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BISEPARATING MAPS BETWEEN SMOOTH VECTOR–VALUED FUNCTIONS ON BANACH MANIFOLDS

CHING-JOU LIAO AND YA-SHU WANG

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Abstract. An $\mathscr{S}$-category consists all Banach manifolds as objects and subclasses of continuous functions (with some kind of smoothness) as morphisms. This notion covers, for example, the categories C^∞, C^n, C, and Lip_{loc} of all smooth functions, C^n-functions, continuous functions, and local Lipschitz functions. It is shown by Garrido, Jaramillo and Prieto in 2000 that two C^∞-smooth Banach manifolds X and Y are C^∞-diffeomorphic to each other if and only if there is an algebra isomorphism from $C^\infty(X,\mathbb{R})$ onto $C^\infty(Y,\mathbb{R})$. We extend this result to general abstract $\mathscr{S}$-categories, and from algebra isomorphisms of scalar functions to the maps which are linear, bijective and separating, between vector-valued functions.

1. Introduction

It is a classical result that the algebra structure of the algebra $C(X)$ of continuous functions determines the topological structure of a completely regular space X. More precisely, if there exists a ring isomorphism $T: C(X) \to C(Y)$ then the realcompactifications of X and Y are homeomorphic [15, pp. 115–118]. If we consider the algebra $C^\infty(X)$ of smooth functions on a smooth Banach manifold X, we know that the algebra structure determines even the smooth structure of X. Indeed, Garrido, Jaramillo and Prieto [13] showed that C^∞-smooth Banach manifolds X and Y are C^∞-diffeomorphic if and only if there is an algebra isomorphism from $C^\infty(X,\mathbb{R})$ onto $C^\infty(Y,\mathbb{R})$.

In the vector-valued case, although there is no multiplicative structure equipping $C(X,E)$ when X is a topological space and E is a general Banach space, we can still consider its disjointness structure. We say that f,g in $C(X,E)$ are *disjoint*, denoted by $fg = 0$, if they have disjoint cozero sets, that is, $\|f(x)\|\|g(x)\| = 0, \forall x \in X$. A linear map T between spaces of vector-valued functions is said to be *disjointness preserving* or *separating* if

$$fg = 0 \quad \Longrightarrow \quad TfTg = 0.$$

T is *biseparating* if it is bijective and both T and T^{-1} are separating.

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In [4], Araujo showed that: Assume $X \subset \mathbb{R}^p$ and $Y \subset \mathbb{R}^q$ are open subsets, and E and F are Banach spaces. If $T : C^n(X,E) \to C^m(Y,F)$ is a linear biseparating map, then $p = q$, $n = m$, and

$$Tf(y) = h(y)(f(\varphi(y))) \quad \forall f \in C^n(X,E),\ \forall y \in Y,$$

where $\varphi : Y \to X$ is a diffeomorphism of class C^n and $h(y) : E \to F$ is a linear bijection for all y in Y.

To set up a general framework for smoothness, Bonic and Frampton [8] gave an abstract theory, called $\mathscr{S}$-category, consisting of all Banach manifolds X, Y as objects and subclasses $S(X,Y)$ of continuous functions as morphisms. This notion covers as morphisms in the category Lip_{loc} of local Lipschitz functions, the category C^n of C^n-functions, and the category D^n_α of C^n-functions with Hölder continuous n-derivatives of order α, where $n \in \mathbb{N} \cup \{\infty\}$ and $0 < \alpha < 1$. See Section 2 for more details.

In this paper, S_1 and S_2 denote any $\mathscr{S}$-categories, G_1 is an S_1-smooth Banach space, and G_2 is an S_2-smooth Banach space. Suppose X is a separable S_1-smooth G_1-manifold and Y is a separable S_2-smooth G_2-manifold, and E and F are general Banach spaces.

We will show in Section 3 that every algebra isomorphism $T : S_1(X,\mathbb{R}) \to S_2(Y,\mathbb{R})$ induces a homeomorphism $\varphi : Y \to X$ such that $Tf = f \circ \varphi$ for all f in $S_1(X)$. With a mild continuity assumption, we will also see that a similar conclusion holds for biseparating linear maps from $S_1(X,E)$ onto $S_2(Y,F)$. As shown in Theorem 3.7 below, we will even see that $X \cong Y$ as smooth Banach manifolds in many interesting cases.

Disjointness preserving linear maps between function spaces or vector-valued function spaces are well-studied (see, e.g., [7, 1, 18, 9, 11, 17, 19, 25, 14]). In particular, Araujo and Jarosz ([2, 3, 4, 6]) investigated separating maps between spaces of vector-valued uniformly continuous functions on complete metric spaces, and spaces of vector-valued differentiable functions on open subsets of $\mathbb{R}^n$. Dubarbie [10] studied these maps between spaces of vector-valued absolutely continuous functions on compact subsets of the real line. Moreover, Leung, Araujo and Dubarbie ([23, 5]) worked on these maps between spaces of generalized Lipschitz vector-valued functions which include Lipschitz, little Lipschitz and local Lipschitz functions. In this paper, we work in a general framework, i.e., $\mathscr{S}$-categories, which include all function spaces mentioned above.

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2. $\mathscr{S}$-categories and S-smooth manifolds

Denote by $C^1(U,V)$ the family of all Frechet differentiable maps between Banach manifolds U,V with continuous derivatives. Similarly, we can define the notions of $C^k(U,V)$ for $k = 1,2,\dots,\infty$.

DEFINITION 2.1. ([8]) An $\mathscr{S}$-category is a category S whose objects consist of all open subsets of Banach spaces. For any pair of objects U and V, the morphism set $S(U,V)$ consists of continuous functions from U into V, and the product of two morphisms is taken as the usual composition. We also require that $C^\infty(U,V) \subseteq S(U,V)$ and that the following conditions are satisfied:

(S1) $f \in S(U,W)$ whenever $f \in S(U,V)$ and W is an open subset of V containing $f(U)$.

(S2) $f \in S(U,V)$ whenever $f \in C(U,V)$ and for each x in U there is an open set W with $x \in W \subseteq U$ such that $f|_W \in S(W,V)$.

(S3) If $f_1 \in S(U_1, V_1)$ and $f_2 \in S(U_2, V_2)$, then $f_1 \times f_2 \in S(U_1 \times U_2, V_1 \times V_2)$.

If V is a Banach space, it follows from (S3) that $S(U,V)$ is a vector space, and indeed, an $S(U)$-module. Moreover, $S(U,V)$ is an algebra if V is a Banach algebra. In general, we have $f+g$ and fg being S-smooth whenever the algebraic operations make sense. A morphism f in $S(U,V)$ will be called an S-*smooth function* or a function of class S. We write $S(U)$ for $S(U,V)$ if V is the scalar field, i.e., $\mathbb{R}$ or $\mathbb{C}$.

A Banach space G is said to be S-*smooth* if there is a nonzero S-smooth function with bounded support. It amounts to say that for any open neighborhood V of any point x in G, there is an f in $S(G)$ such that

$$f(x) \neq 0 \qquad \text{and} \qquad \operatorname{supp}(f) = \overline{\{x \in G : f(x) \neq 0\}} \subset V.$$

In other words, G is an S-smooth Banach space if and only if the norm topology on G is equivalent to the projective topology $\sigma(G, S(G))$, i.e., the weakest topology on G in which all S-smooth functions in $S(G)$ are continuous.

Let X be a Hausdorff space. A pair (U, φ) is called a *chart* if U is open in X, the image set $\varphi(U)$ is open in a Banach space E_φ, and $\varphi : U \to \varphi(U)$ is a homeomorphism. X is called a *manifold of class* S if there is a collection of charts $\{(U_\alpha, \varphi_\alpha)\}$ such that $\{U_\alpha\}$ is a covering of X, and $\varphi_\alpha \circ \varphi_\beta^{-1} \in S(\varphi_\beta(U_\alpha \cap U_\beta), \varphi_\alpha(U_\alpha \cap U_\beta))$ for all α, β. If all Banach spaces E_{φ_α} are equivalent to a single Banach space E, we call X an E-*manifold of class* S. If in addition E is S-smooth, X will be called a *smooth* E-*manifold of class* S, or simply an S-*smooth* E-*manifold*.

The notion of S-category can be extended to the ones including as objects of all Banach manifolds of class S. Given manifolds X and Y of class S, the morphism set is defined to be

$$S(X,Y) = \{f \in C(X,Y) : \psi \circ f \circ \varphi^{-1} \in S(\varphi(U), \psi(V)) \\ \text{for every charts } (U, \varphi) \text{ of } X \text{ and } (V, \psi) \text{ of } Y \text{ with } f(U) \subseteq V\}$$

It is easy to check that this enlarged version of $\mathscr{S}$-category satisfies the conditions stated in Definition 2.1.

Suppose X is a manifold of class S. A family $\{\varphi_\alpha \in S(X)\}$ of non-negative functions is said to be an *S-partition of unity* if every point in X has a neighborhood on which all but a finite number of φ_α vanish, and $\sum \varphi_\alpha(x) = 1$ for all x in X. We say that X *admits S-partitions of unity* if for every open covering $\{V_\beta\}$ of X there is an S-partition of unity $\{\varphi_\alpha\}$ in which each φ_α is supported in some V_β.

THEOREM 2.2. ([8]) *Let X be a separable smooth E-manifold of class S. Then X admits S-partitions of unity. Consequently, if A and B are disjoint closed sets in X, then there is an $0 \leqslant f \leqslant 1$ in $S(X)$ such that $f = 0$ in some neighborhood of A and $f = 1$ in some neighborhood of B.*

3. Recovering smooth structures of Banach manifolds from smooth vector-valued functions

THEOREM 3.1. *Let X,Y be separable smooth G_1,G_2-manifolds of class S_1,S_2, respectively. If $T : S_1(X,\mathbb{R}) \to S_2(Y,\mathbb{R})$ is an algebra isomorphism, there is an homeomorphism $\varphi : Y \to X$ such that $Tf = f \circ \varphi$ for all f in $S_1(X,\mathbb{R})$.*

Proof. Note that X is a realcompact space, and $S(X,\mathbb{R}) \subset C(X,\mathbb{R})$ is a uniformly dense ([8, Theorem 2]), and a unital inverse-closed subalgebra of $C(X,\mathbb{R})$ ([22, pp. 153]), i.e., $\frac{1}{f} \in S(X,\mathbb{R})$ whenever f in $S(X,\mathbb{R})$ is non-vanishing on X. By [12, Theorem 21 and Corollary 24], there is a homeomorphism $\varphi : Y \to X$ such that $Tf = f \circ \varphi, \forall f \in S_1(X,\mathbb{R})$. $\square$

To extend Theorem 3.1 to the case of vector-valued functions with the disjointness structure instead of the multiplicative structure, we introduce the notations

$$I_x = \{f \in S_1(X,E) : x \notin \operatorname{supp}(f)\} \quad \text{and} \quad M_x = \{f \in S_1(X,E) : f(x) = 0\}.$$

A sequence $\{f_n\}$ is said to be *locally uniformly convergent* to an f in $S(X,E)$ if for each x in X there is a neighborhood of x in which f_n converges uniformly to f. A topology on $S(X,E)$ is said to be *locally determined* if every locally uniformly convergent sequence converges. For example, the usual topologies of $C(X,E)$ and $Lip_{loc}(X,E)$ are locally determined.

LEMMA 3.2. *Let X be a separable smooth E-manifold of class S and $x' \in X$. For all g in $M_{x'}$, there is a sequence $\{g_n\}$ in $I_{x'}$ locally uniformly converging to g. In particular, in every locally determined topology of $S(X,E)$, we have*

$$\overline{I_{x'}} = M_{x'}, \quad \forall x' \in X.$$

Proof. Given f in $M_{x'}$. Let $U_n = \{x \in X : \|f(x)\|_E < \frac{1}{n}\}$ for all n in $\mathbb{N}$. By Theorem 2.2, for any n in $\mathbb{N}$, there exists an f_n in $S_1(X,E)$ with $\|f_n(x)\|_E \leqslant 1$ for all

x in X such that $f_n|_{X\setminus U_n} = 0$ and $f_n|_{U_{2n}} = 1$. For any x in X, if $\|f(x)\| \neq 0$, choose an integer N such that $N > 2/\|f(x)\|$. Let $U = \{u \in X : \|f(u)\| > \|f(x)\|/2\}$. Then $U \cap U_n = \emptyset$ whenever $n > N$. Hence, $f_n f = 0$ on U for all $n > N$. If $f(x) = 0$, we set $U = \{u \in X : \|f(u)\| < 1\}$. Then $\|f_n f\| < 1/n$ on U for all $n = 1, 2, \ldots$. In both cases, we have $(1 - f_n)f$ converging uniformly to f on a neighborhood U of each x in X. □

Let $T : S_1(X,E) \to S_2(Y,F)$ be a linear map. We say that T is *locally uniform-pointwisely continuous* if T sends every sequence $\{f_n\}$ locally uniformly converging to f in $S_1(X,E)$ to a sequence $\{Tf_n\}$ pointwisely convergent to Tf in $S_2(Y,F)$, i.e., $Tf(y) = \lim_n Tf_n(y), \forall y \in Y$. This is the case, for example, if T is continuous when $S_1(X,E)$ is equipped with a locally determined topology and $S_2(Y,F)$ with the topology of pointwise convergence.

In the following, we write $B^{-1}(E,F)$ for the set of all invertible bounded linear operator from E onto F equipped with the strong operator topology.

THEOREM 3.3. *Let X,Y be separable smooth S_1,S_2-manifolds, and E,F be Banach spaces, respectively. Suppose $T : S_1(X,E) \to S_2(Y,F)$ is a bijective linear map such that both T, T^{-1} are separating and locally uniform-pointwisely continuous. Then T is a weighted composition operator of the form*

$$Tf(y) = h(y)(f(\varphi(y))), \quad \forall f \in S_1(X,E), \forall y \in Y.$$

Here, $\varphi : Y \to X$ is a homeomorphism, and $h : Y \to (B^{-1}(E,F), SOT)$ is a continuous map.

Proof. We divide the proof into several claims. For each x in X and y in Y, define

$$S_y = \{x \in X : TI_x \subseteq M_y\}.$$

CLAIM 1. *$S_y \neq \emptyset$ for each y in Y.*

Suppose on the contrary that for each x in X, there exists an f_x in I_x such that $Tf_x(y) \neq 0$. Let U_x be an open neighborhood of x on which $f_x = 0$. By the separability of X, we have $X = \bigcup_{n=1}^{\infty} U_{x_n}$ for at most countably many points x_n in X. By Theorem 2.2, we can assume there is an S_1-partition of unity $\{h_n : n \in \mathbb{N}\}$ such that $\operatorname{supp}(h_n) \subseteq U_{x_n}$ for all n in $\mathbb{N}$.

For any f in $S_1(X,E)$ and each n in $\mathbb{N}$, observe that

$$(fh_n)f_{x_n} = 0 \quad \Longrightarrow \quad T(fh_n)Tf_{x_n} = 0 \quad \Longrightarrow \quad T(fh_n)(y) = 0.$$

Since $\sum_{n=1}^{\infty} fh_n$ locally uniformly converges to f, we have $Tf(y) = 0$. This contradicts to the surjectivity of T and Theorem 2.2.

CLAIM 2. *S_y consists of exactly one point for all y in Y.*

Assume $x_1, x_2 \in S_y$. If $x_1 \neq x_2$, by Theorem 2.2, there exists an f in $S_1(X)$ such that $f(x) = 0$ in a neighborhood of x_1 and $f(x) = 1$ in a neighborhood of x_2. For any g in $S_1(X,E)$, we have

$$g = fg + (1-f)g, \quad fg \in I_{x_1}, \quad \text{and} \quad (1-f)g \in I_{x_2}.$$

Therefore $Tg(y) = 0$ for all g in $S_1(X,E)$ by the definition of S_y, a contradiction.

By Claim 2, we can define a map $\varphi : Y \to X$ by $S_y = \{\varphi(y)\}$.

CLAIM 3. *$\varphi : Y \to X$ is continuous.*

Suppose on the contrary that there exists a net $\{y_\lambda\}$ in Y converging to y in Y, but $\{\varphi(y_\lambda)\}$ does not converge to $\varphi(y)$. Passing to a subnet, we can assume by the complete regularity of X that there exists an open set V not containing $\varphi(y)$, but $\varphi(y_\lambda) \in V$ for all λ. Let f be in $S_1(X,E)$ with $f|_V = 0$. Then, $f|_V = 0$ implies $f \in I_{\varphi(y_\lambda)}$, and hence $Tf \in M_{y_\lambda}$. By the continuity of Tf and that $Tf(y_\lambda) = 0$ for all λ, we conclude $Tf(y) = 0$. By Theorem 2.2, there is an k in $S_1(X)$ such that $k(x) = 1$ in a neighborhood of $\varphi(y)$ and $k|_V = 0$. For each f in $S_1(X,E)$, we have $f = kf + (1-k)f$. Thus $Tf(y) = 0$ since $(kf)|_V = 0$ and $(1-k)f \in I_{\varphi(y)}$. This conflicts with Theorem 2.2 and the surjectivity of T.

By Claim 3, there is a continuous map $\varphi : Y \to X$ such that $TI_{\varphi(y)} \subseteq M_y$ for all y in Y. It follows from the locally uniform-pointwise continuity of T and Lemma 3.2 that $TM_{\varphi(y)} \subseteq M_y$ for all y in Y. In other words, $\ker(\delta_{\varphi(y)}) \subseteq \ker(\delta_y \circ T), \forall y \in Y$. Hence for each y in Y, there exists a linear map $h(y) : E \to F$ such that $\delta_y \circ T = h(y)\delta_{\varphi(y)}$. In other words,

$$Tf(y) = h(y)(f(\varphi(y))), \quad \forall y \in Y, \forall f \in S_1(X,E). \tag{3.1}$$

CLAIM 4. *$h(y)$ is bounded for all y in Y. Moreover, $h : Y \to (B(E,F), SOT)$ is continuous.*

Let $\{e_n\}$ be a sequence in E converging to e in norm. By Theorem 2.2, we can choose a function g from $S(X)$ such that $g(\varphi(y)) = 1$. Note that as a continuous function, g is locally bounded on X. Then ge_n in $S_1(X,E)$ converges locally uniformly to ge in $S_1(X,E)$, and

$$h(y)(e_n) = T(ge_n)(y) \longrightarrow T(ge)(y) = h(y)(e)$$

since T is locally uniform-pointwisely continuous. Therefore, $h(y)$ is continuous for all y in Y.

For every e in E, set $\widetilde{e}(x) := e$ constantly on X. By the continuity of $T\widetilde{e}$, the map $y \longmapsto h(y)e$ from Y into F is continuous. That is, h is continuous in the strong operator topology.

CLAIM 5. *$\varphi : Y \to X$ is a homeomorphism, and $h(y)$ is invertible for all y in Y.*

Applying the above arguments to T^{-1}, we shall obtain a continuous map $k : X \to (B(F,E), SOT)$ and a continuous map $\mu : Y \to X$ such that

$$T^{-1}g(x) = k(x)g(\mu(x)), \quad \forall x \in X.$$

It is then easy to see that $\varphi = \mu^{-1}$ is a homeomorphism from Y onto X, and $h(y)^{-1} = k(\varphi(x)), \forall y \in Y$. $\square$

Let $f : U \to V$ be a function from an open set U of a Banach space E into an open set V of a Banach space F. We call f a *weak $S(U,V)$-morphism* if $v^* \circ f \in S(E)$ for all v^* in the Banach dual space F^* of F.

DEFINITION 3.4. A component $S(U,V)$ of an $\mathscr{S}$-category S is called weakly determined if every weak $S(U,V)$-morphism is an $S(U,V)$-morphism.

In [16, Example 3.9], Gutiérrez and Llavona show that not all weak $S(U,V)$-morphisms are $S(U,V)$-morphisms. However, they also show that $C^\infty(U,V)$ is weakly determined for any Banach spaces E and F, and $C^n(U,V)$ is also weakly determined when F is a reflexive C^n-smooth Banach space for $1 \leqslant n < \infty$.

LEMMA 3.5. *For any $\mathscr{S}$-category S and finite dimensional Banach space E, the morphism set $S(X,E)$ is weakly determined.*

Proof. Let $\{e_1, \ldots, e_n\}$ be a Hamel basis for E. Write any weak $S(X,E)$-morphism $f : X \to E$ as

$$f(x) = \sum_{i=1}^{n} f_i(x)e_i,$$

where $f_1, \ldots, f_n$ are the coordinate functions, which are all in $S(E)$. Since $S(X,E)$ is an $S(X)$-module, f is in $S(X,E)$. $\square$

Recall that a map $f : X \to Y$ between metric spaces is called *locally Lipschitz* if at each point of X, there is a neighborhood on which f is Lipschitz. Scanlon [26, Theorem 2.1] shows that $f : X \to Y$ is locally Lipschitz if and only if f is Lipschitz on each compact subset of X.

LEMMA 3.6. *Let X and E be open subsets in Banach spaces. Then the local Lipschitz function space $Lip_{loc}(X,E)$ is weakly determined.*

Proof. Without loss of generality, we can assume E is a Banach space. Let $f : X \to E$ with $\psi \circ f$ in $Lip_{loc}(X,\mathbb{R})$ for all ψ in E^*. Suppose f is not in $Lip_{loc}(X,E)$. So there exist a nonempty compact subset K of X and sequences $\{x_n\}$ and $\{y_n\}$ in K such that

$$\|f(x_n) - f(y_n)\| \geqslant n\|x_n - y_n\|, \quad \forall n = 1, 2, \ldots.$$

For every n in $\mathbb{N}$, define $T_n : E^* \to \mathbb{R}$ by

$$T_n(\psi) = \psi\left(\frac{f(x_n) - f(y_n)}{\|x_n - y_n\|}\right).$$

Hence $\|T_n(\psi)\| \leqslant L_\psi$ for some constant L_ψ and for all n in $\mathbb{N}$ since $\psi \circ f \in Lip_{loc}(X,\mathbb{R})$. By the Principle of Uniform Boundedness, there is a constant L such that $\|T_n\| \leqslant L$. Therefore,

$$n \leqslant \frac{\|f(x_n) - f(y_n)\|}{\|x_n - y_n\|} \leqslant L, \quad n = 1, 2, \ldots.$$

This is a contradiction. □

THEOREM 3.7. *Let X, Y be separable smooth G_1, G_2-manifolds of class S, respectively. Assume there is any linear biseparating map $T : S(X,E) \to S(Y,F)$, which is locally uniform-pointwisely continuous in two directions. Then*

$$Tf(y) = h(y)(f(\varphi(y))), \quad \forall f \in S(X,E), \forall y \in Y.$$

Here, $\varphi : Y \to X$ is a homeomorphism and $h : Y \to (B^{-1}(E,F), SOT)$ is a continuous map.

(1) *If $S(G_2, G_1)$ is weakly determined, φ is in $S(Y,X)$.*

(2) *If $S(G_1, G_2)$ is weakly determined, φ^{-1} is in $S(X,Y)$.*

(3) *If both $S(G_1, G_2)$ and $S(G_2, G_1)$ are weakly determined, $X \cong Y$ as S-smooth manifolds.*

Proof. The first part of the assertions follows from Theorem 3.3.

Let $y_0 \in Y$, and consider $\phi : U \to G_2$ and $\psi : V \to G_1$, the S-charts around y_0 and $\varphi(y_0)$, respectively. We can assume $\varphi(U) = V$. Since G_1 is an S-smooth Banach space and φ is a homeomorphism, we can find open neighborhoods $U_0 \subseteq U$ and $V_0 \subseteq V$, of y_0 and $\varphi(y_0)$, respectively, and θ in $S(X)$ such that $\varphi(U_0) = V_0$, $\theta|_{V_0} = 1$ and $\operatorname{supp}(\theta) \subseteq V$. Given a continuous linear functional v^* in G_1^*, we define

$$g(x) = \begin{cases} \theta(x) v^*(\psi(x)), & x \in V; \\ 0, & x \in X \setminus V. \end{cases}$$

By (S2), $g \in S(X)$.

As in Claim 5 in the proof of Theorem 3.3,

$$T^{-1}f(x) = k(x)(f(\mu(x))), \quad \forall f \in S(Y,F), \forall x \in X.$$

Here $\mu = \varphi^{-1}$ and $k(\varphi(y)) = h(y)^{-1}$. Let $e \in F$ and define

$$\widetilde{f}(x) = g(x)k(x)e, \quad \forall x \in X.$$

We see that $\widetilde{f} = gT^{-1}(\widetilde{e}) \in S(X,E)$, since the constant function $\widetilde{e}(y) = e$ is in $S(Y,F)$. Then $(T\widetilde{f})(y) = g(\varphi(y))e \in S(Y,F)$. Let $e^* \in F^* \subset C^\infty(F,\mathbb{K}) \subset S(F)$ with $e^*(e) = 1$. Since $g(\varphi(y)) = e^*(g(\varphi(y))e)$, we have $g \circ \varphi \in S(Y)$. Thus $v^* \circ \psi \circ \varphi \circ \phi^{-1} \in S(\phi(U_0))$ for each v^* in G_1^*, and hence $\psi \circ \varphi \circ \phi^{-1} \in S(\phi(U_0), \psi(V_0))$ since $S(G_2, G_1)$ is weakly determined. Therefore, $\varphi \in S(Y,X)$. In a similar way, we can prove that $\varphi^{-1} \in S(X,Y)$, provided $S(G_1, G_2)$ is weakly determined. □

COROLLARY 3.8. *Suppose X,Y are separable Lipschitz smooth manifolds, and E,F are Banach spaces. If there exists a linear biseparating map, locally uniform-pointwisely continuous in two directions, from $Lip_{loc}(X,E)$ onto $Lip_{loc}(Y,F)$, then $X \cong Y$ as Lipschitz manifolds.*

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Ching-Jou Liao
Department of Applied Mathematics
National Sun Yet-Sen University
80424 Kaoshiung, Taiwan
e-mail: liaocj@math.nsysu.edu.tw

Ya-Shu Wang
Department of Mathematics
National Central University
32054 Chungli, Taiwan
e-mail: wangys@mx.math.ncu.edu.tw