1\}. \tag{1}$$

It is well known that $W(A)$ is a compact convex subset of $\mathbb{C}$ containing the spectrum of A; see, e.g., monographs [3, 6] for these and other properties, as well as for the history of the subject. In this short note we give an answer to the question of exactly which sets W actually are the numerical range of some matrix A, see Theorem 2. Our second main result (Theorem 4) shows that arbitrary matrices generate the same class of numerical ranges as do symmetric matrices of the same size.

This question was originally raised in Kippenhahn's 1951 article [7] (see also a more accessible English translation [8]) which gave several non-trivial necessary conditions on the "geometrical shape" of a numerical range.

However, a necessary and sufficient condition remained unknown[1]. One can be obtained by the observation that curves critical to the problem were effectively classified in [4]. Didier Henrion in [5] makes such a connection[2] and more, and states explicitly one side (necessary) of the characterization of numerical range. While all components of our paper can easily be extracted from [5] by those comfortable with the theory in [4] , we think our short note will nevertheless be useful to the numerical range community, at least for expository purposes. In particular, our Theorem 2 explicitly states a necessary and sufficient condition.

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[1] We would like to thank P. Y. Wu for discussion of this issue during XXVII South Eastern Analysis Meeting in Gainesville, FL, as well as for pointing out to us that more details were needed in the original version of the proof of Theorem 4.

[2] We are especially grateful to Bernd Sturmfels for bringing [5] to our attention.

Paper OaM-06-41

Our characterization of numerical ranges is in terms of a type of dual convex set. For any set $S \subset \mathbb{R}^n$ its *polar* is defined as

$$S_* = \{x \in \mathbb{R}^n : \sup_{y \in S} \langle x, y \rangle \leqslant 1\} \tag{2}$$

(see, e.g., [1, 9]). The set S_* is closed, convex, and contains 0. Clearly (see also [9, Corollary 14.5.1]), 0 is an interior point of S_* if and only if S is bounded. Directly from the definition it follows that $S_* = (\operatorname{conv}\{S, 0\})_*$, where $\operatorname{conv}\{S, 0\}$ stands for the convex hull of S and 0. If S is closed, convex and contains 0, then

$$S_{**} := (S_*)_* = S \tag{3}$$

[9, Theorem 14.5]. So, if S is closed, convex but does not necessarily contain 0, then

$$S_{**} = (\operatorname{conv}\{S, 0\})_{**} = \operatorname{conv}\{S, 0\}. \tag{4}$$

The next result provides an explicit description of polar sets of numerical ranges. In some form it goes back many years, at least to §3 [7]. A different point of view (in a more general setting) is presented in [10, Section 5] (there the term dual is used in place of polar).

LEMMA 1. *Let $A \in \mathbb{C}^{d \times d}$. Then*

$$W(A)_* = \{z = \xi + i\eta : I - \xi H - \eta K \text{ is positive semi-definite}\}. \tag{5}$$

Here H and K are hermitian matrices from the representation

$$A = H + iK. \tag{6}$$

Proof. Directly from the definitions (1) and (2) it follows that

$$\begin{aligned} W(A)_* &= \{z : \operatorname{Re}(\langle Av, v \rangle \bar{z}) \leqslant 1 \text{ for all } v \in \mathbb{C}^d \text{ with } \|v\| = 1\} \\ &= \{z : \langle (\operatorname{Re}(\bar{z}A)v, v \rangle) \leqslant 1 \text{ for all } v \in \mathbb{C}^d \text{ with } \|v\| = 1\} \\ &= \{z : I - \operatorname{Re}(\bar{z}A) \text{ is positive semi-definite}\} \\ &= \{\xi + i\eta : I - \xi H - \eta K \text{ is positive semi-definite}\}. \qquad \square \end{aligned}$$

Common terminology is that (5) is a *linear matrix inequality* (LMI for short) *representation* for $W(A)_*$ and the lemma says that if a set $W \subset \mathbb{C}$ is a numerical range, then its polar has an LMI representation. The paper [4] describes precisely the sets $\mathscr{C}$ in $\mathbb{R}^2$, hence in $\mathbb{C}$, which have an LMI representation. It characterizes them as "rigidly convex" a term we set about to define. An *algebraic interior* $\mathscr{C}$ has a defining polynomial q, namely $\mathscr{C}$ is the closure of the connected component of $\mathscr{C} := \{z : q(z) > 0\}$ containing 0. A minimum degree defining polynomial for $\mathscr{C}$ is unique (up to a constant), see Lemma 2.1 [4] and its degree we call the degree of $\mathscr{C}$. A convex set $\mathscr{C}$ is called *rigidly convex*[3] provided it is an algebraic interior and it has a defining polynomial q which satisfies the *real zero* (RZ) condition, namely,

$$\text{if } \mu \in \mathbb{C}, z \in \mathbb{R}^2 \text{ and } q(\mu \cdot z) = 0, \text{ then } \mu \in \mathbb{R}.$$

Our characterization of numerical ranges is:

THEOREM 2. *A subset W of $\mathbb{C}$ is the numerical range of some $d \times d$ matrix A if and only if its polar W_* is rigidly convex of degree less than or equal to d.*

Proof. Given $A = H + iK$, observe that p defined by

$$p(z) = \det(I - \xi H - \eta K) \tag{7}$$

is an RZ polynomial, since all eigenvalues of a symmetric matrix are real. Moreover, $W(A)_*$ coincides with the closure of the connected component of $\{z\colon p(z) > 0\}$ containing zero. Thus the set $W(A)_*$ is rigidly convex.

However, Theorem 3.1 of [4] says that converse also holds : if V is rigidly convex, then there exist real symmetric matrices H, K such that

$$V = \{z = \xi + i\eta : I - \xi H - \eta K \text{ is positive semi-definite}\}. \tag{8}$$

Consequently, $V = W(B)_*$ for $B = H + iK$. Moreover, we can do this with an H, K whose dimension is the degree of V. □

The forward side of Theorem 2 is in [5] (stated in the language of homogeneous coordinates, and emphasizing that numerical ranges are affine projections of semi-definite cones). The converse follows easily from ingredients there, though it is not stated explicitly.

Duality (3) allows us to restate Theorem 2 in the following form.

COROLLARY 3. *A subset W of $\mathbb{C}$ is the numerical range of some $d \times d$ matrix A if and only if it is a translation of the polar of a rigidly convex set of degree less than or equal to d.*

Proof. For a given $d \times d$ matrix A, pick $\lambda \in W(A)$ and let $A_0 = A - \lambda I$. By Theorem 2, the polar set V of $W(A_0)$ is rigidly convex and has degree not exceeding d. But $0 \in W(A_0)$, so that due to (3) we have $W(A_0) = V_*$. Consequently, $W(A) = W(A_0) + \lambda$ is a translation of V_*

Conversely, if W is a translation of V_* for some rigidly convex set V of degree not exceeding d, then $W - \lambda = V_*$ for some $\lambda \in \mathbb{C}$. Applying (3) to $S = V$, we conclude that $(W - \lambda)_* = V$. By Theorem 2, $W - \lambda = W(A_0)$ for some $d \times d$ matrix A_0, so that $W = W(A_0 + \lambda I)$. □

[3] The term rigid convexity of $\mathscr{C}$ reflects the property that for $\mathscr{C}$ with algebraically nonsingular boundary, any small enough perturbation of q produces a polynomial $\widetilde{q}$ also defining a convex set $\widetilde{\mathscr{C}}$, the latter also being automatically rigidly convex. See [4, §5.3].

Note that the matrix B constructed in the proof of Theorem 2 is symmetric along with H,K. This yields an affirmative answer to the question stated in [2] (raised by the referee of the latter):

THEOREM 4. *For every $d \times d$ matrix A there exists a symmetric $d \times d$ matrix B such that $W(B) = W(A)$.*

Proof. Note first of all that scalar multiples of the identity are symmetric. So: (i) there is nothing to prove if A is such a scalar multiple, and (ii) otherwise, we may consider $A - \lambda I$ in place of A and therefore suppose without loss of generality that 0 lies in $W(A)$ but is not its corner point.

Let B be as constructed in the proof of Theorem 2. Then $W(A)_* = W(B)_*$, and therefore $W(A)_{**} = W(B)_{**}$.

Since 0 is in $W(A)$, the latter coincides with $W(A)_{**}$. Consequently, $W(B)_{**} = W(A)$, and therefore 0 is not a corner point of $W(B)_{**}$. From (4) with $S = W(B)$ it follows then that $W(B)$ contains 0. Thus, $W(B) = W(B)_{**} = W(A)$. $\square$

Note that for $d = 2$ the statement of Corollary 4 can be established by a direct computation, but even for $d = 3$ an elementary proof evades us.

REMARK 5. If the matrices H,K from representation (6) are linearly dependent with I, then the set V in (8) is unbounded. Moreover, V stays unbounded under translations of A. In other words, $W(A)$ in this case has empty interior. This agrees with the fact that A in this (and only this) case has the form $\alpha R + \beta I$ for some hermitian R and $\alpha, \beta \in \mathbb{C}$, and $W(A)$ is therefore a (closed) line segment. In all other cases the interior of $W(A)$ is non-empty, and $W(A - \lambda I)_*$ is bounded for any λ lying in the interior of $W(A)$. One such value of λ is $\lambda = \mathrm{tr}(A)/d$.

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