Operators and Matrices
Volume 6, Number 4 (2012), 681–714
doi:10.7153/oam-06-45

CLOSED LINEAR RELATIONS AND THEIR REGULAR POINTS

J.-PH. LABROUSSE, A. SANDOVICI, H.S.V. DE SNOO AND H. WINKLER

(*Communicated by A. C. M. Ran*)

Abstract. For a closed linear relation A in a Hilbert space $\mathfrak{H}$ the notions of resolvent set and set of points of regular type are extended to the set of regular points. Such points are defined in terms of quasi-Fredholm relations of degree 0. The set of regular points is open and for $\lambda \in \mathbb{C}$ in this set the spaces $\ker(A-\lambda)$ and $\operatorname{ran}(A-\lambda)$ are continuous in the gap metric. Several characterizations of regular points are presented, in terms of the gap metric between corresponding null spaces, and in terms of generalized resolvents of the linear relation A.

1. Introduction

Let A be a closed linear relation in a Hilbert space $\mathfrak{H}$. A point $\lambda \in \mathbb{C}$ is said to belong to the *resolvent set* $\rho(A)$ of A if

(R1) $\operatorname{ran}(A-\lambda) = \mathfrak{H}$;

(R2) $\ker(A-\lambda) = \{0\}$.

The set $\rho(A)$ is open and $(A-\lambda)^{-1}$, $\lambda \in \rho(A)$, is a holomorphic family of bounded everywhere defined linear operators on $\mathfrak{H}$. Furthermore, $\lambda \in \mathbb{C}$ is said to belong to the set of *points of regular type* $\gamma(A)$ of A if

(T1) $\operatorname{ran}(A-\lambda)$ is closed in $\mathfrak{H}$;

(T2) $\ker(A-\lambda) = \{0\}$.

The set $\gamma(A)$ is open and $(A-\lambda)^{-1}$, $\lambda \in \gamma(A)$, is a family of bounded linear operators on $\operatorname{ran}(A-\lambda)$; see for instance [7].

The purpose of the present paper is to extend the notion of points of regular type. A point $\lambda \in \mathbb{C}$ is said to belong to the set $\operatorname{reg}(A)$ of *regular points* of A if

(F1) $\operatorname{ran}(A-\lambda)$ is closed in $\mathfrak{H}$;

(F2) $\ker(A-\lambda) \subset \operatorname{ran}(A-\lambda)^n$, $n \in \mathbb{N}$.

Mathematics subject classification (2010): Primary 47A57, 47B25; Secondary 47A55, 47B65.
Keywords and phrases: Quasi-Fredholm relation, ascent, descent, generalized resolvent, gap metric, range space.
The research of the first two authors was supported by NWO. The work of the second author was supported by AM-POSDRU, project number: POSDRU/89/1.5/S/49944. The third author thanks the Deutsche Forschungsgemeinschaft (DFG) for the Mercator visiting professorship at the Technische Universität Berlin.

Paper OaM-06-45

Here $\mathbb{N}$ stands for the positive natural numbers. It wil be shown that the set $\operatorname{reg}(A)$ is open and that for $\lambda \in \operatorname{reg}(A)$ the mapping $\lambda \to \ker(A-\lambda)$ is continous in the gap metric (for closed linear subspaces of $\mathfrak{H}$). Moreover, it will be shown that $\lambda \in \operatorname{reg}(A)$ if and only if $\operatorname{ran}(A-\lambda)$ is closed and there exists a neighborhood $\mathscr{U}$ of λ such that $\ker(A-\zeta)$ is close to $\ker(A-\lambda)$ in the gap metric for all $\zeta \in \mathscr{U}$. Finally, a characterization of $\operatorname{reg}(A)$ is given in terms of generalized resolvents of A. For the case where A is an operator, these results can be found in Labrousse's paper [10], and it turns out that the results in [10] remain valid in the context of relations. However, all the previous arguments require an interpretation and an adaptation to make them work for relations.

The present paper can be seen as a natural continuation of [10] and [11]. The notations introduced in [11] will be used here as well. Recall that for a closed linear relation A with $\operatorname{ran} A$ closed the following statements are equivalent:

(i) $\ker A \subset \operatorname{ran} A^n$, $n \in \mathbb{N}$;

(ii) $\ker A^m \subset \operatorname{ran} A$, $m \in \mathbb{N}$;

(iii) $\ker A^m \subset \operatorname{ran} A^n$, $m,n \in \mathbb{N}$,

cf. [11, Lemma 2.7]. A closed linear relation A is said to be a *quasi-Fredholm relation of degree* 0, if $\operatorname{ran} A$ is closed and one of these equivalent conditions is satisfied. Hence, $\lambda \in \operatorname{reg}(A)$ if and only if the closed linear relation $A-\lambda$ is quasi-Fredholm of degree 0. In [11] it has been shown that A is quasi-Fredholm of degree 0 if and only if the adjoint A^* is quasi-Fredholm of degree 0. Hence, if A is a nondensely defined quasi-Fredholm operator of degree 0, then A^* is a multivalued quasi-Fredholm relation of degree 0; this provides already examples of quasi-Fredholm relations which are not operators.

The paper is organized as follows. Section 2 contains a short introduction to linear relations in Hilbert spaces. In particular, the notions of operator part and minimum modulus are introduced. Furthermore, there is a brief review of the opening and gap between closed linear subspaces of a Hilbert space, which play a fundamental role in the later arguments. Section 3 presents the definition of regular points for a closed linear relation. Various estimates are presented in a neighborhood of a regular point. Section 4 contains the characterization of points in $\operatorname{reg}(A)$ in terms of a gap estimate. The regular points of the adjoint relation A^* are studied in Section 5, which leads to another characterization of $\operatorname{reg}(A)$. In Section 6 it is shown that the set $\operatorname{reg}(A)$ is open and that various spaces are continuous on $\operatorname{reg}(A)$ in terms of the gap metric. In Section 7 there is a characterization of $\operatorname{reg}(A)$ in terms of *generalized resolvents* of A. For the convenience of the reader Section 8 returns to the notions of the opening and gap between closed linear subspaces of a Hilbert space. The various connections for gaps are illustrated.

The authors are grateful to a referee for constructive criticism, which gave rise to an improved presentation.

2. Preliminaries

In this section some basic material is presented concerning closed linear relations in Hilbert spaces, their orthogonal operator parts, and their minimum modulus. For general facts concerning relations in linear spaces and in Hilbert spaces, see for instance [7], [14].

2.1. Relations, minimum moduli, and operator parts

Let A be a closed linear relation from a Hilbert space $\mathfrak{H}$ to a Hilbert space $\mathfrak{K}$; i.e., A is a closed linear subspace of the product space $\mathfrak{H}\times\mathfrak{K}$ (in case $\mathfrak{K}=\mathfrak{H}$ one speaks of A as a relation in $\mathfrak{H}$). Then A is the graph of a linear operator if and only if $\mathrm{mul}A=\{0\}$. Here $\mathrm{mul}A$ stands for the multivalued part of A; since A is closed, it is automatically closed. The *orthogonal operator part* A_{s} of A is defined by

$$A_{\mathrm{s}}=\{\{f,g\}:\{f,g\}\in A,\ (I-Q)g=0\}=A\cap(\mathfrak{H}\oplus(\mathrm{mul}A)^{\perp}),$$

where Q be the orthogonal projection from $\mathfrak{K}$ onto $(\mathrm{mul}A)^{\perp}$. In the sense of relations one then has $A_{\mathrm{s}}=QA$. Clearly A_{s} is a closed operator contained in A. Note that it follows from the closed graph theorem that

$$\mathrm{dom}A \quad \text{closed} \quad \Leftrightarrow \quad A_{\mathrm{s}} \quad \text{bounded.}$$

The adjoint A^* of A is a closed linear relation from $\mathfrak{K}$ to $\mathfrak{H}$, defined by

$$A^*=\{\{f,f'\}\in\mathfrak{K}\times\mathfrak{H}:(f',h)=(f,h'),\ \{h,h'\}\in A\}.$$

The orthogonal operator part $(A^*)_{\mathrm{s}}$ of A^* is defined as above. Then A_{s} is a densely defined operator from the Hilbert space $\overline{\mathrm{dom}A}$ to the Hilbert space $\overline{\mathrm{dom}A^*}$. Likewise $(A^*)_{\mathrm{s}}$ is a densely defined operator from $\overline{\mathrm{dom}A^*}$ to $\overline{\mathrm{dom}A}$. It is clear that

$$(A_{\mathrm{s}})^{\times}=(A^*)_{\mathrm{s}},$$

where $A^{\times}$ denotes the adjoint of the densely defined operator A_{s} (as defined between $\overline{\mathrm{dom}A}$ and $\overline{\mathrm{dom}A^*}$). It is obvious that A_{s} is bounded if and only if $(A^*)_{\mathrm{s}}$ is bounded, and in this case

$$\|A_{\mathrm{s}}\|=\|(A^*)_{\mathrm{s}}\|, \tag{2.1}$$

which follows from the usual identity $\|A_{\mathrm{s}}\|=\|(A_{\mathrm{s}})^{\times}\|$. Equivalently one has

$$\mathrm{dom}A \quad \text{closed} \quad \Leftrightarrow \quad \mathrm{dom}A^* \quad \text{closed.} \tag{2.2}$$

For different proofs of this equivalence, see [7].

Let A be a closed linear relation from $\mathfrak{H}$ to $\mathfrak{K}$. Then the *minimum modulus* of A is defined by

$$r(A)=\inf\left\{\frac{\|h'\|}{\|h\|}:\{h,h'\}\in A,\ h\perp\ker A\right\}.$$

This number belongs to $[0,\infty]$. Note that $r(A)>0$ if and only if $(A^{-1})_{\mathrm{s}}$ is bounded, in which case

$$r(A)=\frac{1}{\|(A^{-1})_{\mathrm{s}}\|},$$

cf. [5]. Moreover, it is clear from (2.1) that $r(A)=r(A^*)$, and that

$$\operatorname{ran} A \quad \text{closed} \quad \Leftrightarrow \quad \operatorname{ran} A^* \quad \text{closed},$$

which of course is also clear from (2.2) by going over to inverses.

The multivalued part $\operatorname{mul} A$ is a closed linear subspace of $\mathfrak{H}$ which induces the following closed restriction of A:

$$A_{\mathrm{mul}}=\{0\}\times \operatorname{mul} A.$$

An *operator part* B of A is a linear operator from $\mathfrak{H}$ to $\mathfrak{K}$ which satisfies

$$A=B\,\widehat{+}\,A_{\mathrm{mul}}, \quad \text{direct sum},$$

where $\widehat{+}$ stands for a componentwise sum. The orthogonal operator part A_{s} of A is an example of an operator part. Recall that A_{s} and A are related by $A_{\mathrm{s}}=QA$, where the product is in the sense of relations. The orthogonal operator part is based on the orthogonal decomposition $\mathfrak{K}=(\operatorname{mul} A)^{\perp}\oplus \operatorname{mul} A$. For a different approach to operator parts, see [7]. Now consider a closed linear subspace $\mathfrak{X}$ of $\mathfrak{K}$, such that

$$\mathfrak{K}=\mathfrak{X}+\operatorname{mul} A, \quad \text{direct sum}, \tag{2.3}$$

and let $Q_{\mathfrak{X}}$ be the projection onto $\mathfrak{X}$ parallel to $\operatorname{mul} A$.

LEMMA 2.1. *The relation $A_{\mathfrak{X}}$ defined by*

$$A_{\mathfrak{X}}=\{\,\{f,g\}:\{f,g\}\in A,\ g\in \mathfrak{X}\,\}=A\cap(\mathfrak{H}\oplus\mathfrak{X}) \tag{2.4}$$

is a closed operator part of A and $A_{\mathfrak{X}}=Q_{\mathfrak{X}}A$, so that

$$A=A_{\mathfrak{X}}\,\widehat{+}\,A_{\mathrm{mul}}, \quad \textit{direct sum}. \tag{2.5}$$

Moreover, $A_{\mathfrak{X}}$ is bounded if and only if $\operatorname{dom} A$ *is closed.*

Proof. The identity (2.4) shows that $A_{\mathfrak{X}}$ is closed. Furthermore $A_{\mathfrak{X}}\subset A$ and $A_{\mathrm{mul}}\subset A$ show that $A_{\mathfrak{X}}\,\widehat{+}\,A_{\mathrm{mul}}\subset A$. For the converse inclusion take $\{h,h'\}\in A$. Then according to (2.3) $h'=k+\varphi$ with $k\in\mathfrak{X}$ and $\varphi\in\operatorname{mul} A$, so that

$$\{h,h'\}=\{h,k\}+\{0,\varphi\}.$$

This shows that $\{h,k\}\in A$, since $\{0,\varphi\}\in A_{\mathrm{mul}}\subset A$. Hence $\{h,k\}\in A_{\mathfrak{X}}$ and thus $A\subset A_{\mathfrak{X}}\,\widehat{+}\,A_{\mathrm{mul}}$. To see that $A_{\mathfrak{X}}$ is an operator, let $\{0,k\}\in A_{\mathfrak{X}}$, so that $k\in\mathfrak{X}\cap\operatorname{mul} A$ and $k=0$; cf. (2.3). The representation $A_{\mathfrak{X}}=Q_{\mathfrak{X}}A$ is straightforward. Finally, the last statement follows from the closed graph theorem. $\square$

2.2. Projections associated with relations

Let A be a closed relation in a Hilbert space $\mathfrak{H}$ and let $\lambda \in \mathbb{C}$. Then the formal inverse $(A-\lambda)^{-1}$ is a closed relation in $\mathfrak{H}$ defined by

$$(A-\lambda)^{-1} = \{\, \{h'-\lambda h, h\} : \{h,h'\} \in A \,\}.$$

Clearly $\operatorname{mul}(A-\lambda)^{-1} = \ker(A-\lambda)$ and the orthogonal operator part $((A-\lambda)^{-1})_s$ of $(A-\lambda)^{-1}$ is given by

$$((A-\lambda)^{-1})_{\mathrm{s}} = \{\, \{h'-\lambda h, h\} : \{h,h'\} \in A, h \perp \ker(A-\lambda) \,\}.$$

The *minimum modulus* of $A-\lambda$ is given by

$$r(A-\lambda) = \inf\left\{ \frac{\|h'-\lambda h\|}{\|h\|} : \{h,h'\} \in A,\ h \perp \ker(A-\lambda),\ h \neq 0 \right\}. \tag{2.6}$$

Hence, $\operatorname{ran}(A-\lambda)$ is closed if and only if $r(A-\lambda) > 0$, and in this case

$$r(A-\lambda) = \frac{1}{\|((A-\lambda)^{-1})_{\mathrm{s}}\|}.$$

In order to associate an everywhere defined closed operator with $(A-\lambda)^{-1}$ some direct sum decompositions of the Hilbert space $\mathfrak{H}$ will be introduced.

Let $\mathfrak{X}(\lambda)$ be a closed linear subspace of $\mathfrak{H}$ such that

$$\mathfrak{H} = \mathfrak{X}(\lambda) + \ker(A-\lambda), \quad \text{direct sum.} \tag{2.7}$$

Note that the special choice $\mathfrak{X}(\lambda) = \overline{\operatorname{ran}}\,(A^*-\bar{\lambda})$ corresponds to an orthogonal decomposition. Let Q_λ be the projection onto $\mathfrak{X}(\lambda)$ parallel to $\ker(A-\lambda)$. Clearly, $\ker Q_\lambda = \ker(A-\lambda)$ and Q_λ maps $\operatorname{dom} A$ into itself. The relation $Q_\lambda(A-\lambda)^{-1}$ corresponding to the decomposition (2.7) is a closed operator and it satisfies

$$Q_\lambda(A-\lambda)^{-1}(k-\lambda h) = Q_\lambda h, \quad \{h,k\} \in A. \tag{2.8}$$

Moreover, parallel to (2.5) one has the direct sum decomposition

$$(A-\lambda)^{-1} = Q_\lambda(A-\lambda)^{-1} \,\widehat{+}\, (\{0\} \times \ker(A-\lambda)), \quad \text{direct sum.} \tag{2.9}$$

Hence if $r(A-\lambda) > 0$ or, equivalently, if $\operatorname{ran}(A-\lambda)$ is closed, then $Q_\lambda(A-\lambda)^{-1}$ is a bounded operator; cf. Lemma 2.1.

Now assume that $r(A-\lambda) > 0$ or, equivalently, that $\operatorname{ran}(A-\lambda)$ is closed. Let $\mathfrak{Y}(\lambda)$ be a closed linear subspace of $\mathfrak{H}$ for which

$$\mathfrak{H} = \mathfrak{Y}(\lambda) + \operatorname{ran}(A-\lambda), \quad \text{direct sum.} \tag{2.10}$$

Note that the special choice $\mathfrak{Y}(\lambda) = \ker(A^*-\bar{\lambda})$ corresponds to an orthogonal decomposition. Let P_λ be the projection onto $\operatorname{ran}(A-\lambda)$ parallel to $\mathfrak{Y}(\lambda)$. Clearly $\ker P_\lambda = \mathfrak{Y}(\lambda)$.

Corresponding to the direct sum decompositions (2.7) and (2.10) the operator $\mathscr{R}(\lambda)$ is defined by

$$\mathscr{R}(\lambda) = Q_\lambda (A-\lambda)^{-1} P_\lambda. \tag{2.11}$$

Clearly, it belongs to $\mathbf{B}(\mathfrak{H})$, the Hilbert space of all bounded linear operators defined on all of $\mathfrak{H}$. Note that if $\lambda \in \rho(A)$, then $\operatorname{ran}(A-\lambda) = \mathfrak{H}$ and $\ker(A-\lambda) = \{0\}$, and $\mathscr{R}(\lambda)$ coincides with the usual resolvent of A. For $\lambda \in \mathbb{C}$ the following notation is useful:

$$\mathfrak{N}_\lambda(A) = \ker(A-\lambda), \quad \widehat{\mathfrak{N}}_\lambda(A) = \{\,\{h, \lambda h\} : h \in \mathfrak{N}_\lambda(A)\,\}.$$

LEMMA 2.2. *Let A be a closed relation in a Hilbert space $\mathfrak{H}$ and let $\lambda \in \mathbb{C}$. Assume that $\operatorname{ran}(A-\lambda)$ is closed and that there are direct sum decompositions as in (2.7) and (2.10). Let P_λ be the projection onto $\operatorname{ran}(A-\lambda)$ parallel to $\mathfrak{Y}(\lambda)$, let Q_λ be the projection onto $\mathfrak{X}(\lambda)$ parallel to $\ker(A-\lambda)$, and let $\mathscr{R}(\lambda)$ be defined by (2.11). Then*

$$A = \{\,\{\mathscr{R}(\lambda)\varphi, P_\lambda\varphi + \lambda\mathscr{R}(\lambda)\varphi\} : \varphi \in \mathfrak{H}\,\} \,\widehat{+}\, \widehat{\mathfrak{N}}_\lambda(A), \quad \textit{direct sum}, \tag{2.12}$$

and

$$\mathscr{R}(\lambda)(k-\lambda h) = Q_\lambda h, \quad \{h,k\} \in A. \tag{2.13}$$

Moreover,

$$\operatorname{mul} A = \{\,P_\lambda\varphi : \mathscr{R}(\lambda)\varphi \in \ker(A-\lambda)\,\}. \tag{2.14}$$

Proof. First it will be shown that the righthand side of (2.12) belongs to A or, equivalently, it will be shown that

$$\{\mathscr{R}(\lambda)\varphi, P_\lambda\varphi + \lambda\mathscr{R}(\lambda)\varphi\} \in A, \quad \varphi \in \mathfrak{H}. \tag{2.15}$$

Clearly, $P_\lambda\varphi = k - \lambda h$ for some $\{h,k\} \in A$. With the projection P_λ (2.8) reads as

$$Q_\lambda(A-\lambda)^{-1}P_\lambda(k-\lambda h) = Q_\lambda h, \quad \{h,k\} \in A.$$

Next observe that $\ker Q_\lambda = \ker(A-\lambda)$ implies

$$\{0, (I-Q_\lambda)h\} \in (A-\lambda)^{-1}.$$

Together with $\{k-\lambda h, h\} \in (A-\lambda)^{-1}$, this gives

$$\{k-\lambda h, Q_\lambda h\} \in (A-\lambda)^{-1}$$

But with (2.8) this shows

$$\{k-\lambda h, Q_\lambda(A-\lambda)^{-1}P_\lambda(k-\lambda h\} \in (A-\lambda)^{-1}$$

or, equivalently,

$$\{P_\lambda\varphi, \mathscr{R}(\lambda)\varphi\} \in (A-\lambda)^{-1}, \quad \varphi \in \mathfrak{H},$$

which is equivalent to (2.15).

Next it will be shown that

$$A \subset \{\{\mathscr{R}(\lambda)\varphi, P_\lambda\varphi + \lambda\mathscr{R}(\lambda)\varphi\} : \varphi \in \mathfrak{H}\} \widehat{+} \mathfrak{N}_\lambda(A). \tag{2.16}$$

Let $\{h,k\} \in A$. Then $\ker Q_\lambda = \ker(A-\lambda)$ implies that

$$\{h,k\} - \{(I-Q_\lambda)h, \lambda(I-Q_\lambda)h\} = \{Q_\lambda h, k - \lambda h + \lambda Q_\lambda h\} \in A,$$

and observe that with $\varphi = k - \lambda h$

$$\{Q_\lambda h, k - \lambda h + \lambda Q_\lambda h\} = \{\mathscr{R}(\lambda)\varphi, P_\lambda\varphi + \lambda\mathscr{R}(\lambda)\varphi\}.$$

Hence (2.15) and (2.16) show that (2.12) has been established.

The identity (2.13) follows immediately from (2.8).

Finally, write (2.12) as

$$A = \{\{\mathscr{R}(\lambda)\varphi + h, \lambda(\mathscr{R}(\lambda)\varphi + h) + P_\lambda\varphi\} : \varphi \in \mathfrak{H}, h \in \ker(A-\lambda)\}.$$

Then it is clear that $P_\lambda\varphi \in \operatorname{mul} A$ if and only $\mathscr{R}(\lambda)\varphi + h = 0$. This completes the proof of (2.14). □

2.3. Special properties of the corresponding projections

Assume that there is an open set $\mathscr{U} \subset \mathbb{C}$, such that for all $\lambda \in \mathscr{U}$ the subspace $\operatorname{ran}(A-\lambda)$ is closed. Furthermore, assume that there exist closed linear subspaces $\mathfrak{X}$ and $\mathfrak{Y}$ of $\mathfrak{H}$, such that the following decompositions hold for all $\lambda \in \mathscr{U}$:

$$\mathfrak{H} = \ker(A-\lambda) + \mathfrak{X}, \quad \text{direct sum,} \tag{2.17}$$

and

$$\mathfrak{H} = \operatorname{ran}(A-\lambda) + \mathfrak{Y}, \quad \text{direct sum.} \tag{2.18}$$

In other words, it is assumed that the closed linear subspaces $\mathfrak{X}(\lambda)$ and $\mathfrak{Y}(\lambda)$ in (2.7) and (2.10) are independent of λ. The projection Q_λ onto $\mathfrak{X}$ parallel to $\ker(A-\lambda)$ and the projection P_λ onto $\operatorname{ran}(A-\lambda)$ parallel to $\mathfrak{Y}$ then satisfy some special properties.

LEMMA 2.3. *Let A be a closed relation in a Hilbert space $\mathfrak{H}$. Assume that for all λ in an open set $\mathscr{U}$ $\operatorname{ran}(A-\lambda)$ is closed and that the direct sum decompositions (2.17) and (2.18) hold. Then for all $\lambda, \mu \in \mathscr{U}$ one has for the corresponding projections:*

$$Q_\lambda Q_\mu = Q_\mu, \quad P_\mu P_\lambda = P_\mu.$$

Proof. Let $h \in \mathfrak{H}$, then

$$h = (I-Q_\lambda)h + Q_\lambda h = (I-Q_\mu)h + Q_\mu h,$$

so that

$$(I-Q_\mu)h = (I-Q_\lambda)h + [Q_\lambda h - Q_\mu h] \in \ker(A-\lambda) + \mathfrak{X}.$$

Hence

$$(I-Q_\lambda)(I-Q_\mu)=I-Q_\lambda,$$

which leads to $Q_\lambda Q_\mu = Q_\mu$. Similarly, for $h \in \mathfrak{H}$,

$$h = P_\lambda h + (I-P_\lambda)h = P_\mu h + (I-P_\mu)h,$$

so that

$$P_\lambda h = P_\mu h + [(I-P_\mu)h - (I-P_\lambda)h] \in \operatorname{ran}(A-\mu) + \mathfrak{Y},$$

which leads to $P_\mu P_\lambda = P_\mu$. □

REMARK 2.4. Let P_λ be the projection onto $\operatorname{ran}(A-\lambda)$ as in (2.10) and assume that $P_\mu P_\lambda = P_\mu$, $\lambda,\mu \in \mathscr{U}$. Then

$$\mathfrak{Y}(\lambda) = \mathfrak{Y}(\mu), \quad \lambda,\mu \in \mathscr{U},$$

and (2.18) is satisfied. To see this, note that $(I-P_\mu)(I-P_\lambda) = I-P_\lambda$. Now observe that for any $h \in \mathfrak{H}$ there exists $k \in \mathfrak{H}$ such that

$$(I-P_\lambda)h = P_\mu k + (I-P_\mu)k \in \operatorname{ran}(A-\mu) + \mathfrak{Y}(\mu).$$

This implies that

$$(I-P_\lambda)h = (I-P_\mu)(I-P_\lambda)h = (I-P_\mu)k.$$

Hence, $\mathfrak{Y}(\lambda) \subset \mathfrak{Y}(\mu)$. Symmetry leads to equality.

There is a similar result for the other projections. Let Q_λ be the projection onto $\mathfrak{X}(\lambda)$ as in (2.7) and assume that $Q_\lambda Q_\mu = Q_\mu$, $\lambda,\mu \in \mathscr{U}$. Then

$$\mathfrak{X}(\lambda) = \mathfrak{X}(\mu), \quad \lambda,\mu \in \mathscr{U},$$

and (2.17) is satisfied. This follows in an analogous way.

COROLLARY 2.5. *Let A be a closed relation in a Hilbert space $\mathfrak{H}$. Assume that for all λ in an open set $\mathscr{U}$ $\operatorname{ran}(A-\lambda)$ is closed and that the direct sum decompositions (2.17) and (2.18) hold. Then for all $\lambda,\mu \in \mathscr{U}$, $\lambda \neq \mu$, one has*

$$\mathscr{R}(\lambda) - \mathscr{R}(\mu) = (\lambda-\mu)\mathscr{R}(\lambda)\mathscr{R}(\mu). \tag{2.19}$$

Proof. Let $h \in \mathfrak{H}$, then $\{\mathscr{R}(\mu)h, P_\mu h + \mu\mathscr{R}(\mu)h\} \in A$ by (2.12). An application of (2.13) gives

$$\mathscr{R}(\lambda)(P_\mu h + \mu\mathscr{R}(\mu)h - \lambda\mathscr{R}(\mu)h) = Q_\lambda\mathscr{R}(\mu)h,$$

or, equivalently,

$$(\lambda-\mu)\mathscr{R}(\lambda)\mathscr{R}(\mu) = \mathscr{R}(\lambda)P_\mu - Q_\lambda\mathscr{R}(\mu) = \mathscr{R}(\lambda) - \mathscr{R}(\mu),$$

which follows from Lemma 2.3. □

2.4. Bounds in the graph norm

Let A be a closed linear relation in a Hilbert space $\mathfrak{H}$ with multivalued part $\mathrm{mul}\,A$ and operator part A_{s}. Let P be the orthogonal projection from $\mathfrak{H}$ onto $\mathrm{mul}\,A$. The results in Lemma 2.2 can be rephrased as follows.

COROLLARY 2.6. *Let A be a closed relation in a Hilbert space $\mathfrak{H}$, let $\lambda \in \mathbb{C}$, and assume that $\mathrm{ran}(A-\lambda)$ is closed. Let P be the orthogonal projection onto $\mathrm{mul}\,A$. Then the orthogonal operator part A_{s} acts as follows:*

$$(A_{\mathrm{s}}-\lambda)\mathscr{R}(\lambda) = (I-P)P_{\lambda} - \lambda P\mathscr{R}(\lambda). \tag{2.20}$$

and

$$\mathscr{R}(\lambda)(A_{\mathrm{s}}-\lambda)h = Q_{\lambda}h, \quad h \in \mathrm{dom}\,A. \tag{2.21}$$

The orthogonal operator part A_{s} of a closed linear relation A induces a "graph norm" on $\mathrm{dom}\,A$:

$$\|h\|_{\mathfrak{D}}^2 = \|h\|^2 + \|A_{\mathrm{s}}h\|^2, \quad h \in \mathrm{dom}\,A,$$

so that the pair $(\mathrm{dom}\,A, \|\cdot\|_{\mathfrak{D}})$ is a Hilbert space. An operator $T \in \mathbf{B}(\mathfrak{H})$, mapping $\mathfrak{H}$ into $\mathrm{dom}\,A$, is said to belong to $\mathbf{D}(\mathfrak{H})$ if

$$\|T\|_{\mathfrak{D}} = \sup\{\, \|Th\|_{\mathfrak{D}} : \|h\| = 1\,\} < \infty.$$

Observe that $T \in \mathbf{D}(\mathfrak{H})$ satisfies

$$\|T\| \leqslant \|T\|_{\mathfrak{D}}, \quad \|A_{\mathrm{s}}T\| \leqslant \|T\|_{\mathfrak{D}}. \tag{2.22}$$

In the situation of Corollary 2.5 the question arises about the continuity of the family $\mathscr{R}(\lambda)$, $\lambda \in \mathscr{U}$. Here this question is addressed under the assumption of a uniform bound in the graph norm.

LEMMA 2.7. *Let A be a closed linear relation in a Hilbert space $\mathfrak{H}$. Assume that for all λ in an open set $\mathscr{U}$ $\mathrm{ran}(A-\lambda)$ is closed and that the direct sum decompositions (2.17) and (2.18) hold. Let P_{λ} be the projection onto $\mathrm{ran}(A-\lambda)$, let $I-Q_{\lambda}$ be the projection onto $\ker(A-\lambda)$, and let $\mathscr{R}(\lambda)$ be defined as in (2.11). Assume that there is a constant $K \geqslant 0$ such that for all $\lambda \in \mathscr{U}$*

$$\|\mathscr{R}(\lambda)h\|_{\mathfrak{D}} \leqslant K\|h\|, \quad h \in \mathfrak{H}. \tag{2.23}$$

Then $\mathscr{R}(\lambda)$ satisfies $\|\mathscr{R}(\lambda)\|_{\mathfrak{D}} \leqslant K$ for all $\lambda \in \mathscr{U}$ and, moreover,

$$\|\mathscr{R}(\lambda) - \mathscr{R}(\mu)\|_{\mathfrak{D}} \leqslant K'|\lambda-\mu|, \quad \lambda,\mu \in \mathscr{U}, \tag{2.24}$$

so that $\mathscr{R}(\lambda)$ is continuous with respect to the $\|\cdot\|_{\mathfrak{D}}$ norm.

Proof. It clearly follows from the assumption (2.23) that $\|\mathscr{R}(\lambda)\|_{\mathfrak{D}} \leqslant K$. Furthermore (2.23) implies that

$$\|\mathscr{R}(\lambda)\mathscr{R}(\mu)h\|_{\mathfrak{D}} \leqslant K\|\mathscr{R}(\mu)h\|, \quad h \in \mathfrak{H},$$

so that, in particular,

$$\|\mathscr{R}(\lambda)\mathscr{R}(\mu)\|_{\mathfrak{D}} \leqslant K\|\mathscr{R}(\mu)\|. \tag{2.25}$$

Since $\mathscr{R}(\lambda)$ satisfies the resolvent identity (cf. (2.19)), it follows that

$$\|\mathscr{R}(\lambda)h - \mathscr{R}(\mu)h\|_{\mathfrak{D}} = |\lambda - \mu|\|\mathscr{R}(\lambda)\mathscr{R}(\mu)h\|_{\mathfrak{D}}, \quad h \in \mathfrak{H}.$$

Hence (2.25) gives the estimate

$$\|\mathscr{R}(\lambda) - \mathscr{R}(\mu)\|_{\mathfrak{D}} \leqslant K|\lambda - \mu|\|\mathscr{R}(\mu)\|,$$

which leads to (2.24). □

2.5. The opening between subspaces

This subsection contains a collection of results concerning the various openings between closed linear subspaces of a Hilbert space. A further discussion and proofs can be found in Section 6.

Let $\mathfrak{H}$ be a Hilbert space and let $\mathfrak{M}$ and $\mathfrak{N}$ be closed subspaces of $\mathfrak{H}$. Denote the corresponding orthogonal projections by $P_{\mathfrak{M}}$ and $P_{\mathfrak{N}}$. Define the *opening* $\delta(\mathfrak{M},\mathfrak{N})$ between $\mathfrak{M}$ and $\mathfrak{N}$ by

$$\delta(\mathfrak{M},\mathfrak{N}) = \|(I - P_{\mathfrak{N}})P_{\mathfrak{M}}\|, \tag{2.26}$$

so that clearly $0 \leqslant \delta(\mathfrak{M},\mathfrak{N}) \leqslant 1$ and also

$$\delta(\mathfrak{N}^{\perp},\mathfrak{M}^{\perp}) = \delta(\mathfrak{M},\mathfrak{N}). \tag{2.27}$$

Moreover, observe that

$$\delta(\mathfrak{M},\mathfrak{N}) < 1 \quad \Leftrightarrow \quad \mathfrak{M} + \mathfrak{N}^{\perp} \text{ closed}, \quad \mathfrak{M} \cap \mathfrak{N}^{\perp} = \{0\}. \tag{2.28}$$

The *opening* $\varepsilon(\mathfrak{M},\mathfrak{N})$ between $\mathfrak{M}$ and $\mathfrak{N}$ is defined by

$$\varepsilon(\mathfrak{M},\mathfrak{N}) = \|(I - P_{\mathfrak{N}})P_{\mathfrak{M}\ominus(\mathfrak{M}\cap\mathfrak{N}^{\perp})}\|. \tag{2.29}$$

This leads to $\varepsilon(\mathfrak{M},\mathfrak{N}) = \delta(\mathfrak{M} \ominus (\mathfrak{M} \cap \mathfrak{N}^{\perp}),\mathfrak{N})$ and also to

$$\varepsilon(\mathfrak{M},\mathfrak{N}) = \varepsilon(\mathfrak{N},\mathfrak{M}), \quad \varepsilon(\mathfrak{M}^{\perp},\mathfrak{N}^{\perp}) = \varepsilon(\mathfrak{M},\mathfrak{N}). \tag{2.30}$$

Due to the symmetry in (2.30) it follows that

$$\varepsilon(\mathfrak{M},\mathfrak{N}) < 1 \quad \Leftrightarrow \quad \mathfrak{M} + \mathfrak{N}^{\perp} \text{ closed} \quad \Leftrightarrow \quad \mathfrak{M}^{\perp} + \mathfrak{N} \text{ closed}. \tag{2.31}$$

The *gap* $g(\mathfrak{M},\mathfrak{N})$ between $\mathfrak{M}$ and $\mathfrak{N}$ is defined by

$$g(\mathfrak{M},\mathfrak{N}) = \|P_{\mathfrak{M}} - P_{\mathfrak{N}}\|, \tag{2.32}$$

so that $g(\mathfrak{M},\mathfrak{N}) \leqslant 1$. Moreover, it is clear that

$$g(\mathfrak{M},\mathfrak{N}) = g(\mathfrak{N},\mathfrak{M}), \quad g(\mathfrak{M}^\perp,\mathfrak{N}^\perp) = g(\mathfrak{M},\mathfrak{N}). \tag{2.33}$$

The gap in (2.32) provides a metric on the space $\mathbf{S}(\mathfrak{H})$ of all closed linear subspaces of $\mathfrak{H}$.

Recall that for any pair of not necessarily orthogonal projections $Q_{\mathfrak{M}}$ and $Q_{\mathfrak{N}}$ in $\mathbf{B}(\mathfrak{H})$ such that $\operatorname{ran} Q_{\mathfrak{M}} = \mathfrak{M}$ and $\operatorname{ran} Q_{\mathfrak{N}} = \mathfrak{N}$ one has

$$g(\mathfrak{M},\mathfrak{N}) \leqslant \|Q_{\mathfrak{M}} - Q_{\mathfrak{N}}\|. \tag{2.34}$$

There is a chain of (in)equalities satisfied by the gap and the openings between the subspaces $\mathfrak{M}$ and $\mathfrak{N}$:

$$\varepsilon(\mathfrak{M},\mathfrak{N}) \leqslant \min(\delta(\mathfrak{M},\mathfrak{N}),\delta(\mathfrak{N},\mathfrak{M})) \leqslant \max(\delta(\mathfrak{M},\mathfrak{N}),\delta(\mathfrak{N},\mathfrak{M})) = g(\mathfrak{M},\mathfrak{N}). \tag{2.35}$$

Observe that

$$g(\mathfrak{M},\mathfrak{N}) < 1 \Leftrightarrow \mathfrak{H} = \mathfrak{M} + \mathfrak{N}^\perp \text{ direct sum} \Leftrightarrow \mathfrak{H} = \mathfrak{M}^\perp + \mathfrak{N}, \text{ direct sum.} \tag{2.36}$$

If $\mathfrak{M} \cap \mathfrak{N}^\perp = \{0\}$ and $\mathfrak{M}^\perp \cap \mathfrak{N} = \{0\}$, then

$$\varepsilon(\mathfrak{M},\mathfrak{N}) = \delta(\mathfrak{M},\mathfrak{N}) = \delta(\mathfrak{N},\mathfrak{M}) = g(\mathfrak{M},\mathfrak{N}). \tag{2.37}$$

Hence, if $\delta(\mathfrak{M},\mathfrak{N}) < 1$ and $\delta(\mathfrak{M}^\perp,\mathfrak{N}^\perp) < 1$, and, in particular, if $g(\mathfrak{M},\mathfrak{N}) < 1$, then the identities in(2.37) are satisfied.

Finally, note that if $\mathfrak{H} = \mathfrak{M} + \mathfrak{N}^\perp$, $\mathfrak{M} \cap \mathfrak{N}^\perp = \{0\}$, and P is the projection onto $\mathfrak{M}$ parallel to $\mathfrak{N}^\perp$, then

$$\|P\| = \frac{1}{\sqrt{1 - g(\mathfrak{M},\mathfrak{N})^2}}, \tag{2.38}$$

cf. Corollary 8.11. Furthermore, if $g(\mathfrak{M},\mathfrak{N}) < 1$, then

$$\dim \mathfrak{M} = \dim \mathfrak{N},$$

see [6] or [8].

3. Minimum moduli, openings, gaps, and regular points

Let A be a closed linear relation in a Hilbert space $\mathfrak{H}$ and let $\lambda_0 \in \mathbb{C}$. Then $\operatorname{ran}(A - \lambda_0)$ is closed if and only if $r(A - \lambda_0) > 0$. In this case $\lambda_0 \in \mathbb{C}$ will be called a regular point if $A - \lambda_0$ additionally satisfies (F2). The disk $|\lambda - \lambda_0| < r(A - \lambda_0)$ will play an important role in estimating openings and gaps of closed linear subspaces associated with A. In this section some useful facts and estimates are collected.

LEMMA 3.1. *Let A be a closed linear relation in a Hilbert space $\mathfrak{H}$. Assume that $r(A - \lambda_0) > 0$ for some $\lambda_0 \in \mathbb{C}$. Then for all $\lambda \in \mathbb{C}$:*

$$\delta(\ker (A - \lambda), \ker (A - \lambda_0)) \leqslant \frac{|\lambda - \lambda_0|}{r(A - \lambda_0)}. \tag{3.1}$$

Proof. Observe that for all $\lambda \in \mathbb{C}$:

$$\|(I - P_{\ker(A-\lambda_0)})h\| \leqslant \frac{|\lambda - \lambda_0|}{r(A-\lambda_0)}\,\|h\|, \quad h \in \ker(A-\lambda). \tag{3.2}$$

To see this, let $h \in \ker(A-\lambda)$, so that $\{h, \lambda h\} \in A$. Decompose the element h as follows:

$$h = h_0 + h_1, \quad h_0 \in \ker(A-\lambda_0), \quad h_1 \perp \ker(A-\lambda_0).$$

Then $\{h_0, \lambda_0 h_0\} \in A$ and clearly

$$\{h_1, \lambda h - \lambda_0 h_0\} = \{h, \lambda h\} - \{h_0, \lambda_0 h_0\} \in A.$$

Since $\lambda h - \lambda_0 h_0 - \lambda_0 h_1 = (\lambda - \lambda_0)h$, it follows from (2.6) (with λ_0 instead of λ) that

$$r(A-\lambda_0) \leqslant \frac{|\lambda - \lambda_0|\,\|h\|}{\|h_1\|}.$$

Observe that $h_1 = (I - P_{\ker(A-\lambda_0)})h$, where $P_{\ker(A-\lambda_0)}$ is the orthogonal projection onto $\ker(A-\lambda_0)$. Hence (3.2) follows. Next apply (3.2) to $h = P_{\ker(A-\lambda)}\varphi$, $\varphi \in \mathfrak{H}$, where $P_{\ker(A-\lambda)}$ stands for the orthogonal projection onto $\ker(A-\lambda)$. This gives

$$\begin{aligned} \|(I - P_{\ker(A-\lambda_0)})P_{\ker(A-\lambda)}\varphi\| &\leqslant \frac{|\lambda - \lambda_0|}{r(A-\lambda_0)}\,\|P_{\ker(A-\lambda)}\varphi\| \\ &\leqslant \frac{|\lambda - \lambda_0|}{r(A-\lambda_0)}\,\|\varphi\|, \quad \varphi \in \mathfrak{H}. \end{aligned}$$

Therefore the definition in (2.26) implies (3.1). □

DEFINITION 3.2. Let A be a closed linear relation in a Hilbert space $\mathfrak{H}$. A point $\lambda_0 \in \mathbb{C}$ is said to be a *regular point* of A if

$$\text{ran}(A-\lambda_0) \text{ is closed and } \ker(A-\lambda_0) \subset \text{ran}(A-\lambda_0)^n \text{ for all } n \in \mathbb{N}. \tag{3.3}$$

The set of regular points of A is denoted by $\text{reg}(A)$.

In other words, $\lambda_0 \in \mathbb{C}$ is a regular point of A if and only if $A - \lambda_0$ is a quasi-Fredholm relation of degree 0. Therefore, $\lambda_0 \in \mathbb{C}$ is a regular point of A if and only if

$$\text{ran}(A-\lambda_0) \text{ is closed and } \ker(A-\lambda_0)^n \subset \text{ran}(A-\lambda_0) \text{ for all } n \in \mathbb{N}, \tag{3.4}$$

or, equivalently,

$$\text{ran}(A-\lambda_0) \text{ is closed and } \ker(A-\lambda_0)^n \subset \text{ran}(A-\lambda_0)^m \text{ for all } n, m \in \mathbb{N},$$

cf. [11, Lemma 2.7].

LEMMA 3.3. *Let A be a closed linear relation in a Hilbert space $\mathfrak{H}$ and let $\lambda_0 \in$* reg (A). *Then $r(A-\lambda_0)>0$ and for all $\lambda \in \mathbb{C}$ for which $|\lambda-\lambda_0|<r(A-\lambda_0)$*

$$g(\ker(A-\lambda),\ker(A-\lambda_0))<1, \tag{3.5}$$

or, equivalently,

$$\mathfrak{H}=\ker(A-\lambda)+(\ker(A-\lambda_0))^{\perp}, \quad \textit{direct sum.} \tag{3.6}$$

Proof. Assume that $\lambda_0 \in \operatorname{reg}(A)$. Then $A-\lambda_0$ is quasi-Fredholm of degree 0; in other words, $\operatorname{ran}(A-\lambda_0)$ is closed and

$$\ker(A-\lambda_0)^n \subset \operatorname{ran}(A-\lambda_0), \quad n \in \mathbb{N}; \tag{3.7}$$

see (3.4). Then the condition that $\operatorname{ran}(A-\lambda_0)$ is closed is equivalent to the condition $r(A-\lambda_0)>0$.

First it will be shown that for all $\lambda \in \mathbb{C}$ with $|\lambda-\lambda_0|<r(A-\lambda_0)$ the identity in (3.6) holds or that for all $\lambda \in \mathbb{C}$ with $|\lambda-\lambda_0|<r(A-\lambda_0)$ the inclusion

$$\ker(A-\lambda_0) \subset \ker(A-\lambda)+(\ker(A-\lambda_0))^{\perp}, \tag{3.8}$$

holds. In order to show (3.8), let $h_0 \in \ker(A-\lambda_0)$ or, equivalently, $\{h_0,0\} \in A-\lambda_0$, and assume $|\lambda-\lambda_0|<r(A-\lambda_0)$. Since by (3.7) $\ker(A-\lambda_0) \subset \operatorname{ran}(A-\lambda_0)$, there exists an element $h_1 \in \mathfrak{H}$ such that $\{h_1,h_0\} \in A-\lambda_0$. In fact, one may choose h_1 such that $h_1 \perp \ker(A-\lambda_0)$. Note that $\{h_1,0\} \in (A-\lambda_0)^2$ or, equivalently, $h_1 \in \ker(A-\lambda_0)^2$. Thus

$$\{h_1,h_0\} \in A-\lambda_0, \quad h_1 \in \ker(A-\lambda_0)^2 \cap (\ker(A-\lambda_0))^{\perp}.$$

In fact, there exists a sequence of elements (h_n), $n \in \mathbb{N}$, which satisfies

$$\{h_{n+1},h_n\} \in A-\lambda_0, \quad h_{n+1} \in \ker(A-\lambda_0)^{n+2} \cap (\ker(A-\lambda_0))^{\perp}, \tag{3.9}$$

for $n \in \mathbb{N} \cup \{0\}$. This claim will be verified. The existence of h_1 such that (3.9) is satisfied for $n=0$ has been shown above. Now assume that there are elements $h_1,\dots,h_{m+1}$ such that (3.9) is satisfied for $n=0,1,\dots,m$. It follows from the assumption (3.7) that $h_{m+1} \in \operatorname{ran}(A-\lambda_0)$, and hence there exists an element $h_{m+2} \in \mathfrak{H}$ with $\{h_{m+2},h_{m+1}\} \in A-\lambda_0$. Note that one may choose h_{m+2} such that $h_{m+2} \perp \ker(A-\lambda_0)$. From $h_{m+1} \in \ker(A-\lambda_0)^{m+2}$ it follows that $h_{m+2} \in \ker(A-\lambda_0)^{m+3}$. Clearly, for the elements $h_1,\dots,h_{m+2}$ the statements in (3.9) are satisfied for $n=0,1,\dots,m,m+1$. Thus the claim has been verified.

It follows from (3.9) and (2.6) that $r(A-\lambda_0)\|h_{n+1}\| \leqslant \|h_n\|$, and therefore

$$\|h_n\| \leqslant \frac{\|h_0\|}{(r(A-\lambda_0))^n}, \quad n \in \mathbb{N} \cup \{0\}. \tag{3.10}$$

Hence it follows from (3.9) and (3.10) that for $|\lambda-\lambda_0|<r(A-\lambda_0)$ the sequence in

$$\left\{ \sum_{n=1}^{m} (\lambda-\lambda_0)^n h_n, -\sum_{n=1}^{m} (\lambda-\lambda_0)^n h_{n-1} \right\} \in A-\lambda_0, \tag{3.11}$$

converges for $m \to \infty$ to some element

$$\{\varphi, (\lambda-\lambda_0)(\varphi+h_0)\} \in \mathfrak{H}\times\mathfrak{H},$$

where the element φ is defined by the convergent series

$$\varphi = \sum_{n=1}^{\infty} (\lambda-\lambda_0)^n h_n \perp \ker(A-\lambda_0), \tag{3.12}$$

cf. (3.9). Since A is closed, it follows that $\{\varphi, (\lambda-\lambda_0)(\varphi+h_0)\} \in A-\lambda_0$. Due to $\{h_0, 0\} \in A-\lambda_0$ it follows that

$$\{\varphi+h_0, (\lambda-\lambda_0)(\varphi+h_0)\} \in A-\lambda_0,$$

or, in other words,

$$\{\varphi+h_0, 0\} \in A-\lambda \quad \text{or} \quad \varphi+h_0 \in \ker(A-\lambda).$$

Recall that $\varphi \perp \ker(A-\lambda_0)$, so that

$$h_0 \in \ker(A-\lambda) + (\ker(A-\lambda_0))^{\perp}.$$

This proves (3.8). Hence the identity in (3.6) has been established.

Next it will be shown that the sum in (3.6) is direct. Since $r(A-\lambda_0) > 0$, it follows from Lemma 3.1 that with $|\lambda-\lambda_0| < r(A-\lambda_0)$:

$$\delta(\ker(A-\lambda), \ker(A-\lambda_0)) < 1. \tag{3.13}$$

Hence (3.13) together with (2.28) lead to

$$\ker(A-\lambda) \cap (\ker(A-\lambda_0))^{\perp} = \{0\}. \tag{3.14}$$

Thus the sum in (3.6) is direct.

Therefore the direct sum decomposition in (3.6) has been established. The equivalence between (3.5) and (3.6) follows from (2.36). □

LEMMA 3.4. *Let A be a closed linear relation in a Hilbert space $\mathfrak{H}$ and assume that $\lambda_0 \in \operatorname{reg}(A)$. Then $r(A-\lambda_0) > 0$ and for all $\lambda \in \mathbb{C}$ which satisfy $|\lambda-\lambda_0| < r(A-\lambda_0)$:*

$$r(A-\lambda) \geqslant r(A-\lambda_0) - |\lambda-\lambda_0| > 0, \tag{3.15}$$

so that, in particular, $\operatorname{ran}(A-\lambda)$ is closed, and

$$\delta(\operatorname{ran}(A-\lambda_0), \operatorname{ran}(A-\lambda)) \leqslant \frac{|\lambda-\lambda_0|}{r(A-\lambda_0)}. \tag{3.16}$$

Proof. Since $\lambda_0 \in \operatorname{reg}(A)$, it follows from Lemma 3.3 that $r(A-\lambda_0) > 0$ and that the equivalent statements (3.5) and (3.6) are valid.

First (3.15) will be shown. For this purpose, let $\{h,k\} \in A-\lambda$ and assume that $h \perp \ker(A-\lambda)$. Then, according to the direct sum decomposition (3.6) one has

$$h = h_1 + h_2, \quad h_1 \in \ker(A-\lambda), \quad h_2 \perp \ker(A-\lambda_0),$$

and since $h \perp \ker(A-\lambda)$ it follows for $h_2 = h - h_1$ that

$$\|h_2\|^2 = \|h\|^2 + \|h_1\|^2 \geqslant \|h\|^2. \tag{3.17}$$

Due to $\{h,k\} \in A-\lambda$ and $\{h_1,0\} \in A-\lambda$ it follows that

$$\{h_2,k\} \in A-\lambda \quad \text{or} \quad \{h_2, k+\lambda h_2\} \in A.$$

Hence, one sees that

$$\{h_2, k+(\lambda-\lambda_0)h_2\} \in A-\lambda_0, \quad h_2 \perp \ker(A-\lambda_0),$$

so that from (2.6) (with λ_0 instead of λ) it follows that

$$\|k+(\lambda-\lambda_0)h_2\| \geqslant r(A-\lambda_0)\|h_2\|. \tag{3.18}$$

Therefore, via the triangle inequality, (3.17), and (3.18) one obtains

$$\begin{aligned} \|k\| &\geqslant \big| \|k+(\lambda-\lambda_0)h_2\| - |\lambda-\lambda_0| \, \|h_2\| \big| \\ &\geqslant (r(A-\lambda_0) - |\lambda-\lambda_0|) \, \|h_2\| \\ &\geqslant (r(A-\lambda_0) - |\lambda-\lambda_0|) \, \|h\|, \end{aligned} \tag{3.19}$$

where use has been made of $|\lambda-\lambda_0| < r(A-\lambda_0)$. Since the inequality (3.19) holds for all $\{h,k\} \in A-\lambda$ with $h \perp \ker(A-\lambda)$, it follows that (3.15) holds.

It follows from (3.15) that $\operatorname{ran}(A-\lambda)$ is closed for $|\lambda-\lambda_0| < r(A-\lambda_0)$, so that the lefthand side of (3.16) is well defined.

Next (3.16) will be shown. For this purpose, let $k \in \operatorname{ran}(A-\lambda_0)$. Then $\{h,k\} \in A-\lambda_0$ for some $h \in \mathfrak{H}$ and one may choose $h \perp \ker(A-\lambda_0)$. Hence

$$\{h,k\} \in A-\lambda_0, \quad h \perp \ker(A-\lambda_0),$$

so that, by (2.6) (with λ_0 instead of λ)

$$r(A-\lambda_0) \leqslant \frac{\|k\|}{\|h\|}. \tag{3.20}$$

From $k \in \operatorname{ran}(A-\lambda_0)$ and $\{h, k+(\lambda_0-\lambda)h\} \in A-\lambda$ it follows that

$$\begin{aligned} &(I-P_{\operatorname{ran}(A-\lambda)})P_{\operatorname{ran}(A-\lambda_0)}k \\ &\qquad = (I-P_{\operatorname{ran}(A-\lambda)})k \\ &\qquad = (\lambda-\lambda_0)(I-P_{\operatorname{ran}(A-\lambda)})h, \end{aligned}$$

where $P_{\mathrm{ran}(A-\lambda)}$ and $P_{\mathrm{ran}(A-\lambda_0)}$ are the orthogonal projections onto $\mathrm{ran}(A-\lambda)$ and $\mathrm{ran}(A-\lambda_0)$, respectively. Therefore, with (3.20), one obtains

$$\begin{aligned} \|(I-P_{\mathrm{ran}(A-\lambda)})P_{\mathrm{ran}(A-\lambda_0)}k\| &\leqslant |\lambda-\lambda_0|\,\|h\| \\ &\leqslant \frac{|\lambda-\lambda_0|}{r(A-\lambda_0)}\|k\|, \end{aligned} \tag{3.21}$$

for all $k \in \mathrm{ran}(A-\lambda_0)$ and hence for all $k \in \mathfrak{H}$. Therefore the definition in (2.26) shows that (3.16) holds. □

4. A characterization of regular points

The following theorem is one of the basic results of this paper. It characterizes regular points of a closed linear relation in a Hilbert space, as defined in Definition 3.2, in terms of the gap metric between appropriate null spaces.

THEOREM 4.1. *Let A be a closed linear relation in a Hilbert space $\mathfrak{H}$ and let $\lambda_0 \in \mathbb{C}$. Then $\lambda_0 \in \mathrm{reg}\,(A)$ if and only if $r(A-\lambda_0) > 0$ and there exists a neighborhood $\mathscr{U}(\lambda_0)$ of λ_0 such that*

$$g(\ker\,(A-\lambda), \ker\,(A-\lambda_0)) < 1, \quad \lambda \in \mathscr{U}(\lambda_0). \tag{4.1}$$

The neighborhood $\mathscr{U}(\lambda_0)$ may be chosen so as to contain the disk

$$\{\,\lambda \in \mathbb{C} : |\lambda-\lambda_0| < r(A-\lambda_0)\,\}, \tag{4.2}$$

and on that disk

$$g(\ker\,(A-\lambda), \ker\,(A-\lambda_0)) \leqslant \frac{|\lambda-\lambda_0|}{r(A-\lambda_0)}. \tag{4.3}$$

Proof. ($\Rightarrow$) This implication has been shown in Lemma 3.3.

($\Leftarrow$) For this implication assume that $r(A-\lambda_0) > 0$ and that (4.1) holds for all λ in a neighborhood $\mathscr{U}(\lambda_0)$ of λ_0. It is useful to observe that (2.36) and (2.37) then imply that

$$g(\ker\,(A-\lambda), \ker\,(A-\lambda_0)) = \delta(\ker\,(A-\lambda), \ker\,(A-\lambda_0)). \tag{4.4}$$

By assumption $\mathrm{ran}(A-\lambda_0)$ is closed and it suffices to show that

$$\ker\,(A-\lambda_0) \subset \mathrm{ran}(A-\lambda_0)^n, \quad n \in \mathbb{N}, \tag{4.5}$$

cf. (3.3).

First observe that if $\mathrm{ran}(A-\lambda_0)$ is closed and (4.1) holds, then

$$\ker\,(A-\lambda_0) \subset \mathrm{ran}(A-\lambda_0). \tag{4.6}$$

To see this, let $h \in \ker(A-\lambda_0)$. Then $h = P_{\ker(A-\lambda_0)}h$, where $P_{\ker(A-\lambda_0)}$ stands for the orthogonal projection onto $\ker(A-\lambda_0)$. Furthermore, one has

$$P_{\ker(A-\lambda)}h \in \operatorname{ran}(A-\lambda_0), \quad \lambda \neq \lambda_0, \tag{4.7}$$

since $\ker(A-\lambda) \subset \operatorname{ran}(A-\lambda_0)$, $\lambda \neq \lambda_0$, as is easily verified. Hence, it is clear from (4.7) that

$$h - (P_{\ker(A-\lambda_0)} - P_{\ker(A-\lambda)})h = P_{\ker(A-\lambda)}h \in \operatorname{ran}(A-\lambda_0). \tag{4.8}$$

Let $P_{\operatorname{ran}(A-\lambda_0)}$ be the orthogonal projection onto the closed subspace $\operatorname{ran}(A-\lambda_0)$. From (4.8) it follows that

$$(I - P_{\operatorname{ran}(A-\lambda_0)})h = (I - P_{\operatorname{ran}(A-\lambda_0)})(P_{\ker(A-\lambda_0)} - P_{\ker(A-\lambda)})h. \tag{4.9}$$

Hence, (4.1), (4.9), and (2.32) give

$$\begin{aligned} \|(I - P_{\operatorname{ran}(A-\lambda_0)})h\| &\leqslant \|(P_{\ker(A-\lambda_0)} - P_{\ker(A-\lambda)})h\| \\ &\leqslant \|P_{\ker(A-\lambda_0)} - P_{\ker(A-\lambda)}\| \, \|h\| \\ &= g(\ker(A-\lambda), \ker(A-\lambda_0)) \, \|h\|. \end{aligned} \tag{4.10}$$

Therefore (4.10), (4.4), and Lemma 3.1 give

$$\|(I - P_{\operatorname{ran}(A-\lambda_0)})h\| \leqslant \frac{|\lambda-\lambda_0|}{r(A-\lambda_0)} \|h\| \tag{4.11}$$

for all $\lambda \in \mathscr{U}(\lambda_0)$ with $|\lambda-\lambda_0| < r(A-\lambda_0)$. Thus the lefthand side of inequality (4.11) vanishes, and it follows that

$$h = P_{\operatorname{ran}(A-\lambda_0)}h \in \operatorname{ran}(A-\lambda_0).$$

Hence (4.6) has been shown.

With (4.6) established, the following statement will be proved by induction:

$$\operatorname{ran}(A-\lambda_0)^n \text{ is closed}, \quad \ker(A-\lambda_0) \subset \operatorname{ran}(A-\lambda_0)^n, \quad n \in \mathbb{N}. \tag{4.12}$$

For $n=1$ this is clearly satisfied. Assume that (4.12) is valid for some $n \in \mathbb{N}$. The statements in (4.12) will be shown with n replaced by $n+1$.

First it will be shown that

$$\operatorname{ran}(A-\lambda_0)^{n+1} \text{ is closed.} \tag{4.13}$$

Let $k \in \overline{\operatorname{ran}}(A-\lambda_0)^{n+1}$. Then there exist elements $k_j \in \operatorname{ran}(A-\lambda_0)^{n+1}$ such that $k_j \to k$ in $\mathfrak{H}$, and there are elements $h_j \in \mathfrak{H}$ such that

$$\{h_j, k_j\} \in (A-\lambda_0)^{n+1}.$$

Since $(A-\lambda_0)^{n+1} = (A-\lambda_0)(A-\lambda_0)^n$, there are elements $\chi_j \in \mathfrak{H}$ such that

$$\{h_j, \chi_j\} \in (A-\lambda_0)^n, \quad \{\chi_j, k_j\} \in A-\lambda_0. \tag{4.14}$$

Decompose these elements χ_j by

$$\chi_j = \varphi_j + \psi_j, \quad \varphi_j \in \ker\,(A-\lambda_0), \quad \psi_j \perp \ker\,(A-\lambda_0). \tag{4.15}$$

Note that $\{\varphi_j, 0\} \in A-\lambda_0$, and it follows from (4.14) and (4.15) that

$$\{\psi_j, k_j\} = \{\chi_j, k_j\} - \{\varphi_j, 0\} \in A-\lambda_0, \quad \psi_j \perp \ker\,(A-\lambda_0). \tag{4.16}$$

Therefore it follows from (4.16) and (2.6) that

$$\|\psi_j - \psi_l\| \leqslant r(A-\lambda_0)\|k_j - k_l\|.$$

Hence (ψ_j) is a Cauchy sequence, and thus $\psi_j \to \psi$ for some $\psi \in \mathfrak{H}$. Therefore $\{\psi_j, k_j\}$ is a sequence in $A-\lambda_0$ with the property

$$\{\psi_j, k_j\} \to \{\psi, k\} \in A-\lambda_0,$$

since A is closed.

Now recall that $\psi_j = \chi_j - \varphi_j$. Here by (4.14) one has $\chi_j \in \operatorname{ran}(A-\lambda_0)^n$ and by (4.15) and the induction hypothesis one has

$$\varphi_j \in \ker\,(A-\lambda_0) \subset \operatorname{ran}(A-\lambda_0)^n.$$

Therefore $\psi_j \in \operatorname{ran}(A-\lambda_0)^n$ and by the induction hypothesis that $\operatorname{ran}(A-\lambda_0)^n$ is closed, it follows that $\psi \in \operatorname{ran}(A-\lambda_0)^n$. Together with $\{\psi, k\} \in A-\lambda_0$ this shows that $k \in \operatorname{ran}(A-\lambda_0)^{n+1}$. Hence $\overline{\operatorname{ran}}\,(A-\lambda_0)^{n+1} \subset \operatorname{ran}(A-\lambda_0)^{n+1}$ and thus (4.13) has been shown.

Secondly, it will be shown that

$$\ker\,(A-\lambda_0) \subset \operatorname{ran}(A-\lambda_0)^{n+1}. \tag{4.17}$$

The argument involves the same principle as used for the earlier step in (4.6). Let $h \in \ker\,(A-\lambda_0)$. Then $h = P_{\ker\,(A-\lambda_0)}h$, where $P_{\ker\,(A-\lambda_0)}$ stands for the orthogonal projection onto $\ker\,(A-\lambda_0)$. Furthermore, one has

$$P_{\ker\,(A-\lambda)}h \in \operatorname{ran}(A-\lambda_0)^{n+1}, \quad \lambda \neq \lambda_0, \tag{4.18}$$

since $\ker\,(A-\lambda) \subset \operatorname{ran}(A-\lambda_0)^n$ for $\lambda \neq \lambda_0$, $n \in \mathbb{N}$, as is easily verified. Hence, it is clear from (4.18) that

$$h - (P_{\ker\,(A-\lambda_0)} - P_{\ker\,(A-\lambda)})h = P_{\ker\,(A-\lambda)}h \in \operatorname{ran}(A-\lambda_0)^{n+1}. \tag{4.19}$$

It has been shown in (4.13) that the space $\operatorname{ran}(A-\lambda_0)^{n+1}$ is closed, let $P_{\operatorname{ran}(A-\lambda_0)^{n+1}}$ be the corresponding orthogonal projection. From (4.19) it follows that

$$(I - P_{\operatorname{ran}(A-\lambda_0)^{n+1}})h = (I - P_{\operatorname{ran}(A-\lambda_0)^{n+1}})(P_{\ker\,(A-\lambda_0)} - P_{\ker\,(A-\lambda)})h. \tag{4.20}$$

Hence, (4.1), (4.20), and (2.32) give

$$\begin{aligned} \|(I - P_{\operatorname{ran}(A-\lambda_0)^{n+1}})h\| &\leqslant \|(P_{\ker\,(A-\lambda_0)} - P_{\ker\,(A-\lambda)})h\| \\ &\leqslant \|P_{\ker\,(A-\lambda_0)} - P_{\ker\,(A-\lambda)}\|\,\|h\| \\ &= g(\ker\,(A-\lambda), \ker\,(A-\lambda_0))\,\|h\|. \end{aligned} \tag{4.21}$$

Therefore (4.21) , (4.4), and Lemma 3.1 give

$$\|(I - P_{\mathrm{ran}(A-\lambda_0)^{n+1}})h\| \leqslant \frac{|\lambda - \lambda_0|}{r(A-\lambda_0)} \|h\| \tag{4.22}$$

for all $\lambda \in \mathscr{U}(\lambda_0)$ with $|\lambda - \lambda_0| < r(A-\lambda_0)$. Thus the lefthand side of inequality (4.22) vanishes, and it follows that

$$h = P_{\mathrm{ran}(A-\lambda_0)^{n+1}} h \in \mathrm{ran}(A-\lambda_0)^{n+1}.$$

Hence (4.17) has been shown.

With (4.13) and (4.17) the assertion (4.12) has been established. In particular, the inclusions in (4.17) are valid. Hence, it follows that $\lambda_0 \in \mathrm{reg}\,(A)$.

If $r(A-\lambda_0) > 0$ and $\mathscr{U}(\lambda_0)$ is a neighborhood of λ_0 on which (4.1) holds then $\lambda_0 \in \mathrm{reg}\,(A)$. Furthermore, it follows from Lemma 3.3 that (4.1) holds for all $\lambda \in \mathbb{C}$ for which $|\lambda - \lambda_0| < r(A-\lambda_0)$. □

5. Regular points for adjoint relations

The regular points of adjoint relations are described in the following theorem. For completeness, a short proof is included; see [11]. For the operator case, see [10, Corollaire 4.12]; due to the formal level of relations there is no need anymore to require the operator A to be densely defined.

Recall that for a closed linear relation A with $\lambda_0 \in \mathrm{reg}\,(A)$ the following identity holds:

$$(\ker\,(A-\lambda_0)^n)^\perp = \mathrm{ran}(A^* - \bar{\lambda}_0)^n, \quad n \in \mathbb{N}. \tag{5.1}$$

For an elementary proof, see [11, Theorem 7.1].

THEOREM 5.1. *Let A be a closed linear relation in a Hilbert space $\mathfrak{H}$. Then*

$$\lambda_0 \in \mathrm{reg}\,(A) \quad \Leftrightarrow \quad \bar{\lambda}_0 \in \mathrm{reg}\,(A^*).$$

Proof. By symmetry it suffices to show the implication $(\Rightarrow)$. Let $\lambda_0 \in \mathrm{reg}\,(A)$, so that $A - \lambda_0$ is a quasi-Fredholm relation of order 0. Hence, $\mathrm{ran}(A-\lambda_0)$ is closed and

$$\ker\,(A-\lambda_0)^n \subset \mathrm{ran}(A-\lambda_0), \quad n \in \mathbb{N}, \tag{5.2}$$

cf. (3.4). Take orthogonal complements in (5.2) and use (5.1) to obtain

$$\ker\,(A^* - \bar{\lambda}_0) \subset \mathrm{ran}(A^* - \bar{\lambda}_0)^n, \quad n \in \mathbb{N}.$$

Hence $A^* - \bar{\lambda}_0$ is quasi-Fredholm of degree 0 and thus $\bar{\lambda}_0 \in \mathrm{reg}\,(A^*)$. □

This result leads to an analog of Lemma 3.3 involving ranges instead of kernels.

LEMMA 5.2. *Let A be a closed linear relation in a Hilbert space $\mathfrak{H}$ and assume that $\lambda_0 \in \operatorname{reg}(A)$. Then $r(A-\lambda_0) > 0$ and for all $\lambda \in \mathbb{C}$ which satisfy $|\lambda-\lambda_0| < r(A-\lambda_0)$*

$$g(\operatorname{ran}(A-\lambda), \operatorname{ran}(A-\lambda_0)) < 1, \tag{5.3}$$

or, equivalently,

$$\mathfrak{H} = \operatorname{ran}(A-\lambda) + (\operatorname{ran}(A-\lambda_0))^{\perp}, \quad \textit{direct sum}. \tag{5.4}$$

Proof. Let $\lambda \in \mathbb{C}$ satisfy $|\lambda-\lambda_0| < r(A-\lambda_0)$. By Theorem 5.1 it follows that $\bar{\lambda}_0 \in \operatorname{reg}(A^*)$. Since $|\bar{\lambda}-\bar{\lambda}_0| < r(A-\lambda_0) = r(A^*-\bar{\lambda}_0)$, it follows from Lemma 3.3 that

$$g(\ker(A^*-\bar{\lambda}), \ker(A^*-\bar{\lambda}_0)) < 1,$$

or, equivalently, by (2.33)

$$g(\overline{\operatorname{ran}}(A-\lambda), \overline{\operatorname{ran}}(A-\lambda_0)) < 1.$$

Moreover, recall from Lemma 3.4 that $\operatorname{ran}(A-\lambda)$ is closed, which now leads to (5.3). The equivalence between (5.3) and (5.4) follows from (2.36). □

It is now clear that there exists an analog of the description in Theorem 4.1, which extends Lemma 5.2.

COROLLARY 5.3. *Let A be a closed linear relation in a Hilbert space $\mathfrak{H}$ and let $\lambda_0 \in \mathbb{C}$. Then $\lambda_0 \in \operatorname{reg}(A)$ if and only if $r(A-\lambda_0) > 0$ and there exists a neighborhood $\mathscr{U}(\lambda_0)$ of λ_0 such that*

$$\operatorname{ran}(A-\lambda) \textit{ is closed}, \quad \lambda \in \mathscr{U}(\lambda_0), \tag{5.5}$$

and

$$g(\operatorname{ran}(A-\lambda), \operatorname{ran}(A-\lambda_0)) < 1, \quad \lambda \in \mathscr{U}(\lambda_0). \tag{5.6}$$

The neighborhood $\mathscr{U}(\lambda_0)$ may be chosen so as to contain the disk

$$\{\lambda \in \mathbb{C} : |\lambda-\lambda_0| < r(A-\lambda_0)\}, \tag{5.7}$$

and on that disk

$$g(\operatorname{ran}(A-\lambda), \operatorname{ran}(A-\lambda_0)) \leqslant \frac{|\lambda-\lambda_0|}{r(A-\lambda_0)}. \tag{5.8}$$

Let A be a closed linear relation in a Hilbert space $\mathfrak{H}$. Recall that $\lambda \in \rho(A)$ if and only if $\bar{\lambda} \in \rho(A^*)$. However, when $\lambda \in \gamma(A)$, then, in general, only $\bar{\lambda} \in \operatorname{reg}(A^*)$. A direct consequence of Theorem 4.1 is the following result, cf. [10, Corollaire 4.11], which is applicable to normal relations, cf. [7].

COROLLARY 5.4. *Let A be a closed linear relation in a Hilbert space for which*

$$\ker(A^*-\bar{\lambda}) = \ker(A-\lambda), \quad \lambda \in \mathbb{C}. \tag{5.9}$$

Then $\operatorname{reg}(A) = \rho(A)$.

Proof. Since $\rho(A) \subset \operatorname{reg}(A)$, it suffices to show $\operatorname{reg}(A) \subset \rho(A)$. Let $\lambda \in \operatorname{reg}(A)$, then (5.9) implies that

$$\ker(A^* - \bar{\lambda}) = \ker(A - \lambda) \subset \operatorname{ran}(A - \lambda) = \ker(A^* - \bar{\lambda})^{\perp},$$

so that $\ker(A^* - \bar{\lambda}) = \{0\}$. Hence $\ker(A - \lambda) = \{0\}$ and $\operatorname{ran}(A - \lambda) = \mathfrak{H}$, which shows that $\lambda \in \rho(A)$. □

6. Regular points and continuity

Let A be a closed linear relation in a Hilbert space $\mathfrak{H}$ and assume that $\lambda_0 \in \operatorname{reg}(A)$. Then by Lemma 3.3, Lemma 3.4, and Lemma 5.2 there exists a neighborhood $\mathscr{U}$ of λ_0, defined by

$$\mathscr{U} = \{\lambda \in \mathbb{C} : |\lambda - \lambda_0| < r(A - \lambda_0)\}, \tag{6.1}$$

for which

$$r(A - \lambda) \geqslant r(A - \lambda_0) - |\lambda - \lambda_0| > 0, \quad \lambda \in \mathscr{U}, \tag{6.2}$$

so that $\operatorname{ran}(A - \lambda)$, $\lambda \in \mathscr{U}$, is closed; and, moreover, for $\lambda \in \mathscr{U}$ the inequalities (3.5) and (5.3) are valid. Recall from Lemma 3.1 and Lemma 3.3 that, due to (2.36) and (2.37), in fact,

$$g(\ker(A - \lambda), \ker(A - \lambda_0)) \leqslant \frac{|\lambda - \lambda_0|}{r(A - \lambda_0)}, \quad \lambda \in \mathscr{U}, \tag{6.3}$$

and

$$g(\operatorname{ran}(A - \lambda), \operatorname{ran}(A - \lambda_0)) \leqslant \frac{|\lambda - \lambda_0|}{r(A - \lambda_0)}, \quad \lambda \in \mathscr{U}. \tag{6.4}$$

For all $\lambda \in \mathscr{U}$ one has the direct sum decompositions

$$\mathfrak{H} = \ker(A - \lambda) + \ker(A - \lambda_0)^{\perp}, \quad \text{direct sum}, \tag{6.5}$$

and

$$\mathfrak{H} = \operatorname{ran}(A - \lambda) + \operatorname{ran}(A - \lambda_0)^{\perp}, \quad \text{direct sum}. \tag{6.6}$$

Likewise it follows from (6.5), (6.6), and (2.36) that

$$\mathfrak{H} = (\ker(A - \lambda))^{\perp} + \ker(A - \lambda_0), \quad \text{direct sum}, \tag{6.7}$$

and

$$\mathfrak{H} = (\operatorname{ran}(A - \lambda))^{\perp} + \operatorname{ran}(A - \lambda_0), \quad \text{direct sum}. \tag{6.8}$$

The dependence on λ of the first summands in the direct sum decompositions (6.5), (6.6), (6.7), and (6.8) is studied in the following proposition.

PROPOSITION 6.1. *Let A be a closed linear relation in a Hilbert space $\mathfrak{H}$. Then the set $\operatorname{reg}(A)$ is open, and the mappings*

(i) $\operatorname{reg}(A) \ni \lambda \mapsto \ker(A - \lambda)$*;*

(ii) $\operatorname{reg}(A) \ni \lambda \mapsto (\ker (A-\lambda))^{\perp}$;

(iii) $\operatorname{reg}(A) \ni \lambda \mapsto \operatorname{ran}(A-\lambda)$;

(iv) $\operatorname{reg}(A) \ni \lambda \mapsto \operatorname{ran}((A-\lambda))^{\perp}$,

from $\operatorname{reg}(A)$ *into the space* $\mathbf{S}(\mathfrak{H})$ *of closed linear subspaces of* $\mathfrak{H}$ *provided with the gap metric, are continuous.*

Proof. In order to show that the set $\operatorname{reg}(A)$ is open, let $\lambda_0 \in \operatorname{reg}(A)$. Then let the point $\lambda \in \mathbb{C}$ satisfy

$$|\lambda-\lambda_0| < r(A-\lambda_0). \tag{6.9}$$

It has been shown in Lemma 3.4 that $\operatorname{ran}(A-\lambda)$ is closed. Now it will be shown that there exists a neighborhood $\mathscr{V}$ of λ so that for all μ in that neighborhood one has $g(\ker (A-\mu), \ker (A-\lambda)) < 1$; in other words that λ is also a regular point of the relation A; cf. Theorem 4.1.

Let $\lambda \in \mathbb{C}$ satisfy (6.9). The neighborhood $\mathscr{V}$ of λ is defined by

$$\mathscr{V} = \{\, \mu \in \mathbb{C} : 2|\mu-\lambda| < r(A-\lambda_0) - |\lambda-\lambda_0|\,\}. \tag{6.10}$$

For any $\mu \in \mathscr{V}$ it follows from the definition in (6.10) and the assumption (6.9) that

$$\begin{aligned} |\mu-\lambda_0| &\leqslant |\mu-\lambda| + |\lambda-\lambda_0| \\ &< (r(A-\lambda_0) - |\lambda-\lambda_0|)/2 + |\lambda-\lambda_0| \\ &= (r(A-\lambda_0) + |\lambda-\lambda_0|)/2. \end{aligned} \tag{6.11}$$

Due to (6.9), the inequality (6.11) shows that any $\mu \in \mathscr{V}$ also satisfies:

$$|\mu-\lambda_0| < r(A-\lambda_0), \tag{6.12}$$

Hence $\mathscr{V}$ is contained in the disk in (6.9). In particular, it follows from Lemma 3.4 that $\operatorname{ran}(A-\mu)$ is closed for all $\mu \in \mathscr{V}$.

For $\mu \in \mathscr{V}$ the definition in (6.10) and the inequality in (3.15) imply that

$$|\mu-\lambda| < 2|\mu-\lambda| \leqslant r(A-\lambda_0) - |\lambda-\lambda_0| \leqslant r(A-\lambda). \tag{6.13}$$

Since $\operatorname{ran}(A-\lambda)$ is closed, Lemma 3.1 may be applied, which gives with (6.13)

$$\delta(\ker (A-\mu), \ker (A-\lambda)) \leqslant \frac{|\mu-\lambda|}{r(A-\lambda)} < 1, \quad \mu \in \mathscr{V}. \tag{6.14}$$

Furthermore, (6.12) shows that (3.15) holds with λ replaced by μ:

$$r(A-\mu) \geqslant r(A-\lambda_0) - |\mu-\lambda_0| > 0, \tag{6.15}$$

Hence, in (6.15) an application of (6.11) and the definition of $\mathscr{V}$ lead to

$$\begin{aligned} r(A-\mu) &\geqslant r(A-\lambda_0) - |\mu-\lambda_0| \\ &> (r(A-\lambda_0) - |\lambda-\lambda_0|)/2 \\ &> |\lambda-\mu|. \end{aligned} \tag{6.16}$$

Since $\operatorname{ran}(A-\mu)$ is closed, Lemma 3.1 may be applied, which gives with (6.16)

$$\delta(\ker(A-\lambda),\ker(A-\mu)) \leqslant \frac{|\lambda-\mu|}{r(A-\mu)} < 1, \quad \mu \in \mathscr{V}. \tag{6.17}$$

It follows from (6.14), (6.17), and (2.35) that $g(\ker(A-\lambda),\ker(A-\mu)) < 1$ for $\mu \in \mathscr{V}$. In particular, this leads to $\lambda \in \operatorname{reg}(A)$.

Therefore it has been shown that $\operatorname{reg}(A)$ is open in $\mathbb{C}$. The continuity of the mappings in the theorem follows from the majorization of the corresponding gap norms in (6.3) and (6.4). □

7. Regular points and continuous generalized resolvents

Let A be a closed linear relation with multivalued part $\operatorname{mul} A$ and corresponding orthogonal operator part A_s. Let P be the orthogonal projection onto $\operatorname{mul} A$. Assume that there is an open set $\mathscr{U} \subset \mathbb{C}$ such that $\operatorname{ran}(A-\lambda)$ is closed for all $\lambda \in \mathscr{U}$. Let P_λ and Q_λ be projections with

$$\operatorname{ran} P_\lambda = \operatorname{ran}(A-\lambda), \quad \ker Q_\lambda = \ker(A-\lambda), \quad \lambda \in \mathscr{U}.$$

Let $\mathscr{R}(\lambda) \in \mathbf{D}(\mathfrak{H})$ be a family of operators from $\mathfrak{H}$ to $\operatorname{dom} A$ (with the graph norm), which satisfy for $\lambda \in \mathscr{U}$:

$$(A_s-\lambda)\mathscr{R}(\lambda)h = (I-P)P_\lambda h - \lambda P\mathscr{R}(\lambda)h, \quad h \in \mathfrak{H},$$

and

$$\mathscr{R}(\lambda)(A_s-\lambda)h = Q_\lambda h, \quad h \in \operatorname{dom} A.$$

The family $\mathscr{R}(\lambda)$, $\lambda \in \mathscr{U}$, will be called a *generalized resolvent* of A. The generalized resolvent $\mathscr{R}(\lambda)$ will be called *continuous*, in the sense of the graph norm, if the mapping $\lambda \in \mathscr{U} \to \mathscr{R}(\lambda) \in \mathbf{D}(\mathfrak{H})$ is continuous, i.e., if

$$\|\mathscr{R}(\lambda)-\mathscr{R}(\mu)\|_{\mathfrak{D}} \to 0 \quad \text{as} \quad \lambda \to \mu, \quad \lambda,\mu \in \mathscr{U}.$$

The following theorem is another one of the main results of this paper. It characterizes the regular points $\operatorname{reg}(A)$ of a closed linear relation A in a Hilbert space $\mathfrak{H}$ in terms of the existence of a continuous generalized resolvent $\mathscr{R}(\lambda)$ as defined above.

THEOREM 7.1. *Let A be a closed linear relation in a Hilbert space $\mathfrak{H}$ and let $\lambda_0 \in \mathbb{C}$. Then the following statements are equivalent:*

(i) $\lambda_0 \in \operatorname{reg}(A)$;

(ii) *there is a generalized resolvent of A, continuous in a neighborhood of λ_0, in the sense of the graph norm.*

Proof. (i) $\Rightarrow$ (ii) Assume that $\lambda_0 \in \operatorname{reg}(A)$. With the neighborhood

$$\mathscr{U} = \{ \lambda \in \mathbb{C} : |\lambda - \lambda_0| < r(A - \lambda_0) \}$$

of λ_0 one obtains the estimate (6.2), so that $\operatorname{ran}(A - \lambda)$, $\lambda \in \mathscr{U}$, is closed. Moreover, one obtains the direct sum decompositions (6.5) and (6.6), and the corresponding estimates (6.3) and (6.4).

For $\lambda \in \mathscr{U}$ let Q_λ be the projection from $\mathfrak{H}$ onto $(\ker(A - \lambda_0))^\perp$ parallel to $\ker(A - \lambda)$

$$\ker Q_\lambda = \ker(A - \lambda),$$

and let P_λ be the projection from $\mathfrak{H}$ onto $\operatorname{ran}(A - \lambda)$ parallel to $(\operatorname{ran}(A - \lambda_0))^\perp$

$$\operatorname{ran} P_\lambda = \operatorname{ran}(A - \lambda).$$

With these projections define the generalized resolvent $\mathscr{R}(\lambda)$, $\lambda \in \mathscr{U}$:

$$\mathscr{R}(\lambda) = Q_\lambda (A - \lambda)^{-1} P_\lambda, \quad \lambda \in \mathscr{U}, \tag{7.1}$$

and recall that $\mathscr{R}(\lambda) \in \mathbf{B}(\mathfrak{H})$; cf. Lemma 2.2. Due to (6.5) and (6.6) the generalized resolvent $\mathscr{R}(\lambda)$ satisfies the resolvent identity (2.19). Then by (2.20)

$$A_s \mathscr{R}(\lambda) = (I - P)P_\lambda + \lambda (I - P)\mathscr{R}(\lambda),$$

where P is the orthogonal projection onto $\operatorname{mul} A$. Hence, for all $h \in \mathfrak{H}$ this leads to

$$\begin{aligned} \|\mathscr{R}(\lambda)h\|_{\mathfrak{D}}^2 &= \|A_s \mathscr{R}(\lambda)h\|^2 + \|\mathscr{R}(\lambda)h\|^2 \\ &= \|(I - P)P_\lambda h + \lambda (I - P)\mathscr{R}(\lambda)h\|^2 + \|\mathscr{R}(\lambda)h\|^2 \\ &\leqslant 2\|P_\lambda h\|^2 + (2|\lambda|^2 + 1)\|\mathscr{R}(\lambda)h\|^2. \end{aligned} \tag{7.2}$$

Each of these terms will be estimated. First observe that

$$\{P_{\ker(A-\lambda)} \mathscr{R}(\lambda)h, 0\} \in A - \lambda,$$

and thus (2.12) leads to

$$\begin{aligned} &\{ (I - P_{\ker(A-\lambda)})\mathscr{R}(\lambda)h, P_\lambda h \} \in A - \lambda, \\ &\qquad (I - P_{\ker(A-\lambda)})\mathscr{R}(\lambda)h \perp \ker(A - \lambda). \end{aligned} \tag{7.3}$$

It follows from (7.3) and (2.6) that

$$r(A - \lambda)\|(I - P_{\ker(A-\lambda)})\mathscr{R}(\lambda)h\| \leqslant \|P_\lambda h\|, \quad h \in \mathfrak{H}. \tag{7.4}$$

Furthermore, it follows from the definition of Q_λ that for all $h \in \mathfrak{H}$:

$$\begin{aligned} \|P_{\ker(A-\lambda)} \mathscr{R}(\lambda)h\| &= \|P_{\ker(A-\lambda)}(I - P_{\ker(A-\lambda_0)})\mathscr{R}(\lambda)h\| \\ &\leqslant \|P_{\ker(A-\lambda)}(I - P_{\ker(A-\lambda_0)})\| \, \|\mathscr{R}(\lambda)h\| \\ &= g(\ker(A - \lambda), \ker(A - \lambda_0)) \, \|\mathscr{R}(\lambda)h\|. \end{aligned} \tag{7.5}$$

Note that (7.5) shows

$$\begin{aligned}(1-g(\ker(A-\lambda),\ker(A-\lambda_0))^2)\|\mathscr{R}(\lambda)h\|^2 \\ \leqslant \|\mathscr{R}(\lambda)h\|^2-\|P_{\ker(A-\lambda)}\mathscr{R}(\lambda)h\|^2 \\ =\|(I-P_{\ker(A-\lambda)})\mathscr{R}(\lambda)h\|^2,\end{aligned} \tag{7.6}$$

where the first term in the inequality is postive due to the direct sum decomposition in (6.5). Combine (7.4) and (7.6) to obtain

$$\|\mathscr{R}(\lambda)h\|^2 \leqslant \frac{1}{(1-g(\ker(A-\lambda),\ker(A-\lambda_0))^2)(r(A-\lambda)^2)}\|P_\lambda h\|^2. \tag{7.7}$$

Recall that

$$\|P_\lambda\|^2=\frac{1}{1-g(\mathrm{ran}(A-\lambda),\mathrm{ran}(A-\lambda_0))^2}, \tag{7.8}$$

as follows from (2.38).

Now choose $0<c<r(A-\lambda_0)$ and consider a compact disk $\mathscr{U}_c$ of the form

$$\mathscr{U}_c=\{\lambda\in\mathscr{U}:|\lambda-\lambda_0|\leqslant c\}$$

inside $\mathscr{U}$. Then one obtains from (6.2), (6.3), and (6.4) the uniform bounds

$$r(A-\lambda)\geqslant r(A-\lambda_0)-c>0, \quad \lambda\in\mathscr{U}_c, \tag{7.9}$$

$$g(\ker(A-\lambda),\ker(A-\lambda_0))\leqslant\frac{c}{r(A-\lambda_0)}<1, \quad \lambda\in\mathscr{U}_c, \tag{7.10}$$

and

$$g(\mathrm{ran}(A-\lambda),\mathrm{ran}(A-\lambda_0))\leqslant\frac{c}{r(A-\lambda_0)}<1, \quad \lambda\in\mathscr{U}_c. \tag{7.11}$$

Hence (7.2), (7.7), and (7.8) together with (7.9), (5.8), and (7.11) lead to the existence of K_c for which

$$\|\mathscr{R}(\lambda)h\|_{\mathfrak{D}}\leqslant K_c\|h\|, \quad h\in\mathfrak{H}, \tag{7.12}$$

for all $\lambda\in\mathscr{U}_c$. Now apply Lemma 2.7 to obtain the desired result.

(ii) $\Rightarrow$ (i) Assume that there exists a generalized resolvent $\mathscr{R}(\lambda)$ of A which is continuous in a neighborhood $\mathscr{V}$ of λ_0. By definition

$$\mathrm{ran}(A-\lambda) \text{ is closed}, \quad \lambda\in\mathscr{V}. \tag{7.13}$$

In particular, $\mathrm{ran}(A-\lambda_0)$ is closed and in order to show that $\lambda_0\in\mathrm{reg}(A)$ it suffices to show that for all $n\in\mathbb{N}$

$$\ker(A-\lambda_0)^n\subset\mathrm{ran}(A-\lambda_0), \tag{7.14}$$

see (3.4).

First it will be shown by induction that for all $n\in\mathbb{N}$

$$\ker(A-\lambda_0)^n\subset\mathrm{ran}(A-\lambda), \quad \lambda\neq\lambda_0. \tag{7.15}$$

If $h \in \ker(A-\lambda_0)$ then $\{h, (\lambda_0-\lambda)h\} \in A-\lambda$, and it follows that $h \in \operatorname{ran}(A-\lambda)$ when $\lambda \neq \lambda_0$. This proves (7.15) for $n=1$. Now assume for some $n \in \mathbb{N}$ that

$$\ker(A-\lambda_0)^{n-1} \subset \operatorname{ran}(A-\lambda), \quad \lambda \neq \lambda_0.$$

If $h \in \ker(A-\lambda_0)^n$ then $\{h, \varphi\} \in A-\lambda_0$ for some $\varphi \in \ker(A-\lambda_0)^{n-1}$. Therefore $\varphi + (\lambda_0-\lambda)h \in \operatorname{ran}(A-\lambda)$ with $\varphi \in \operatorname{ran}(A-\lambda)$, which implies that $h \in \operatorname{ran}(A-\lambda)$, $\lambda \neq \lambda_0$. Hence, (7.15) has been established.

Now (7.14) will be shown. Recall from (7.13) that $\operatorname{ran}(A-\lambda)$ is closed for λ in a neighborhood $\mathscr{V}$ of λ_0. Let $P_{\operatorname{ran}(A-\lambda)}$ be the orthogonal projection onto $\operatorname{ran}(A-\lambda)$ and let P denote the orthogonal projection onto $\operatorname{mul} A$. Since $\operatorname{mul} A \subset \operatorname{ran}(A-\lambda)$ for all $\lambda \in \mathbb{C}$, it follows for the orthogonal projection $P_{\operatorname{ran}(A-\lambda)}$ that

$$PP_{\operatorname{ran}(A-\lambda)} = P_{\operatorname{ran}(A-\lambda)}P = P, \quad \lambda \in \mathbb{C}, \tag{7.16}$$

and for the projection P_λ associated with the generalized resolvent $\mathscr{R}(\lambda)$ that

$$P_\lambda P = P, \quad \lambda \in \operatorname{reg}(A). \tag{7.17}$$

Observe that $(I-P)P_{\operatorname{ran}(A-\lambda)}$ is a projection by (7.16), which is orthogonal since it is selfadjoint, and that $(I-P)P_\lambda$ is a bounded projection by (7.17). Moreover, it is easily checked that each of these projections has the same range $\operatorname{ran}(A-\lambda) \ominus \operatorname{mul} A$.

Finally, let $h \in \ker(A-\lambda_0)^n$, then clearly by (7.15) $P_{\operatorname{ran}(A-\lambda)}h = h$. Hence

$$\begin{aligned} h - P_{\operatorname{ran}(A-\lambda_0)}h &= P_{\operatorname{ran}(A-\lambda)}h - P_{\operatorname{ran}(A-\lambda_0)}h \\ &= (I-P)(P_{\operatorname{ran}(A-\lambda)} - P_{\operatorname{ran}(A-\lambda_0)}), \end{aligned}$$

where (7.16) has been used. Therefore it follows that

$$\begin{aligned} \|h - P_{\operatorname{ran}(A-\lambda_0)}h\| &= \|(I-P)(P_{\operatorname{ran}(A-\lambda)} - P_{\operatorname{ran}(A-\lambda_0)})h\| \\ &\leqslant \|(I-P)(P_{\operatorname{ran}(A-\lambda)} - P_{\operatorname{ran}(A-\lambda_0)})\|\,\|h\| \\ &= g((I-P)P_{\operatorname{ran}(A-\lambda)}, (I-P)P_{\operatorname{ran}(A-\lambda_0)})\,\|h\| \\ &\leqslant \|(I-P)P_\lambda - (I-P)P_{\lambda_0})\|\|h\| \\ &= \|(I-P)(P_\lambda - P_{\lambda_0})\|\|h\|, \end{aligned} \tag{7.18}$$

where (2.32) and (2.34) have been used. Now observe that the definition

$$(I-P)P_\lambda = A_{\mathrm{s}}\mathscr{R}(\lambda) - \lambda(I-P)\mathscr{R}(\lambda)$$

implies that

$$(I-P)(P_\lambda - P_{\lambda_0}) = A_{\mathrm{s}}(\mathscr{R}(\lambda) - \mathscr{R}(\lambda_0)) - (I-P)(\lambda\mathscr{R}(\lambda) - \lambda_0\mathscr{R}(\lambda_0)).$$

Since

$$\|A_{\mathrm{s}}(\mathscr{R}(\lambda) - \mathscr{R}(\lambda_0))\| \leqslant \|\mathscr{R}(\lambda) - \mathscr{R}(\lambda_0)\|_{\mathfrak{D}}$$

and $\mathscr{R}(\lambda)$ is continuous in the graph norm, it follows that $\|(I-P)(P_\lambda - P_{\lambda_0})\|$ tends to 0 for $\lambda \to \lambda_0$. Hence one concludes from (7.18) that

$$h = P_{\operatorname{ran}(A-\lambda_0)}h \in \operatorname{ran}(A-\lambda_0).$$

Therefore (7.14) has been established. □

8. On the opening between subspaces

Let $\mathfrak{H}$ be a Hilbert space and let $\mathfrak{M}$ and $\mathfrak{N}$ be closed linear subspaces of $\mathfrak{H}$. In general, the sum $\mathfrak{M}+\mathfrak{N}$ need not be closed (see [15] for an interesting example). This section presents a review of necessary and sufficient conditions under which $\mathfrak{M}+\mathfrak{N}$ is closed.

The intersection $\mathfrak{M}\cap\mathfrak{N}$ is a closed linear subspace. Hence the Hilbert space $\mathfrak{H}$ has the following orthogonal decomposition

$$\mathfrak{H} = (\mathfrak{M}\cap\mathfrak{N})^{\perp} \oplus (\mathfrak{M}\cap\mathfrak{N}). \tag{8.1}$$

Introduce the 'reduced' subspaces $\mathfrak{M}_0$ and $\mathfrak{N}_0$ by

$$\mathfrak{M}_0 = \mathfrak{M}\cap(\mathfrak{M}\cap\mathfrak{N})^{\perp}, \quad \mathfrak{N}_0 = \mathfrak{N}\cap(\mathfrak{M}\cap\mathfrak{N})^{\perp}. \tag{8.2}$$

Then $\mathfrak{M}_0$ and $\mathfrak{N}_0$ are closed linear subspaces of $(\mathfrak{M}\cap\mathfrak{N})^{\perp}$ and

$$\mathfrak{M}_0\cap\mathfrak{N}_0 = \{0\}. \tag{8.3}$$

Denote the orthogonal complements of $\mathfrak{M}_0$ and $\mathfrak{N}_0$ in $(\mathfrak{M}\cap\mathfrak{N})^{\perp}$ by $\mathfrak{M}_0^{\perp}$ and $\mathfrak{N}_0^{\perp}$, respectively.

LEMMA 8.1. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be closed linear subspaces of a Hilbert space $\mathfrak{H}$ and let $\mathfrak{M}_0$ and $\mathfrak{N}_0$ be defined by* (2.26)*. Then, corresponding to* (8.1)*, $\mathfrak{M}$ and $\mathfrak{N}$ have the orthogonal decompositions*

$$\mathfrak{M} = \mathfrak{M}_0\oplus(\mathfrak{M}\cap\mathfrak{N}), \quad \mathfrak{N} = \mathfrak{N}_0\oplus(\mathfrak{M}\cap\mathfrak{N}). \tag{8.4}$$

Moreover, the space $(\mathfrak{M}\cap\mathfrak{N})^{\perp}$ has the following decompositions

$$(\mathfrak{M}\cap\mathfrak{N})^{\perp} = \mathfrak{M}_0\oplus\mathfrak{M}^{\perp}, \quad (\mathfrak{M}\cap\mathfrak{N})^{\perp} = \mathfrak{N}_0\oplus\mathfrak{N}^{\perp}, \tag{8.5}$$

in other words $\mathfrak{M}^{\perp} = \mathfrak{M}_0^{\perp}$ and $\mathfrak{N}^{\perp} = \mathfrak{N}_0^{\perp}$.

COROLLARY 8.2. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be closed linear subspaces of a Hilbert space $\mathfrak{H}$ and let $\mathfrak{M}_0$ and $\mathfrak{N}_0$ be defined by* (8.2)*. Then the following statements are equivalent:*

(i) $\mathfrak{M}+\mathfrak{N}$ *is closed;*

(ii) $\mathfrak{M}_0+\mathfrak{N}_0$ *is closed.*

Moreover, the orthogonal complements satisfy

$$\mathfrak{M}^{\perp}+\mathfrak{N}^{\perp} = \mathfrak{M}_0^{\perp}+\mathfrak{N}_0^{\perp},$$

so that both sums are closed simultaneously.

If the subspace $\mathfrak{M}+\mathfrak{N}$ is closed and $\mathfrak{M}\cap\mathfrak{N}=\{0\}$, then $\mathfrak{M}+\mathfrak{N}$ may be considered as a Hilbert space in its own right with corresponding projections from $\mathfrak{M}+\mathfrak{N}$ onto $\mathfrak{M}$ or $\mathfrak{N}$. This leads to the following simple characterization, based on parallel projections and the closed graph theorem.

LEMMA 8.3. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be closed linear subspaces of a Hilbert space $\mathfrak{H}$. Then the following statements are equivalent:*

(i) $\mathfrak{M}+\mathfrak{N}$ *is closed and* $\mathfrak{M}\cap\mathfrak{N}=\{0\}$*;*

(ii) *there exists* $\rho>0$ *such that*

$$\rho\sqrt{\|h\|^2+\|k\|^2}\leqslant\|h+k\|,\quad h\in\mathfrak{M},\ k\in\mathfrak{N}. \tag{8.6}$$

Let $\mathfrak{M}$ and $\mathfrak{N}$ be closed linear subspaces of a Hilbert space $\mathfrak{H}$ and let $P_{\mathfrak{M}}$ and $P_{\mathfrak{N}}$ denote the corresponding orthogonal projections. The *opening* $c_0(\mathfrak{M},\mathfrak{N})$ between $\mathfrak{M}$ and $\mathfrak{N}$ is defined as

$$c_0(\mathfrak{M},\mathfrak{N})=\sup\{\,|(h,k)|:h\in\mathfrak{M},\ \|h\|\leqslant 1,\ k\in\mathfrak{N},\ \|k\|\leqslant 1\,\}. \tag{8.7}$$

It is clear from this definition that $c_0(\mathfrak{M},\mathfrak{N})=c_0(\mathfrak{N},\mathfrak{M})$. Moreover, since

$$c_0(\mathfrak{M},\mathfrak{N})=\sup\{\,|(P_{\mathfrak{M}}h,P_{\mathfrak{N}}k)|:\ \|h\|\leqslant 1,\ \|k\|\leqslant 1\,\},$$

it follows that

$$c_0(\mathfrak{M},\mathfrak{N})=\|P_{\mathfrak{M}}P_{\mathfrak{N}}\|,$$

which characterizes $c_0(\mathfrak{M},\mathfrak{N})$ in terms of the orthogonal projections $P_{\mathfrak{M}}$ and $P_{\mathfrak{N}}$.

PROPOSITION 8.4. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be closed linear subspaces of a Hilbert space $\mathfrak{H}$. Then the following statements are equivalent:*

(i) $c_0(\mathfrak{M},\mathfrak{N})<1$*;*

(ii) $\mathfrak{M}+\mathfrak{N}$ *is closed and* $\mathfrak{M}\cap\mathfrak{N}=\{0\}$*.*

Proof. (i) $\Rightarrow$ (ii) Assume that $c_0(\mathfrak{M},\mathfrak{N})<1$. It is clear that $\mathfrak{M}\cap\mathfrak{N}=\{0\}$. In order to see that $\mathfrak{M}+\mathfrak{N}$ is closed, observe that the identity

$$\|h+k\|^2=\|h\|^2+\|k\|^2+2\mathrm{Re}\,(h,k),\quad h,k\in\mathfrak{H},$$

leads to the following inequalities

$$\begin{aligned}\|h\|^2+\|k\|^2&\leqslant\|h+k\|^2+2|(h,k)|\\&\leqslant\|h+k\|^2+2c_0(\mathfrak{M},\mathfrak{N})\|h\|\|k\|\\&\leqslant\|h+k\|^2+c_0(\mathfrak{M},\mathfrak{N})(\|h\|^2+\|k\|^2),\quad h\in\mathfrak{M},\quad k\in\mathfrak{N}.\end{aligned} \tag{8.8}$$

In particular, it follows that

$$(1-c_0(\mathfrak{M},\mathfrak{N}))(\|h\|^2+\|k\|^2)\leqslant\|h+k\|^2,\quad h\in\mathfrak{M},\quad k\in\mathfrak{N}. \tag{8.9}$$

Hence, by Lemma 8.3 $\mathfrak{M}+\mathfrak{N}$ is closed.

(ii) $\Rightarrow$ (i) Assume that $\mathfrak{M}+\mathfrak{N}$ is closed and $\mathfrak{M}\cap\mathfrak{N}=\{0\}$. According to Lemma 8.3 the inequality (8.6) holds for some $\rho>0$. Now suppose that $c_0(\mathfrak{M},\mathfrak{N})=1$. Then there exist sequences $h_n\in\mathfrak{M}$ and $k_n\in\mathfrak{N}$, such that

$$(h_n,k_n)\to 1, \quad \|h_n\|=\|k_n\|=1.$$

Hence, it follows from (8.6) that

$$\begin{aligned} 2\rho^2 \leqslant \|h_n-k_n\|^2 &= \|h_n\|^2-2\mathrm{Re}\,(h_n,k_n)+\|k_n\|^2 \\ &= 2(1-\mathrm{Re}\,(h_n,k_n))\to 0, \end{aligned}$$

which leads to a contradiction. Thus it follows that $c_0(\mathfrak{M},\mathfrak{N})<1$. □

Let $\mathfrak{M}$ and $\mathfrak{N}$ be closed linear subspaces of a Hilbert space $\mathfrak{H}$. The *opening* $c(\mathfrak{M},\mathfrak{N})$ between $\mathfrak{M}$ and $\mathfrak{N}$ is defined as

$$c(\mathfrak{M},\mathfrak{N})=c_0(\mathfrak{M}_0,\mathfrak{N}_0), \tag{8.10}$$

where $\mathfrak{M}_0$ and $\mathfrak{N}_0$ are defined as in (8.2). It is clear from this definition that $c(\mathfrak{M},\mathfrak{N})=c(\mathfrak{N},\mathfrak{M})$. Moreover, it follows that

$$\begin{aligned} c(\mathfrak{M},\mathfrak{N}) &= \sup\{\,|(P_{\mathfrak{M}\cap(\mathfrak{M}\cap\mathfrak{N})^\perp}h,P_{\mathfrak{N}\cap(\mathfrak{M}\cap\mathfrak{N})^\perp}k)|:\,\|h\|\leqslant 1,\,\|k\|\leqslant 1\,\} \\ &= \sup\{\,|(P_{(\mathfrak{M}\cap\mathfrak{N})^\perp}P_{\mathfrak{M}}h,P_{(\mathfrak{M}\cap\mathfrak{N})^\perp}P_{\mathfrak{N}}k)|:\,\|h\|\leqslant 1,\,\|k\|\leqslant 1\,\} \\ &= \sup\{\,|P_{\mathfrak{M}}h,P_{(\mathfrak{M}\cap\mathfrak{N})^\perp}P_{\mathfrak{N}}k)|:\,\|h\|\leqslant 1,\,\|k\|\leqslant 1\,\} \\ &= \sup\{\,|(P_{\mathfrak{M}}h,P_{\mathfrak{N}\cap(\mathfrak{M}\cap\mathfrak{N})^\perp}k)|:\,\|h\|\leqslant 1,\,\|k\|\leqslant 1\,\}, \end{aligned}$$

which leads to

$$c(\mathfrak{M},\mathfrak{N})=c_0(\mathfrak{M},\mathfrak{N}_0)=c_0(\mathfrak{M}_0,\mathfrak{N}), \tag{8.11}$$

where the last equality follows by symmetry. Observe that

$$\begin{aligned} c(\mathfrak{M},\mathfrak{N}) &= \|P_{\mathfrak{M}\cap(\mathfrak{M}\cap\mathfrak{N})^\perp}P_{\mathfrak{N}\cap(\mathfrak{M}\cap\mathfrak{N})^\perp}\| = \|P_{\mathfrak{M}}P_{(\mathfrak{M}\cap\mathfrak{N})^\perp}P_{\mathfrak{N}}P_{(\mathfrak{M}\cap\mathfrak{N})^\perp}\| \\ &= \|P_{\mathfrak{M}}P_{\mathfrak{N}}P_{(\mathfrak{M}\cap\mathfrak{N})^\perp}\| = \|P_{\mathfrak{M}}P_{\mathfrak{N}}(I-P_{\mathfrak{M}\cap\mathfrak{N}})\| \\ &= \|P_{\mathfrak{M}}P_{\mathfrak{N}}-P_{\mathfrak{M}}P_{\mathfrak{N}}P_{\mathfrak{M}\cap\mathfrak{N}}\| = \|P_{\mathfrak{M}}P_{\mathfrak{N}}-P_{\mathfrak{M}\cap\mathfrak{N}}\|, \end{aligned}$$

which characterizes $c(\mathfrak{M},\mathfrak{N})$ in terms of the orthogonal projections $P_{\mathfrak{M}}$ and $P_{\mathfrak{N}}$.

PROPOSITION 8.5. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be closed linear subspaces of a Hilbert space $\mathfrak{H}$. Then the following statements are equivalent:*

(i) $c(\mathfrak{M},\mathfrak{N})<1$*;*

(ii) $\mathfrak{M}+\mathfrak{N}$ *is closed.*

Proof. The condition $c(\mathfrak{M},\mathfrak{N})<1$ is equivalent to $c_0(\mathfrak{M}_0,\mathfrak{N}_0)<1$, where $\mathfrak{M}_0$ and $\mathfrak{N}_0$ are defined in (8.2) and satisfy (2.27). Hence by Proposition 8.4 the condition

$c(\mathfrak{M},\mathfrak{N})<1$ is equivalent to $\mathfrak{M}_0+\mathfrak{N}_0$ is closed. Recall that $\mathfrak{M}_0+\mathfrak{N}_0$ is closed if and only if $\mathfrak{M}+\mathfrak{N}$ is closed. □

Let $\mathfrak{H}$ be a Hilbert space and let $A\in\mathbf{B}(\mathfrak{H})$ (the bounded linear operators, defined on all of $\mathfrak{H}$). The *minimum modulus* $r(A)$ of A is now

$$r(A)=\inf\left\{\frac{\|Ah\|}{\|h\|}:h\in\mathfrak{H}\ominus\ker A\right\}. \tag{8.12}$$

Then $\operatorname{ran}A$ is closed if and only if $r(A)>0$ and, furthermore, $r(A^*)=r(A)$.

THEOREM 8.6. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be closed linear subspaces of a Hilbert space $\mathfrak{H}$. Then*

$$c(\mathfrak{M},\mathfrak{N})^2+r((I-P_{\mathfrak{N}})P_{\mathfrak{M}})^2=1. \tag{8.13}$$

In particular,

$$c(\mathfrak{M}^\perp,\mathfrak{N}^\perp)=c(\mathfrak{M},\mathfrak{N}). \tag{8.14}$$

Proof. First observe that the following identity holds:

$$\ker\,((I-P_{\mathfrak{N}})P_{\mathfrak{M}})=(\mathfrak{M}\cap\mathfrak{N})\oplus\mathfrak{M}^\perp. \tag{8.15}$$

To see this, note that the righthand side is contained in the lefthand side. For the reverse inclusion, assume that $(I-P_{\mathfrak{N}})P_{\mathfrak{M}}h=0$ and write $h=f+g$ with $f\in\mathfrak{M}$ and $g\in\mathfrak{M}^\perp$. Then $f=P_{\mathfrak{N}}f$, so that $f\in\mathfrak{M}\cap\mathfrak{N}$. Hence, $h\in(\mathfrak{M}\cap\mathfrak{N})\oplus\mathfrak{M}^\perp$. This completes the proof of the reverse inclusion. It follows from (8.15) and (8.5) that

$$(\ker\,((I-P_{\mathfrak{N}})P_{\mathfrak{M}}))^\perp=\mathfrak{M}\cap(\mathfrak{M}\cap\mathfrak{N})^\perp. \tag{8.16}$$

Hence, by means of (8.12) and (8.16), it can be seen that

$$r((I-P_{\mathfrak{N}})P_{\mathfrak{M}})=\inf\left\{\frac{\|(I-P_{\mathfrak{N}})P_{\mathfrak{M}}h\|}{\|h\|}:h\in\mathfrak{M}\cap(\mathfrak{M}\cap\mathfrak{N})^\perp\right\}. \tag{8.17}$$

The following straightforward identity

$$\frac{\|(I-P_{\mathfrak{N}})P_{\mathfrak{M}}h\|^2}{\|h\|^2}=\frac{\|P_{\mathfrak{M}}h\|^2}{\|h\|^2}-\frac{\|P_{\mathfrak{N}}P_{\mathfrak{M}}h\|^2}{\|h\|^2},\qquad h\in\mathfrak{H}\setminus\{0\},$$

and (8.17) lead to

$$r((I-P_{\mathfrak{N}})P_{\mathfrak{M}})^2=1-\sup\left\{\frac{\|P_{\mathfrak{N}}P_{\mathfrak{M}}h\|^2}{\|h\|^2}:h\in\mathfrak{M}\cap(\mathfrak{M}\cap\mathfrak{N})^\perp\right\}. \tag{8.18}$$

It follows from $\mathfrak{M}\cap(\mathfrak{M}\cap\mathfrak{N})^\perp\subset\mathfrak{M}$ and the identity (8.5) that

$$\begin{aligned}
&\sup\left\{\frac{\|P_{\mathfrak{N}}P_{\mathfrak{M}}h\|^2}{\|h\|^2}:h\in\mathfrak{M}\cap(\mathfrak{M}\cap\mathfrak{N})^\perp\right\}\\
&\qquad=\sup\left\{\frac{\|P_{\mathfrak{N}}P_{\mathfrak{M}\cap(\mathfrak{M}\cap\mathfrak{N})^\perp}h\|^2}{\|h\|^2}:h\in\mathfrak{M}\cap(\mathfrak{M}\cap\mathfrak{N})^\perp\right\}\\
&\qquad=\sup\left\{\frac{\|P_{\mathfrak{N}}P_{\mathfrak{M}\cap(\mathfrak{M}\cap\mathfrak{N})^\perp}h\|^2}{\|h\|^2}:h\in\mathfrak{H}\right\}.
\end{aligned} \tag{8.19}$$

Hence, (8.18) and (8.19) show that

$$r((I-P_{\mathfrak{N}})P_{\mathfrak{M}})^2 = 1 - c_0(\mathfrak{N}, \mathfrak{M} \cap (\mathfrak{M} \cap \mathfrak{N})^{\perp})^2. \tag{8.20}$$

This identity (8.20) together with (8.11) and the symmetry of c_0 lead to (8.13).

Since the minimum modulus is invariant under taking adjoints, it follows that

$$r((I-P_{\mathfrak{N}})P_{\mathfrak{M}})) = r(P_{\mathfrak{M}}(I-P_{\mathfrak{N}})). \tag{8.21}$$

The identity (8.21), Theorem 8.6, and the symmetry property of $c(\mathfrak{M}, \mathfrak{N})$ lead to

$$c(\mathfrak{M}, \mathfrak{N}) = c(\mathfrak{N}^{\perp}, \mathfrak{M}^{\perp}) = c(\mathfrak{M}^{\perp}, \mathfrak{N}^{\perp}),$$

in other words (8.14) has been shown. □

The next result is a direct consequence of Theorem 8.6, when it is combined with the characterization in Proposition 8.5.

THEOREM 8.7. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be closed linear subspaces of a Hilbert space $\mathfrak{H}$. Then the following statements are equivalent:*

(i) $\mathfrak{M} + \mathfrak{N}$ *is closed;*

(ii) $\mathfrak{M}^{\perp} + \mathfrak{N}^{\perp}$ *is closed.*

Morover, the following statements are equivalent:

(iii) $\mathfrak{M} + \mathfrak{N}$ *is closed and* $\mathfrak{M} \cap \mathfrak{N} = \{0\}$*;*

(iv) $\mathfrak{M}^{\perp} + \mathfrak{N}^{\perp} = \mathfrak{H}$.

In particular, the following statements are equivalent:

(v) $\mathfrak{M} + \mathfrak{N} = \mathfrak{H}$ *and* $\mathfrak{M} \cap \mathfrak{N} = \{0\}$*;*

(vi) $\mathfrak{M}^{\perp} + \mathfrak{N}^{\perp} = \mathfrak{H}$ *and* $\mathfrak{M}^{\perp} \cap \mathfrak{N}^{\perp} = \{0\}$.

Let $\mathfrak{M}$ and $\mathfrak{N}$ be closed linear subspaces of a Hilbert space $\mathfrak{H}$. The *gap* $g(\mathfrak{M}, \mathfrak{N})$ between $\mathfrak{M}$ and $\mathfrak{N}$ is defined as (2.32), where $P_{\mathfrak{M}}$ and $P_{\mathfrak{N}}$ are the orthogonal projections onto $\mathfrak{M}$ and $\mathfrak{N}$, respectively. The identity

$$P_{\mathfrak{M}} - P_{\mathfrak{N}} = P_{\mathfrak{M}}(I-P_{\mathfrak{N}}) - (I-P_{\mathfrak{M}})P_{\mathfrak{N}}$$

shows that $g(\mathfrak{M}, \mathfrak{N}) \leqslant 1$.

PROPOSITION 8.8. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be closed linear subspaces in $\mathfrak{H}$. Then*

$$\max(c_0(\mathfrak{M}, \mathfrak{N}), c_0(\mathfrak{M}^{\perp}, \mathfrak{N}^{\perp})) = g(\mathfrak{M}, \mathfrak{N}^{\perp}). \tag{8.22}$$

In particular, if $c_0(\mathfrak{M}, \mathfrak{N}) = c_0(\mathfrak{M}^{\perp}, \mathfrak{N}^{\perp})$*, then* $c_0(\mathfrak{M}, \mathfrak{N}) = g(\mathfrak{M}, \mathfrak{N}^{\perp})$.

COROLLARY 8.9. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be closed linear subspaces in $\mathfrak{H}$. Then*

$$\begin{aligned} c(\mathfrak{M},\mathfrak{N}) &\leqslant \min(c_0(\mathfrak{M},\mathfrak{N}), c_0(\mathfrak{M}^\perp,\mathfrak{N}^\perp) \\ &\leqslant \max(c_0(\mathfrak{M},\mathfrak{N}), c_0(\mathfrak{M}^\perp,\mathfrak{N}^\perp)) = g(\mathfrak{M},\mathfrak{N}^\perp). \end{aligned} \tag{8.23}$$

Moreover, if $\mathfrak{M}\cap\mathfrak{N}=\{0\}$ and $\mathfrak{M}^\perp\cap\mathfrak{N}^\perp=\{0\}$, then

$$c(\mathfrak{M},\mathfrak{N}) = c_0(\mathfrak{M},\mathfrak{N}) = c_0(\mathfrak{M}^\perp,\mathfrak{N}^\perp) = g(\mathfrak{M},\mathfrak{N}^\perp). \tag{8.24}$$

THEOREM 8.10. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be closed linear subspaces of a Hilbert space $\mathfrak{H}$. Then the following statements are equivalent:*

(i) $g(\mathfrak{M},\mathfrak{N}^\perp) < 1$*;*

(ii) $\mathfrak{M}+\mathfrak{N}=\mathfrak{H}$ *and* $\mathfrak{M}\cap\mathfrak{N}=\{0\}$.

If either of these equivalent conditions holds, then the chain of equalities in (8.24) *is satisfied.*

COROLLARY 8.11. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be closed linear subspaces of a Hilbert space $\mathfrak{H}$ such that $g(\mathfrak{M},\mathfrak{N}^\perp)<1$, or equivalently, $\mathfrak{H}=\mathfrak{M}+\mathfrak{N}$ and $\mathfrak{M}\cap\mathfrak{N}=\{0\}$. Then*

$$g(\mathfrak{M},\mathfrak{N}^\perp) = \sqrt{1-\frac{1}{\|P\|^2}} \quad (= c(\mathfrak{M},\mathfrak{N}) = c_0(\mathfrak{M},\mathfrak{N}) = c_0(\mathfrak{M}^\perp,\mathfrak{N}^\perp)), \tag{8.25}$$

where P is the projection onto $\mathfrak{M}$, parallel to $\mathfrak{N}$.

Proof. Observe that the condition $\mathfrak{M}\cap\mathfrak{N}=\{0\}$ implies that

$$\begin{aligned} r((I-P_{\mathfrak{N}})P_{\mathfrak{M}}) &= \inf\left\{ \frac{\|(I-P_{\mathfrak{N}})f\|}{\|f\|} : f\in\mathfrak{M} \right\} \\ &= \left(\sup\left\{ \frac{\|f\|}{\|(I-P_{\mathfrak{N}})f\|} : f\in\mathfrak{M} \right\} \right)^{-1}. \end{aligned} \tag{8.26}$$

Hence (8.25) follows from Theorem 8.6, Theorem 8.10, and (8.26), once the following identity has been established:

$$\|P\| = \sup\left\{ \frac{\|f\|}{\|(I-P_{\mathfrak{N}})f\|} : f\in\mathfrak{M} \right\}. \tag{8.27}$$

In order to show (8.27), note that

$$\|P\| = \sup\left\{ \frac{\|f\|}{\|f+g\|} : f\in\mathfrak{M},\, g\in\mathfrak{N} \right\}.$$

The decomposition $f+g=(I-P_{\mathfrak{N}})f+h$ with $h=P_{\mathfrak{N}}f+g$ belonging to $\mathfrak{N}$ gives

$$\|f+g\|^2 = \|(I-P_{\mathfrak{N}})f\|^2 + \|P_{\mathfrak{N}}f+g\|^2, \quad f\in\mathfrak{M}, \quad g\in\mathfrak{N},$$

and it follows that

$$\|P\| = \sup\left\{ \frac{\|f\|}{\sqrt{\|(I-P_{\mathfrak{N}})f\|^2+\|h\|^2}} : f \in \mathfrak{M},\, h \in \mathfrak{N} \right\}.$$

This representation clearly implies (8.27). □

The notion of opening between closed linear subspaces of a Hilbert space is due to various people. The opening $c_0(\mathfrak{M},\mathfrak{N})$ has been introduced by J. Dixmier [3], whereas the opening $c(\mathfrak{M},\mathfrak{N})$ has been introduced by K. Friedrichs [4]. For related treatments, see [2] and [10]; note that in [10] the notations

$$\varepsilon(\mathfrak{M},\mathfrak{N}) = c(\mathfrak{M},\mathfrak{N}^{\perp}) \quad \text{and} \quad \delta(\mathfrak{M},\mathfrak{N}) = c_0(\mathfrak{M},\mathfrak{N}^{\perp})$$

have been used.

The results in Propositions 8.4 and 8.5 go back to J.-Ph. Labrousse [10] and to F. Deutsch [2, Theorem 12]. Theorem 8.6 goes back to Labrousse [10]. According to [2] the identity (8.14) was originally found by D.C. Salmon [13]; a different proof of it was provided in [2]. Note that a similar result does not hold for the opening $c_0(\mathfrak{M},\mathfrak{N})$. Theorem 8.7 can be found, for instance, in [8].

Proposition 8.8 has a long history; see [1] and [10]. The result in Corollary 8.11 goes back to V.E. Lyantse [12]. In this particular case the identity $c_0(\mathfrak{M},\mathfrak{N}) = c_0(\mathfrak{M}^{\perp},\mathfrak{N}^{\perp})\,(<1)$ goes back to M.G. Kreĭn, M.A. Krasnoselskiĭ, and D.P. Milman [9]; for a different proof see [2], [6].

REFERENCES

[1] N. I. ACHIESER AND I. M. GLASMANN, *Theorie der linearen Operatoren im Hilbert-Raum*, 8th edition, Akademie-Verlag, Berlin, 1981.

[2] F. DEUTSCH, *The angle between subspaces of a Hilbert space*, in S. P. Singh (editor) Approximation Theory, Wavelets and Applications, Kluwer Academic Publishers, 1995, 107–130.

[3] J. DIXMIER, *Études sur les varietés et les opérateurs de Julia avec quelques applications*, Bull. Soc. Math. France **77** (1949), 11–101.

[4] K. FRIEDRICHS, *On certain inequalities and characteristic value problems for analytic functions and for functions of two variables*, Trans. Amer. Math. Soc. **41** (1937), 321–364.

[5] S. GOLDBERG, *Unbounded linear operators*, Mc Graw Hill, New York, 1966.

[6] I. C. GOHBERG AND M. G. KREĬN, *Introduction to the theory of linear non-selfadjoint operators*, Transl. Math. Monographs, Vol. 18, Amer. Math. Soc., 1969.

[7] S. HASSI, H. S. V. DE SNOO, AND F. H. SZAFRANIEC, *Componentwise and canonical decompositions of linear relations*, Dissertationes Mathematicae, **465**, 2009 (59 pages).

[8] T. KATO, *Perturbation theory for linear operators*, Springer-Verlag, Berlin, 1980.

[9] M. G. KREĬN, M. A. KRASNOSELSKIĬ, AND D. P. MILMAN, *On the defect numbers of linear operators in a Banach space and on some geometric questions*, Sb. Trud. Inst. Mat. Akad. Nauk Ukr. SSR **11** (1948), 97–112.

[10] J.-PH. LABROUSSE, *Les opérateurs quasi Fredholm: une généralisation des opérateurs semi Fredholm*, Rend. Circ. Mat. Palermo **29** (1980), 161–258.

[11] J.-PH. LABROUSSE, A. SANDOVICI, H. S. V. DE SNOO, AND H. WINKLER, *The Kato decomposition of quasi-Fredholm relations*, Operators and Matrices **4** (2010), 1–51.

[12] V. E. LYANTSE, *Some properties of idempotent operators*, Teor. i Prikl. Mat. L'vov **1** (1959), 16–22 (Russian).

[13] D. C. SALMON, *Mathematical aspects of computed tomography*, Handwritten lecture notes, 1976–1977.

[14] A. SANDOVICI, H. S. V. DE SNOO, AND H. WINKLER, *Ascent, descent, nullity, and defect for linear relations in linear spaces*, Lin. Alg. Appl. **423** (2007), 456–497.

[15] M. H. STONE, *Linear transformations in Hilbert space and their applications to analysis*, American Mathematical Society Colloquium Publications, 15, Providence, RI, 1932.

(Received January 19, 2011)

J.-Ph. Labrousse
Département de Mathematiques
Université de Nice
Laboratoire J. A. Dieudonné
UMR du CNRS 6621 Parc Valrose
06108 Nice Cedex 02
France
e-mail: labro@math.unice.fr

A. Sandovici
Department of Sciences
University "Al. I. Cuza"
Lascăr Catargi 54, 700107, Iaşi
Romania
e-mail: adrian.sandovici@uaic.ro

H.S.V. de Snoo
Johann Bernoulli Institute for Mathematics and Computer Science
University of Groningen
P.O. Box 407, 9700 AK Groningen
Nederland
e-mail: desnoo@math.rug.nl

H. Winkler
Institut für Mathematik
Technische Universität Ilmenau
Curiebau, Weimarer Str. 25, 98693 Ilmenau
Germany
e-mail: henrik.winkler@tu-ilmenau.de