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HOW TO COMPARE THE ABSOLUTE VALUES OF OPERATOR SUMS AND THE SUMS OF ABSOLUTE VALUES?

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Abstract. We address the problem of comparing $|A+B|$ and $|A|+|B|$, for $A,B\in\mathbb{M}_n(\mathbb{C})$. Some results are obtained by using a technique of positive linear maps and several open questions are proposed.

1. Introduction

In 1975, R. C. Thompson [12] showed that given any $A,B\in\mathbb{M}_n(\mathbb{C})$ there exist unitaries U,V such that

$$|A+B|\leqslant U|A|U^*+V|B|V^*.$$

Recently, it has been noticed by Bourin and Ricard [8] that for normal matrices A and B there exists a unitary V such that

$$|A+B|\leqslant\frac{1}{2}\{|A|+|B|+V(|A|+|B|)V^*\}.$$

These two estimates are matrix versions of the scalar triangle inequality $|a+b|\leqslant|a|+|b|$. However these do not give a direct comparison between $|A+B|$ and $|A|+|B|$. In this short note we establish comparison of them

$$|A+B|\leqslant\kappa(|A|+|B|)$$

and more generally

$$|A_1+\cdots+A_n|\leqslant\kappa(|A_1|+\cdots+|A_n|)$$

for some constant κ depending on only the extreme singular values of $A_1,\dots,A_n$. Thus, up to a scalar multiple, we obtain a substitute to the triangle inequality $|A+B|\leqslant|A|+|B|$. We do not know whether or not the constant κ is the best possible, this is one of the open question that we will propose.

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2. A matrix triangle inequality

It is easy to check that the triangle inequality $|A+B| \leqslant |A|+|B|$ does not hold in $\mathbb{M}_n(\mathbb{C})\,(n \geqslant 2)$ in general. In fact, for

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

we notice that the usual operator norm takes the value $\|\,|A+B|\,\|_\infty = \sqrt{2}$ and $\|\,|A| + |B|\,\|_\infty = 1$. In this example we have $|A|+|B| = I$, the identity, so that

$$|A+B| \leqslant \sqrt{2}(|A|+|B|). \tag{1}$$

However, there is no constant $\kappa > 0$ such that

$$|X+Y| \leqslant \kappa(|X|+|Y|) \tag{2}$$

for all $X, Y \in \mathbb{M}_2(\mathbb{C})$. To check that, take

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{pmatrix}.$$

In despite of the failing of (2.2), it turns out that the norm version of (2.1) holds. More generally, we have the following for the class of unitarily invariant or symmetric norms (i.e., $\|UAV\| = \|A\|$ for all $A \in \mathbb{M}_n(\mathbb{C})$ and all unitaries $U, V \in \mathbb{M}_n(\mathbb{C})$):

PROPOSITION 2.1. *Let $A_1, \ldots, A_m \in \mathbb{M}_n(\mathbb{C})$. Then, for all symmetric norms,*

$$\|A_1 + \cdots + A_m\| \leqslant \sqrt{m}\,\|\,|A_1| + \cdots + |A_m|\,\|.$$

Proof 1. Consider the polar decompositions $A_1 = U_1|A_1|, \ldots, A_m = U_m|A_m|$ and note that

$$A_1 + \cdots + A_m = \begin{pmatrix} |A_1{}^*|^{1/2} & \cdots & |A_m{}^*|^{1/2} \end{pmatrix} \begin{pmatrix} U_1 & & \\ & \ddots & \\ & & U_m \end{pmatrix} \begin{pmatrix} |A_1|^{1/2} \\ \vdots \\ |A_m|^{1/2} \end{pmatrix}$$

so that using the Cauchy-Schwarz inequality for symmetric norms,

$$\|XY\| \leqslant \|X^*X\|^{1/2}\|Y^*Y\|^{1/2}$$

we infer,

$$\|A_1 + \cdots + A_m\| \leqslant \||A_1{}^*| + \cdots + |A_m{}^*|\|^{1/2}\||A_1| + \cdots + |A_m|\|^{1/2}.$$

Combining this with

$$\||A_1{}^*| + \cdots + |A_m{}^*|\,\| \leqslant m \max_{1 \leqslant i \leqslant m} \{\,\||A_i{}^*|\,\} \leqslant m\|\,|A_1| + \cdots + |A_m|\,\|$$

completes the proof. $\square$

The second proof that we propose is based on convexity ideas.

Proof 2. Thanks to the operator convexity of $t \to t^2$, we have for any m-tuple of Hermitian $X_1, \ldots, X_m$,

$$\left(\frac{X_1 + \cdots + X_m}{m}\right)^2 \leqslant \frac{X_1{}^2 + \cdots + X_m{}^2}{m}.$$

Taking

$$X_1 = \begin{pmatrix} 0 & A_1 \\ A_1{}^* & 0 \end{pmatrix}, \ldots, X_m = \begin{pmatrix} 0 & A_m \\ A_m{}^* & 0 \end{pmatrix}$$

and noting that

$$\left|\begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}\right| = \begin{pmatrix} |A^*| & 0 \\ 0 & |A| \end{pmatrix}$$

we obtain

$$\frac{|A_1 + \cdots + A_m|^2}{m^2} \leqslant \frac{|A_1|^2 + \cdots + |A_m|^2}{m},$$

that is

$$|A_1 + \cdots + A_m|^2 \leqslant m(|A_1|^2 + \cdots + |A_m|^2). \tag{3}$$

Hence, by operator monotony of $t \to t^{1/2}$,

$$|A_1 + \cdots + A_m| \leqslant \sqrt{m}(|A_1|^2 + \cdots + |A_m|^2)^{1/2}$$

so that

$$\| \, |A_1 + \cdots + A_m| \, \| \leqslant \sqrt{m} \, \|(|A_1|^2 + \cdots + |A_m|^2)^{1/2}\|$$

for all symmetric norms. To complete the proof, we may use Bourin and Uchiyama's inequality [9],

$$\|f(Z_1 + \cdots + Z_m)\| \leqslant \|f(Z_1) + \cdots + f(Z_m)\| \tag{4}$$

for non-negative concave functions f on the positive half-line and positive operators $Z_1, \ldots, Z_m$. $\square$

So we have in particular:

$$\| \, |A+B| \, \|_\infty \leqslant \sqrt{2} \| \, |A| + |B| \, \|_\infty$$

which is sharp. On the other hand, for the trace norm, we obviously have:

$$\| \, |A+B| \, \|_1 \leqslant \| \, |A| + |B| \, \|_1.$$

What about triangle inequalities with other Schatten p-norms?

QUESTION 1. Let $1 < p < \infty$. What is the best c_p such that

$$\||A+B|\|_p \leqslant c_p \||A| + |B|\|_p$$

for all $A,B \in \mathbb{M}_n(\mathbb{C})$?

Though the Frobenius norm of a matrix, $p=2$, is easily computed as the euclidean norm of the vector of the entries, strangely enough, the case of this norm is already quite intricate.

CONJECTURE 1. For the Frobenius norm, we conjecture that

$$\| |A+B| \|_2 \leqslant \sqrt{\frac{1+\sqrt{2}}{2}} \| |A|+|B| \|_2.$$

The above conjecture is supported by the fact that if A is assumed to have rank one, then the inequality holds and is sharp.

Another recent, and still seemingly simple, estimate for the Frobenius norm is: For all $A,B \in \mathbb{M}_n(\mathbb{C})$,

$$\|AB-BA\|_2 \leqslant \sqrt{2}\|A\|_2\|B\|_2.$$

This non-trivial commutator inequality was conjectured by Böttcher and Wenzel [4] and proved by these authors in [5]. An elegant proof has been obtained by Audenaert [1].

Question 1 can be addressed in the setting of the Schatten q-norm for $0<q<1$. These are actually not norms but quasi-norms.

Proposition 2.1 is a rather weak estimate: it is a norm inequality rather than an operator inequality for the order relation $\leqslant$ and the constant $\sqrt{m}$ increase with the number of operator involved. Is it possible to obtain a more subtle estimate? Note that in the case of operators which are scalar multiple of unitary operators, then we have

$$|A+B| \leqslant |A|+|B|.$$

This simple observation suggests that we should have an estimate of the type

$$|A+B| \leqslant \mathscr{C}(|A|+|B|)$$

where the constant $\mathscr{C}$ depends on how far A and B are from scalar multiples of unitary operators. A good measure for an invertible operator X of its distance to multiple of unitaries is the condition number,

$$\kappa_X = \|X\|_\infty \|X^{-1}\|_\infty^{-1},$$

that is, $\kappa_X = \frac{\mu_1(X)}{\mu_n(X)}$. We have the following result:

THEOREM 2.2. *Let $A_1,\cdots,A_m$ be with condition numbers dominated by $\omega>0$. Then*

$$|A_1+\cdots+A_m| \leqslant \frac{\omega+1}{2\sqrt{\omega}}(|A_1|+\cdots+|A_m|).$$

The theorem will follow from the next lemma about positive maps. A linear map $\Phi: \mathbb{M}_n \to \mathbb{M}_d$ is said *positive* if it preserves positive semi-definiteness. If $\Phi(I)=I$, where I denotes the units of both $\mathbb{M}_n$ and $\mathbb{M}_d$, then Φ is said unital. Furthermore Φ is said *two-positive* if, for all A, B, $C \in \mathbb{M}_n$,

$$\begin{pmatrix} A & B^* \\ B & C \end{pmatrix} \geqslant 0 \quad \Longrightarrow \quad \begin{pmatrix} \Phi(A) & \Phi(B^*) \\ \Phi(B) & \Phi(C) \end{pmatrix} \geqslant 0.$$

Unital positive linear maps, regarded as non-commutative version of Expectations, plays an important role in Matrix Analysis [2]. We may state the following sharp result where $\mathbb{M}_n := \mathbb{M}_n(\mathbb{C})$.

LEMMA 2.3. *Let $\Phi : \mathbb{M}_n \to \mathbb{M}_d$ be a two-positive unital map. Then, for all invertible $A \in \mathbb{M}_n$ with condition number ω,*

$$|\Phi(A)| \leqslant \frac{\omega+1}{2\sqrt{\omega}}\Phi(|A|).$$

Furthermore, the constant is optimal.

Proof 1. We have

$$|\Phi(A)|^2 = \Phi(A^*)\Phi(A) \leqslant \Phi(A^*A) = \Phi(|A|^2)$$

by using Choi's inequality for two-positive unital map. Next, the operator monotony of $t \to t^{1/2}$ and the reverse Cauchy-Schwarz inequality for the matrix geometric mean $(A,B) \mapsto A\sharp B$, see [10, Theorem 4], yield

$$|\Phi(A)| \leqslant \Phi(|A|^2)^{1/2} = \Phi(|A|^2)\sharp\Phi(I) \leqslant \frac{\omega+1}{2\sqrt{\omega}}\Phi(|A|^2\sharp I) = \frac{\omega+1}{2\sqrt{\omega}}\Phi(|A|). \qquad \square$$

Once again, we may propose an alternative proof.

Proof 2. According to [6, Theorem 2.1], if $f(t)$ is operator convex and $A > 0$ has a condition number ω, then we have

$$\Phi(f(A)) \leqslant \frac{(\omega+1)^2}{4\omega} f(\Phi(A)).$$

In particular for $f(t) = t^2$,

$$\Phi(A^2) \leqslant \frac{(\omega+1)^2}{4\omega}(\Phi(A))^2.$$

By using Choi's inequality and the operator monotony of $t \to t^{1/2}$,

$$|\Phi(A)| \leqslant \Phi(A^2)^{1/2} \leqslant \frac{(\omega+1)}{2\sqrt{\omega}}\Phi(A),$$

completing the second proof of Lemma 2.3. $\square$

Proof of Theorem 2.2. Let $A_1,\dots,A_m$ be invertible operators in $\mathbb{M}_n$ with condition numbers $\omega_1,\dots,\omega_m$, respectively and let $\omega=\max\{\omega_1,\dots,\omega_m\}$. For a suitable choice of positive scalars $a_1,\dots,a_m$, the operators $a_i^{-1}|A_i|$ have spectra in $[1,\omega]$. Hence, in the algebra $\mathbb{M}_{mn}$,

$$A=\frac{1}{a_1}A_1\oplus\cdots\oplus\frac{1}{a_m}A_m,$$

has condition number ω. Now, if we apply Lemma 2.3 to A and to the unitaloid (i.e., a scalar multiple of a unital map) completely positive map $\Phi:\mathbb{M}_{mn}\to\mathbb{M}_n$,

$$\Phi([A_{ij}]):=a_1A_{1,1}+\cdots+a_mA_{m,m}$$

then we get the result. $\square$

QUESTION 2. The constant $\frac{\omega+1}{2\sqrt{\omega}}$ in Theorem 2.2 is interesting as it does not depend on the number of operators. It is also a surprisingly small constant, for instance if $w=4$, then it takes the value 1.25. Is it the best possible constant depending only on w ? By using operators of the form

$$\begin{pmatrix}0 & X^*\\ X & 0\end{pmatrix}$$

we note the best constant possible is the same if we confine to Hermitian matrices.

3. Inequalities with concave functions

By using a previous convexity principle and Bourin-Uchiyama's inequality (4) for positive operators, we can obtain a version of this inequality for all operators, solving a conjecture in [11].

PROPOSITION 3.1. *Let $A,B\in\mathbb{M}_n(\mathbb{C})$ and let $f:[0,\infty)\to[0,\infty)$ be a concave function. Then, for all symmetric norms,*

$$\|f(A+B)\|\leqslant\sqrt{2}\,\|\,f(|A|)+f(|B|)\,\|.$$

Proof. By (3)

$$|A+B|^2\leqslant 2(|A|^2+|B|^2)$$

for all operators A,B. Therefore for all positive non-decreasing function $g(t)$ and all symmetric norms

$$||g(|A+B|^2)||\leqslant||g(2|A|^2+2|B|^2)||.$$

Now, if $f:[0,\infty)\to[0,\infty)$ is a concave function, so is $g(t)=f(\sqrt{t})$. Hence we may apply (4) to obtain

$$||f(|A+B|)||=||g(|A+B|^2)||\leqslant||g(2|A|^2+2|B|^2)||\leqslant||g(2|A|^2)+g(2|B|^2)||$$

so that

$$||f(|A+B|)||\leqslant||f(\sqrt{2}|A|)+f(\sqrt{2}|B|)||.$$

Since for a non-negative concave function f and $t \geqslant 1$ we have $f(tA) \leqslant tf(A)$ for all positive operators A we get

$$||f(|A+B|)|| \leqslant \sqrt{2}||f(|A|)+f(|B|)||$$

and the proof is complete. $\square$

REMARK. If A and B are normal we can replace $\sqrt{2}$ by 1, by a recent result of Bourin [7]. Of course there is a version of Proposition 3.1 for m operators. In case of the Frobenius norm. We may wonder whether Conjecture 1 could hold in the following form: Is it true that

$$\| f(|A+B|) \|_2 \leqslant \sqrt{\frac{1+\sqrt{2}}{2}} \| f(|A|)+f(|B|) \|_2$$

for all non-negative concave functions f defined on the positive half-line?

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