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SPECTRAL PROPERTIES OF n–PERINORMAL OPERATORS

SALAH MECHERI AND NAIM L. BRAHA

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Abstract. In this paper we study spectral properties of class (M,n) or n-perinormal operators. It is shown that if T belongs to class (M,n), then its point spectrum and joint point spectrum are identical. As an application we show that the spectral mapping theorem holds for the essential approximate point spectrum and for Weyl spectrum. It is also shown that a-Browder's theorem holds for n-perinormal operators. A general version of the famous Fuglede-Putnam's theorem for n-perinormal operator is also presented.

1. Introduction

Let $B(H)$ be the algebra of all bounded linear operators acting on infinite dimensional separable complex Hilbert space H. An operator $T \in B(H)$ is said to be of class (M,n) or n-perinormal if $T^{*n}T^n - (T^*T)^n \geqslant 0$, for each $n \geqslant 2$ [5]. The operator T is said to be a p-hyponormal operator if and only if $(T^*T)^p \geqslant (TT^*)^p$ for a positive number p. In [17] is defined the class of log-hyponormal operators as follows: T is a log-hyponormal operator if it is invertible and satisfies the following relation $\log T^*T \geqslant \log TT^*$. Class of p-hyponormal operators and class of log-hyponormal operators were defined as extension class of hyponormal operators, i.e, $T^*T \geqslant TT^*$. An operator $T \in B(H)$ is said to be quasihyponormal if $T^*(T^*T - TT^*)T \geqslant 0$.

If $T \in B(H)$, write $N(T)$ and $\operatorname{ran}(T)$ for null space and range of T; $\alpha(T) = \dim N(T)$; $\beta(T) = \dim(T^*)$; $\sigma(T), \sigma_a(T), \sigma_p(T), \sigma_{jp}(T), \sigma_{ja}(T)$ for the spectrum of T, the approximate point spectrum of T, the point spectrum of T, the joint point spectrum, the joint approximate point spectrum of T, respectively.

A complex number $\lambda \in \mathbb{C}$ is said to be in the point spectrum $\sigma_p(T)$ of the operator T if there is a unit vector x satisfying $(T-\lambda)x = 0$. If in addition, $(T^* - \overline{\lambda})x = 0$, then λ is said to be in the joint point spectrum $\sigma_{jp}(T)$ of T. A complex number $\lambda \in \mathbb{C}$ is said to be in the approximate point spectrum $\sigma_a(T)$ of the operator T if there is a sequence $\{x_n\}$ of unit vectors satisfying $(T-\lambda)x_n \to 0$. If in addition, $(T^* - \overline{\lambda})x_n \to 0$, then λ is said to be in the joint approximate point spectrum $\sigma_{ja}(T)$ of T. The boundary $\partial\sigma(T)$ of the spectrum of T of the operator T is known to be

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a subset of $\sigma_a(T)$. Although, in general, one has $\sigma_{jp}(T) \subset \sigma_p(T), \sigma_{ja}(T) \subset \sigma_a(T)$. There are many classes of operators for which

$$\sigma_{jp}(T) = \sigma_p(T), \tag{1}$$

$$\sigma_{ja}(T) = \sigma_a(T). \tag{2}$$

For example , if T is either normal or hyponormal, then T satisfies (1) and (2). More generally, (1) and (2) holds if T is semi-hyponormal (cf. [19]), p-hyponormal (cf. [3]) or log-hyponormal (cf. [17]).

In this paper, we show that n-perinormal operators T satisfy (1). Let $T \in B(H)$. $N(T)$ denotes the null space of T and let $\alpha(T) = \dim N(T)$. ran T denotes the range of T and let

$$\beta(T) = \dim N(T^*) = \dim[\text{ran } T]^{\perp},$$

where $[\text{ran } T]$ denotes the closure of ran T. T is called semi-Fredholm if it has closed range and either $\alpha(T) < \infty$ or $\beta(T) < \infty$. T is called Fredholm if it is semi-Fredholm and both $\alpha(T) < \infty, \beta(T) < \infty$. T is called Weyl if it is Fredholm of index zero, i.e., $i(T) = \alpha(T) - \beta(T) = 0$. The Weyl spectrum of T is defined by

$$w(T) = \{\lambda \in \mathbb{C} \mid T - \lambda \text{ is not Weyl }\}.$$

$\pi_{00}(T)$ denotes the set of all eigenvalues of T such that λ is an isolated point of $\sigma(T)$ and $0 < \alpha(T - \lambda) < \infty$. We say that Weyl's theorem holds for T if

$$\sigma(T) \setminus \omega(T) = \pi_{00}(T).$$

T is said to have the single valued extension property if there exists no nonzero analytic function f such that $(T - z)f(z) \equiv 0$ and T is called isoloid if every isolated point of $\sigma(T)$ is an eigenvalue of T.

H. Weyl [18] studied the spectrum of compact perturbations of self-adjoint operators and proved that Weyl's theorem holds for self-adjoint operators. This result has been extended to hyponormal operators, p-hyponormal operators and to many other non-normal operators (cf. [11]) and references therein. In this paper it is shown that Weyl's theorem holds for n-perinormal operator $T \in B(H)$.

Let C_2 denote the Hilbert-Schmidt class. Let $T \in C_2$ and assume that $\{e_n\}$ is an orthonormal basis for H. We define the Hilbert-Schmidt norm to be

$$||T||_2 = (\sum_{n=1}^{\infty} ||Te_n||^2)^{\frac{1}{2}}.$$

If $||T||_2 < \infty$, then T is said to be a Hilbert-Schmidt operator.

THEOREM 1.1. (Fuglede) *Let X, N be linear bounded operators on a complex Hilbert space and assume that N is normal. If $NX = XN$, then $N^*X = XN^*$.*

Colloquially, the theorem claims that commutativity between operators is transitive under the given assumptions. The claim does not hold in general if N is not normal.

A simple counterexample is provided by letting N be the unilateral shift and $X = N$. Also, when X is self-adjoint, the claim is trivial regardless of whether N is normal: $XN^* = (NX)^* = (XN)^* = N^*X$. In the following theorem Putnam obtained Fuglede's result as a special case.

THEOREM 1.2. (Putnam) *Let M,N,X be linear bounded operators on a complex Hilbert space and assume that M,N are normal. If $MX = XN$, then $M^*X = XN^*$.*

This theorem was originally proved in [7] under the assumption that $M = N$. As stated, the theorem was proved in [14]. Another proof is given by Radjavi and Rosenthal (1973) [16]. In [1] the author observed that Putnam's version can be derived from Fuglede's original theorem by the following matrix trick. If

$$L = \begin{pmatrix} N & 0 \\ 0 & M \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}.$$

Then L is normal on $H \oplus H$ and $LX = XL$. Hence $L^*X = XL^*$, and this gives Putnam's version.

In the past several years, many authors have extended this theorem for several classes of nonnormal operators. In [1], the author has extended the result by assuming N, M^* are hyponormal and X is a Hilbert-Schmidt operator. [4] showed that the hyponormality in the result of [1] can be replaced by the quasihyponormality of N and M^* under some additional conditions. [12] showed that the result of [4] remains true without any additional condition. In this paper we will extended Fuglede-Putnam's theorem to the case in which T and S^* are n-perinormal and X a Hilbert-Schmidt operator.

Let $A, B \in B(H)$. The operator Γ defined on C_2 by $\Gamma X = AXB$ has been studied in [1]. It is easy to see that $||\Gamma|| \leqslant ||A||||B||$ and the adjoint of Γ is given by $\Gamma^* X = A^*XB^*$. Indeed,

$$\begin{aligned} \langle \Gamma^* X, Y\rangle = \langle X, \Gamma\rangle = \langle X, AYB\rangle &= tr((AYB)^*X) \\ &= tr(XB^*Y^*A^*) = tr(A^*XB^*Y^*) = \langle A^*XB^*, Y\rangle. \end{aligned}$$

If $A \geqslant 0$ and $B \geqslant 0$, then $\Gamma \geqslant 0$ and

$$\Gamma^{\frac{1}{2}} X = A^{\frac{1}{2}} X B^{\frac{1}{2}}.$$

Indeed,

$$\langle \Gamma X, X\rangle = tr(AXBX^*) = tr(A^{\frac{1}{2}}XBX^*A^{\frac{1}{2}}) = tr(A^{\frac{1}{2}}XB^{\frac{1}{2}}(A^{\frac{1}{2}}XB^{\frac{1}{2}}))^* \geqslant 0.$$

2. Fuglede-Putnams theorem

In the following lemmas we will present some properties of n-perinormal operators.

LEMMA 2.1. *If A, B^* are n- perinormal operators, then the operator Γ in C_2 defined by $\Gamma X = AXB$ is also n-perinormal operator.*

Proof. We have

$$(\Gamma^{n*}\Gamma^n - (\Gamma^*\Gamma)^n)X = (A^{n*}A^n - (A^*A)^n)XB^nB^{*n} + (A^*A)^nX(B^nB^{*n} - (BB^*)^n) \geqslant 0.$$

Thus $\Gamma^{n*}\Gamma^n \geqslant (\Gamma^*\Gamma)^n$. $\square$

Recall that

$$(T^*T)^\alpha = T^*(TT^*)^{\alpha-1}T, \text{ for all real number } \alpha \geqslant 0. \tag{3}$$

LEMMA 2.2. *If T is an invertible n-perinormal operator for each positive integer $n \geqslant 2$, then T^{-1} is also n-perinormal operator.*

Proof. We will prove the lemma by induction. For $n = 2$, $T^*|T|^2T \geqslant (T^*T)^2$ holds because T is quasihyponormal operator and that every invertible quasihyponormal operator is hyponormal. Suppose that the result holds for some $n \geqslant 2$, that is, $|T^{-n}|^2 \geqslant (T^{*-1}T^{-1})^n$. We have by (3)

$$\begin{aligned}|T^{-1}|^{2n+2} &= (T^{*-1}T^{-1})^{n+1} = T^{*-1}(T^{-1}T^{*-1})^nT^{-1}\\ &= T^{*-1}(T^*T)^{-n}T^{-1}\\ &\leqslant T^{*-1}(T^{*-n}T^{-n})T^{-1}.\end{aligned}$$

Thus

$$(T^{-1*})^{n+1}(T^{-1})^{n+1} \geqslant (T^{-1*}T^{-1})^{n+1}.$$

Hence T^{-1} is n-perinormal operator. $\square$

Before proving the following theorem we need a lemma.

LEMMA 2.3. *(Hölder-McCarthy Inequality). Let P be a positive operator. Then the following inequalities hold for all $x \in H$.*
(i) $\langle P^\alpha x, x\rangle \leqslant \langle Px, x\rangle^\alpha ||x||^{2(1-\alpha)}$, *for* $0 < \alpha \leqslant 1$.
(ii) $\langle P^\alpha x, x\rangle \geqslant \langle Px, x\rangle ||x||^{2(1-\alpha)}$ *for* $\alpha \geqslant 1$.

THEOREM 2.1. *Let $T \in B(H)$ be n-perinormal operator. If $(T-\lambda)x = 0, \lambda \neq 0$, then $(T-\lambda)^*x = 0$.*

Proof. We have $|||T^n|x|| = ||T^nx|| = |\lambda|^n||x||$ and

$$\begin{aligned}|\lambda|^n||x||^n = ||T^nx|| &= \langle Tx, Tx\rangle^{\frac{n}{2}} = \langle |T|^2x, x\rangle^{\frac{n}{2}}\\ &\leqslant \langle |T|^nx, x\rangle ||x||^{n-2} \text{ by Hölder-McCarthy Inequality (i), with } \quad \alpha = \frac{2}{n}\\ &\leqslant |||T^n|x||||x||^{n-1} = |\lambda|^n||x||^n.\end{aligned}$$

It follows that $|T^n|x$ and x are linearly dependent, we have $|T^n|x = |\lambda|^nx$. Now from

$$||(|T^n|^{\frac{2}{n}} - |T|^2)^{\frac{1}{2}}x||^2 = \langle |T^n|^{\frac{2}{n}}x, x\rangle - \langle |T^2|x, x\rangle = 0$$

we get,

$$T^*Tx = |T^n|^{\frac{2}{n}}x = |\lambda|^2 x.$$

Hence $T^*x = \overline{\lambda}x$. $\square$

Now we are ready to prove a generalized Fuglede-Putnam's theorem.

THEOREM 2.2. *Let A be a n-perinormal operator and B^* an invertible n-perinormal operator. If $AX = XB$ for some $X \in C_2$, Then $A^*X = XB^*$.*

Proof. Let Γ be a Hilbert-Schmidt operator defined by $\Gamma X = TXS^{-1}$ for all $X \in C_2$. Since $(S^*)^{-1} = (S^{-1})^*$ is n-perinormal, Lemma 2.1 implies that Γ is n-perinormal. The rest follows as in the proof of [12, Theorem 2.2]. $\square$

3. Some spectral properties

In this section we will show that n-perinormal operator T satisfies equality (1) and a-Browder's theorem holds for n-perinormal operators.

THEOREM 3.1. *It T is n-perinormal operator, then*
(1) $\sigma_{jp}(T) = \sigma_p(T)$,
(2) $Tx = \lambda x$, $Ty = \mu y$ with $\lambda \neq \mu$, then $\langle x,y\rangle = 0$.

Proof. (1) it is obvious from Theorem 2.1.
(2) As $\lambda\langle x,y\rangle = \langle Tx,y\rangle = \langle x,T^*y\rangle$ with $\lambda \neq \mu$, then $\langle x,y\rangle = 0$. $\square$

COROLLARY 3.1. *If T^* is n-perinormal, then $\beta(T-\lambda) \leqslant \alpha(T-\lambda)$ for all $\lambda \in \mathbb{C}$.*

Proof. Obvious from Theorem 2.1 $\square$

Now we will show that the spectral mapping theorem holds for Weyl's spectrum.

THEOREM 3.2. *If T or T^* is n-perinormal, then $w(f(T)) = f(w(T))$ for every $f \in H(\sigma(T))$, where $H(\sigma(T))$ denotes the set of all analytic functions on some open neighborhood of $\sigma(T)$.*

Proof. Since $w(f(T)) \subseteq f(w(T))$ holds for any operator. We need only to prove that

$$f(w(T)) \subseteq w(f(T)). \tag{4}$$

Note that (4) clearly holds if f is constant on some open neighborhood of $\sigma(T)$. Let $\lambda \notin w(f(T))$, we may assume that $f(z)-\lambda$ have only finitely many zeros in some open neighborhood G of $\sigma(T)$. Now write

$$f(z)-\lambda = (z-\lambda_1)...(z-\lambda_n)g(z),$$

where $\lambda_j, j = 1,...,n$ are the zeros of $f(z) - \lambda$ in G, listed according to multiplicity, and $g(z) \neq 0$ for all $z \in G$. Thus

$$f(T) - \lambda = (T - \lambda_1)...(T - \lambda_n)g(T). \tag{5}$$

Clearly, $\lambda \in f(w(T))$ if and only if $\lambda_j \in w(T)$ for some j. Therefore, to prove (4), it suffices to show that $\lambda_j \notin w(T)$ for all j. First, suppose that T is n-perinormal. Since $f(T) - \lambda$ is Weyl and the operators $T - \lambda_1, ..., T - \lambda_n$ commute, each $T - \lambda_j$ is Fredholm. Moreover, since $N(T - \lambda_j) \subseteq N(f(T) - \lambda)$ and $N((T - \lambda_j)^*) \subseteq N((f(T) - \lambda)^*)$, both $N(T - \lambda_j)$ and $N((T - \lambda_j)^*)$ are finite dimensional. Then $i(T - \lambda_j) \leqslant 0$ by Theorem 2.1. Then $i(f(T) - \lambda) = i(g(T)) = 0$, it follows from (5) that $i(T - \lambda_j) = 0$ for all j. Consequently, $T - \lambda_j$ is Weyl, and $\lambda_j \notin w(T)$. Now assume that T^* is n-perinormal. Then by Corollary 3.1 $i(T - \lambda) \geqslant 0$ for each $j = 1, 2, ..., n$. However,

$$\sum_{i=1}^{n} i(T - \lambda_j) = i(f(T) - \lambda) = 0,$$

and so $T - \lambda_j$ is Weyl for each $j = 1, 2, ..., n$. Hence $\lambda \notin f(w(T))$. Therefore $w(f(T)) = f(w(T))$. $\square$

An operator $T \in B(H)$ is said to have finite ascent if $\ker T^m = \ker T^{m+1}$ for some positive integer m, and finite descent if ran $T^n =$ ran T^{n+1} for some positive integer n.

LEMMA 3.1. *If T is n-perinormal operator, then $T - \lambda$ has finite ascent for each λ.*

Proof. Since T is n-perinormal operator, it follows that $N(T - \lambda) \subset N(T^* - \overline{\lambda})$, for each $\lambda \in \mathbb{C}$ by Theorem 2.1. Thus we can represent $T - \lambda$ as the following 2x2 operator matrix with respect to the decomposition $N(T - \lambda) \oplus N(T - \lambda)^{\perp}$:

$$T - \lambda = \begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix}.$$

Let $x \in N((T - \lambda)^2)$. Write $x = y + z$, where $y \in N(T - \lambda)$ and $z \in N(T - \lambda)^{\perp}$. Then $0 = (T - \lambda)^2 x = (T - \lambda)^2 z$, so that $(T - \lambda)z \in N(T - \lambda) \cap N(T - \lambda)^{\perp} = \{0\}$, which implies that $z \in N(T - \lambda)$, and hence $x \in N(T - \lambda)$. Therefore $N(T - \lambda) = N(T - \lambda)^2$. $\square$

Let $T \in B(H)$. The essential approximate point spectrum $\sigma_{ea}(T)$ is defined by

$$\sigma_{ea}(T) = \cap\{\sigma_a(T + K) : K \text{ is a compact operator}\},$$

where $\sigma_a(T)$ is the approximate point spectrum of T. We consider the set

$$\Phi_+^-(H) = \{T \in B(H) : T \text{ is left semi-Fredholm and } \mathrm{ind}(T) \leqslant 0\}.$$

V. Rakočević[15] proved that

$$\sigma_{ea}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin \Phi_+^-(H)\}$$

and the inclusion $\sigma_{ea}(f(T)) \subset f(\sigma_{ea}(T))$ holds for all function f which is analytic on some open neighborhood of $\sigma(T)$ with no restriction on T. The next theorem shows the spectral mapping theorem on the essential approximate point spectrum of n-perinormal operator.

THEOREM 3.3. *Let T or T^* be n-perinormal. Then $\sigma_{ea}(f(T)) = f(\sigma_{ea}(T))$ for every $f \in H(\sigma(T))$.*

Proof. Proof is similar to Theorem 2.9 given in [8], based on Corollary 3.1 and Lemma 3.1 . □

COROLLARY 3.2. *If T is n-perinormal operator, then T has SVEP.*

Proof. Proof of the corollary follows directly from lemma 3.1 and Proposition 1.8 in [9]. □

Recall [10] that $S,T \in B(H)$ are said to be quasisimilar if there exist injections $X,Y \in B(H)$ with dense range such that $XS = TX$ and $YT = SY$, respectively, and this relation is denoted by $S \sim T$.

THEOREM 3.4. *Let $T \in B(H)$ be n-perinormal. If $S \sim T$, then S has SVEP.*

Proof. Since T is n-perinormal, it follows from Corollary 3.2 that T has SVEP. Let U be any open set and $f : U \to H$ be any analytic function such that $(S-\lambda)f(\lambda) = 0$ for all $\lambda \in U$. Since $S \sim T$, there exists an injective operator A with dense range such that $AS = TA$. Thus $A(S-\lambda) = (T-\lambda)A$ for all $\lambda \in U$. Since $(S-\lambda)f(\lambda) = 0$ for all $\lambda \in U$, $A(S-\lambda) = 0 = (T-\lambda)A$ for all $\lambda \in U$. But T has SVEP, hence $Af(\lambda) = 0$ for all $\lambda \in U$. Since A is injective, $f(\lambda) = 0$ for all $\lambda \in U$. Thus S has SVEP. □

Now we will show that a-Browder's theorem holds for n-perinormal operators. For this we need the following definitions. The browder essential approximate point spectrum $\sigma_{ab}(T)$ of T is defined by

$$\sigma_{ab}(T) = \cap\{\sigma_a(T+K) : TK = KT, K \text{ is a compact operator}\}.$$

We say that a-Browder's theorem holds for T if $\sigma_{ea}(T) = \sigma_{ab}(T)$. It is well known that

$$\text{Weyl's theorem} \Rightarrow \text{Browder's theorem},$$

$$a\text{-Browder's theorem} \Rightarrow \text{Browder's theorem}.$$

THEOREM 3.5. *Let $T \in B(H)$ be n-perinormal operator. Then T obeys a-Browder's theorem.*

Proof. Since a n-perinormal operator has SVEP, [11, Theorem 2.8] implies that T obeys a-Browder's theorem. □

THEOREM 3.6. *Let $T \in B(H)$ be n- perinormal operator. Then a-Browder's theorem holds for $f(T)$ for every analytic function f on some open neighborhood of $\sigma(T)$.*

Proof. Since $\sigma_{ea}(f(T)) = f(\sigma_{ea}(T))$, the rest of the proof follows as in the proof of [11, Corollary 2.3]. □

THEOREM 3.7. *Let $T \in B(H)$ be n-perinormal. If $S \sim T$, then a-Browder's theorem holds for $f(S)$ for every analytic function f on some open neighborhood of $\sigma(T)$.*

Proof. Since a-Browder's theorem holds for S, it follows from Theorem 3.3 that

$$\sigma_{ab}(f(s)) = f(\sigma_{ab}(S)) = f(\sigma_{ea}(S)) = \sigma_{ea}(f(s)).$$

Hence a-Browder's theorem holds for $f(s)$. □

4. Examples of n-perinormal operators

In what follows we will give some examples of n-perinormal operators.

EXAMPLE 4.1. *An example of n-perinormal operator which is not normaloid. Suppose H is a direct sum of denumerable copies of two dimensional Hilbert space $\mathbb{R} \times \mathbb{R}$. Let A and B be any two positive operators on $\mathbb{R} \times \mathbb{R}$. For any fixed $n \in \mathbb{N}$ define operator $T = T_{A,B,n}$ on H as follows:*

$$T(x_1, x_2, \cdots,) = (0, Ax_1, Ax_2, \cdots, Ax_n, Bx_{n+1}, \cdots),$$

and from this we have the adjoint operator:

$$T^*(x_1, x_2, \cdots,) = (Ax_2, Ax_3, \cdots, Ax_{n+1}, Bx_{n+2}, \cdots).$$

The operator T belong to class (M,n) if and only if $AB^nA - A^{2n} \geqslant 0$(see Proposition 4.1 in [2]). Let us denote by $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ positive matrices. Then we get

$$AB^nA - A^{2n} \geqslant 0,$$

from which follows that $T \in (M,n)$. Now we will prove that T is not normaloid. From above definition of the operator T we have:

$$T^n(x_1, x_2, \cdots, x_n, \cdots) = (\underbrace{0, 0, \cdots, 0}_{n\text{-times}}, A^n(x_1), BA^{n-1}(x_2), \cdots). \tag{6}$$

From relation (6) it follows that

$$||T^n|| = ||T||^n$$

does not hold. For example, let $e_2 = (0,1,0\cdots)$ *be the second vector on canonical bases, then we get*

$$||Te_2||^n = 1,$$

and after some calculations we get that

$$||T||^n = 1.$$

But

$$||T^n e_2|| \neq 1,$$

and

$$||T^n|| \neq 1.$$

EXAMPLE 4.2. *There is a normaloid operator which is not* n*-perinormal operator. Let* T *be the operator defined by:*

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then it follows that T *is normaloid, because the following relation holds:*

$$||T^n|| = ||T||^n,$$

for all positive integers n*. In what follows we will show that* T *is not* n*-perinormal operator. The operator* T *belongs to class* (M,n) *(for* $n \geqslant 2$*) if and only if*

$$\||T|^n(x)\| \leqslant \|T^n(x)\|, \tag{7}$$

for every $x \in H$ *(see Proposition 4.1 in* [2]*). After some calculations we show that*

$$|T|^n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \textit{and} \quad T^n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For $e_2 = (0,1,0)$ *relation (7) does not hold, and this implies that* $T \notin (M,n)$*.*

We close this paper by asking the following open question. As we have mentioned in the introduction, if T is either normal or hyponormal, then T satisfies $\sigma_{ja}(T) = \sigma_a(T)$. But it is not known that $\sigma_{ja}(T) = \sigma_a(T)$ holds for n-perinormal operator $T \in B(H)$. It is natural to pose the following open question.

Open Question: Does $\sigma_{ja}(T) = \sigma_a(T)$ for n-perinormal operator $T \in B(H)$?

REFERENCES

[1] S. K. BERBERIAN, *An extension of a theorem of Fuglede-Putnam*, Proc. Amer. Math. Soc. **10** (1959), 175–182.

[2] N. L. BRAHA, M. LOHAJ, F. MAREVCI, SH. LOHAJ, *Some properties of paranormal and hyponormal operators*, Bull. Math. Anal. Appl. **1**, 2 (2009), 23–35.

[3] M. CHO AND T. HURUYA, *p-hyponormal operators for $0 < p < \frac{1}{2}$.*, Coment. Nath. **33** (1993), 23–29.

[4] H. K. CHA, *An extension of Fuglede-Putnam theorem to quasihyponormal operators using a Hilbert-Schmidt operator*, Youngman Math. J. **1** (1994), 73–76.

[5] N. CHENNAPPAN, S. KARTHIKEYAN, * *Paranormal composition operators*, Indian J. Pure Appl. Math. **31**, 6 (2000), 591–601.

[6] A. DEFANT, K. FLORET, *Tensor Norms and Operator Ideals*, North-Holland-Amsterdam, Elsevier Science Publishers, 1993.

[7] B. FUGLEDE, *A commutativity theorem for normal operators*, Proc. Nat. Acad. Sci. **36** (1950), 35–40.

[8] YOUNG MIN HAN AND AN-HYUN KIM, *A note on $*-$ paranormal operators*, Integr. equ. oper. theory **49** (2004), 435–444.

[9] K. B. LAURSEN, *Operators with finite ascent*, Pacific J. Math. **152** (1992), 323–336.

[10] K. B. LAURSEN AND M. M. NEUMANN, *An introduction to Local Spectral Theory*, London Mathematical society Monographs, Oxford, 2000.

[11] S. MECHERI AND S. MAKHLOUF, *Weyl Type theorems for posinormal operators*, Math. Proc. Royal Irish. Acad. **108** A (2008), 68–79.

[12] S. MECHERI, *An extension of Fuglede Putnam theorem to (p,k)-quasihyponormal operator*, Scientiae Math. Jap **62** (2005), 259–264.

[13] S. PANAYAPPAN AND A. RADHARAMANI, *A Note on $p-*$ Paranormal Operators and Absolute $k-*$ Paranormal Operators*, Int. J. Math. Anal. (Ruse) **2**, 25–28 (2008), 1257–1261.

[14] C. R. PUTNAM, *On normal operator in Hilbert space.*, Amer. J. Math. **73** (1951), 357–362.

[15] V. RAKOČEVIĆ, *On the essential approximate point spectrum II*, Math Vesnik **36** (1984), 89–97.

[16] H. RADJAVI AND P. ROSENTHAL, *Invariant subspaces*, New York, Springer-Verlag, 1973.

[17] K. TANAHASHI, *Putnam's Inequality for* $\log$*-hyponormal operators*, Integr. equ. oper. theory **43** (2004), 364–372.

[18] H. WEYL, *Über beschränkte quadratische Formen, deren Differenz vollsteig ist*, Rend. Circ. Mat. Palermo **27** (1909), 373–392.

[19] D. XIA, *Spectral Theory of Hyponormal Operators*, Birkhauser Verlag, Basel, 1983.

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Salah Mecheri
Taibah University, College of Science
Department of Mathematics, P.O. Box 30002
Al Madinah Al Munawarah
Saudi Arabia
e-mail: mecherisalah@hotmail.com

Naim L. Braha
University of Prishtina
Department of Mathematics and Computer Sciences
Avenue "Mother Theresa" 5
Prishtinë, 10000
Kosova
e-mail: nbraha@yahoo.com