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SPECTRA AND APPROXIMATIONS OF A CLASS OF SIGN–SYMMETRIC MATRICES

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Abstract. A new class of sign-symmetric matrices is introduced in this paper. Such matrices are called J-sign-symmetric. The spectrum of a J-sign-symmetric irreducible matrix is studied under the assumption that its second compound matrix is also J-sign-symmetric. The conditions for such matrices to have complex eigenvalues on the spectral circle are given. The existence of two positive simple eigenvalues $\lambda_1 > \lambda_2 > 0$ of a J-sign-symmetric irreducible matrix A is proved under some additional conditions. The question when the approximation of a J-sign-symmetric matrix with a J-sign-symmetric second compound matrix by strictly J-sign-symmetric matrices with strictly J-sign-symmetric second compound matrices is possible is also answered in this paper.

1. Introduction

The classical theorem of Gantmacher and Krein (see [1, p. 263, Theorem 9]) allows one to infer the positivity of the first two eigenvalues of a matrix $\mathbf{A} = \{a_{ij}\}_{i,j=1}^n$ from simple positivity properties of $\mathbf{A}$.

A matrix $\mathbf{A}$ is said to be *positive (non-negative)* if all its elements a_{ij} are positive (respectively, nonnegative). A matrix $\mathbf{A}$ is said to be 2*-strictly totally positive (*2*-STP)* if $\mathbf{A}$ is positive and its second compound matrix $\mathbf{A}^{(2)}$ is also positive. Recall that $\mathbf{A}^{(2)}$ is the matrix that consists of all the minors $A\begin{pmatrix} i & j \\ k & l \end{pmatrix}$, where $1 \leqslant i < j \leqslant n$, $1 \leqslant k < l \leqslant n$, of the initial matrix $\mathbf{A}$. The minors are listed in the lexicographic order. The matrix $\mathbf{A}^{(2)}$ is $\binom{n}{2} \times \binom{n}{2}$ dimensional, where $\binom{n}{2} = \dfrac{n(n-1)}{2}$.

We denote by $\rho(A)$ the spectral radius of $\mathbf{A}$. Arrange the eigenvalues $\{\lambda_i\}_{i=1}^n$ of $\mathbf{A}$ into a sequence (taking into account their multiplicities), so that

$$\rho(A) = |\lambda_1| \geqslant |\lambda_2| \geqslant |\lambda_3| \geqslant \cdots \geqslant |\lambda_n|.$$

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THEOREM A. (Gantmacher, Krein [1, p. 263]) *If* $\mathbf{A}$ *is a 2-STP matrix, then*

(a) $\rho(A) = \lambda_1 > \lambda_2 > |\lambda_3| \geqslant \cdots \geqslant |\lambda_n| \geqslant 0$*;*

(b) *both* λ_1 *and* λ_2 *are simple.*

The first result of this paper (Theorem 8) extends the Gantmacher–Krein theorem to a wider class of matrices. To specify this class we take any subset J of $[n] := \{1,2,\dots,n\}$ and a matrix $\mathbf{A} = \{a_{ij}\}_{i,j=1}^n$. As usual, $J^c := [n]\setminus J$. Then

$$[n]\times[n] = (J\times J)\cup(J^c\times J^c)\cup(J\times J^c)\cup(J^c\times J)$$

is a partition of $[n]\times[n]$ into four pairwise disjoint subsets.

DEFINITION 1. A matrix $\mathbf{A} = \{a_{ij}\}_{i,j=1}^n$ is called *strictly J-sign-symmetric (SJS)* if

$$a_{ij} > 0 \quad \text{on} \quad (J\times J)\cup(J^c\times J^c);$$

and

$$a_{ij} < 0 \quad \text{on} \quad (J\times J^c)\cup(J^c\times J).$$

Note, that the subset J is uniquely determined (up to J^c) by $\mathbf{A}$.

A matrix $\mathbf{A}$ is called 2*-strictly totally* J*–sign-symmetric (*2*-STJS)* if $\mathbf{A}$ is SJS, and its second compound matrix $\mathbf{A}^{(2)}$ is also SJS.

THEOREM 8. *If* $\mathbf{A}$ *is a 2-STJS matrix, then*

(a) $\rho(A) = \lambda_1 > \lambda_2 > |\lambda_3| \geqslant \cdots \geqslant |\lambda_n| \geqslant 0$*;*

(b) *both* λ_1 *and* λ_2 *are simple.*

We also extend the second Gantmacher–Krein theorem (see [1, p. 269, Theorem 13]). A matrix $\mathbf{A}$ is said to be 2*-totally positive (*2*-TP)* if $\mathbf{A}$ is nonnegative and its second compound matrix $\mathbf{A}^{(2)}$ is also nonnegative.

THEOREM B. (Gantmacher, Krein [1, p. 269]) *If* $\mathbf{A}$ *is a* 2*-TP matrix, then*

$$\rho(A) = \lambda_1 \geqslant \lambda_2 \geqslant |\lambda_3| \geqslant \cdots \geqslant |\lambda_n| \geqslant 0.$$

Theorem B comes out from Theorem A and from the following statement (see [1, p. 268, Theorem 12].

THEOREM C. (Gantmacher, Krein [1, p. 268]) *If* $\mathbf{A}$ *is a 2-TP matrix, then there exists a sequence* $\{\mathbf{A}\}_{n=1}^\infty$ *of 2-STP matrices which converges to* $\mathbf{A}$.

DEFINITION 2. A matrix $\mathbf{A} = \{a_{ij}\}_{i,j=1}^n$ is called *J-sign-symmetric (JS)* if

$$a_{ij} \geqslant 0 \quad \text{on} \quad (J\times J)\cup(J^c\times J^c);$$

and

$$a_{ij} \leqslant 0 \quad \text{on} \quad (J\times J^c)\cup(J^c\times J).$$

In this case the subset J may not be uniquely determined, but there is a finite number of ways to determine it.

A matrix $\mathbf{A}$ is called *2-totally J–sign-symmetric (2-TJS)* if $\mathbf{A}$ is JS and its second compound matrix $\mathbf{A}^{(2)}$ is also JS.

We show that not every 2-TJS matrix is similar to a 2-TP matrix. So the following results can not be deduced from similarity transformations of the well-known class of 2-TP matrices. We show that, although the set of all 2-STP matrices is dense in the set of all 2-TP matrices, the set of all 2-STJS matrices is not dense in the set of all 2-TJS matrices. So Theorem B can be extended only to a certain subclass of 2-TJS matrices, which can be approximated by 2-STJS matrices. This approximation exists under certain requirements on both sets $J \subseteq [n]$ and $J_2 \subseteq [\binom{n}{2}]$. (The sets J and J_2 are given in Definition 1 for the matrices $\mathbf{A}$ and $\mathbf{A}^{(2)}$, respectively.) These requirements are described in Section 10 in terms of the properties of a special binary relation $W(J,J_2)$ on $[n]$. The obtained conditions are necessary as Example 4 of a 2-TJS matrix, for which such an approximation does not exist, demonstrates.

Our proof of the extension of Theorem B consists of two steps.

First, for a given 2-TJS matrix, we find a 2-TP matrix $\widetilde{\mathbf{A}}$, a permutation matrix $\mathbf{Q}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A} = \mathbf{D}\mathbf{Q}\widetilde{\mathbf{A}}\mathbf{Q}^T\mathbf{D}^{-1}$ (Theorem 10). Note that this construction is not possible for every 2-TJS matrix, but is possible under our assumptions.

Applying Theorem C, we find a sequence $\{\widetilde{\mathbf{A}}_n\}_{n=1}^{\infty}$ of 2-STP matrices that converges to $\widetilde{\mathbf{A}}$. Then each $\mathbf{A}_n = \mathbf{D}\mathbf{Q}\widetilde{\mathbf{A}}_n\mathbf{Q}^T\mathbf{D}^{-1}$ is a 2-STJS matrix and the sequence $\{\mathbf{A}_n\}_{n=1}^{\infty}$ converges to $\mathbf{A}$. Thus we obtain

THEOREM 12. *If $\mathbf{A}$ is a 2-TJS matrix and at least one of the possible binary relations $W(J,J_2)$ is transitive, then*

$$\rho(A) = \lambda_1 \geqslant \lambda_2 \geqslant |\lambda_3| \geqslant \cdots \geqslant |\lambda_n| \geqslant 0.$$

If all the possible binary relations $W(J,J_2)$ are not transitive, the spectral properties of a 2-TJS matrix $\mathbf{A}$ are completely different and the matrix $\mathbf{A}$ cannot be approximated by 2-STJS matrices. However, we can still describe the peripheral spectrum of such a matrix under some additional conditions.

The matrix $\mathbf{A}$ is said to be *reducible* if there is a permutation of coordinates which reduces it to the form $\begin{pmatrix} \mathbf{A}_1 & 0 \\ \mathbf{B} & \mathbf{A}_2 \end{pmatrix}$, where $\mathbf{A}_1$, $\mathbf{A}_2$ are square matrices. Otherwise the matrix $\mathbf{A}$ is said to be *irreducible* [6].

THEOREM 13. *Let $\mathbf{A}$ be an irreducible 2-TJS matrix. Then one of the following two cases occurs:*

(1) *At least one of the possible binary relations $W(J,J_2)$ is transitive. Then $\mathbf{A}$ has a positive simple eigenvalue λ_1 and a nonnegative eigenvalue λ_2:*

$$\rho(A) = \lambda_1 > \lambda_2 \geqslant |\lambda_3| \geqslant \cdots \geqslant |\lambda_n| \geqslant 0.$$

(2) *All $W(J,J_2)$ are not transitive. Then there is an odd number $k \geqslant 1$ of eigenvalues on the spectral circle $|\lambda| = \rho(A)$. Each of them is simple and they coincide with the kth roots of $(\rho(A))^k$.*

A matrix $\mathbf{A}$ is called 2*-totally irreducible* J*-sign-symmetric (2-TIJS)* if $\mathbf{A}$ is irreducible J-sign-symmetric and its second compound matrix $\mathbf{A}^{(2)}$ is also irreducible J-sign-symmetric. In this case both the sets J and J_2 are uniquely determined. Thus the binary relation $W(J,J_2)$ is uniquely determined. So we have the statement

THEOREM 14. *Let* $\mathbf{A}$ *be a 2-TIJS matrix. Then one of the following two cases occurs:*

(1) *The binary relation* $W(J,J_2)$ *is transitive. Then* $\mathbf{A}$ *has two positive simple eigenvalues* λ_1, λ_2*:*

$$\rho(A) = \lambda_1 > \lambda_2 \geqslant |\lambda_3| \geqslant \cdots \geqslant |\lambda_n|.$$

(2) *The binary relation* $W(J,J_2)$ *is not transitive. Then there are exactly three eigenvalues on the spectral circle* $|\lambda| = \rho(A)$*. Each of them is simple and they coincide with the cube roots of* $(\rho(A))^3$.

We also give examples illustrating both cases of Theorem 14.

Then we give a sufficient condition of the existence of the second nonnegative eigenvalue.

THEOREM 15. *Let* $\mathbf{A} = \{a_{ij}\}_{i,j=1}^n$ *be an irreducible 2-TJS matrix. Let at least one entry* a_{ii} $(i = 1, \ldots, n)$ *be nonzero. Then* $\mathbf{A}$ *has a positive simple eigenvalue* $\lambda_1 = \rho(A)$ *and a nonnegative eigenvalue* λ_2*:*

$$\rho(A) = \lambda_1 > \lambda_2 \geqslant |\lambda_3| \geqslant \cdots \geqslant |\lambda_n| \geqslant 0.$$

The following statement generalizes Theorem 13 to the case of arbitrary 2-TJS matrices.

THEOREM 16. *Let* $\mathbf{A}$ *be a 2-TJS matrix with* $\rho(A) > 0$*. Then* $\lambda_1 = \rho(A)$ *is a positive eigenvalue of* $\mathbf{A}$*. Moreover, there are* m *sets of eigenvalues on the spectral circle* $|\lambda| = \rho(A)$*, where* m *is the algebraic multiplicity of* $\lambda_1 = \rho(A)$*. The* j*th set* $(j = 1, \ldots, m)$ *contains an odd number* $k_j \geqslant 1$ *of eigenvalues which coincide with the* k_j*th roots of* $(\rho(A))^{k_j}$.

2. Tensor and exterior powers of $\mathbb{R}^n$

Since tensor and exterior powers of function spaces can be realized also as function spaces, we consider $\mathbb{R}^n$ as the n-dimensional function space $\mathbb{X}$, defined on the discrete set $[n] = \{1,2,\ldots,n\}$. The standard basis of $\mathbb{X}$ is formed by the functions $e_1, e_2, \ldots, e_n$, defined by

$$e_i(j) = \delta_{ij} = \begin{cases} 1, \text{ if } i = j; \\ 0, \text{ if } i \neq j. \end{cases}$$

The tensor square $\otimes^2\mathbb{X}$ of the space $\mathbb{X}$ is the space of all functions defined on the set $[n]\times[n]$, which consists of n^2 pairs of the form (i,j), where $i,j \in [n]$. If $x,y \in \mathbb{X}$, then their tensor product

$$(x \otimes y)(i,j) = x(i)y(j)$$

is a function on $[n]\times[n]$. All the possible tensor products $\{e_i\otimes e_j\}_{i,j=1}^n$ of the initial basis functions form a basis in $\otimes^2\mathbb{X}$ (see [2], [3]). It follows that $\dim(\otimes^2\mathbb{X})=n^2$.

The exterior square $\wedge^2\mathbb{X}$ of the space $\mathbb{X}$ is a subspace of the space $\otimes^2\mathbb{X}$, consisting of antisymmetric functions, i.e. functions $f(i,j)$, satisfying the equality $f(i,j)=-f(j,i)$ on $[n]\times[n]$.

The space $\wedge^2\mathbb{X}$ is spanned by elementary exterior products $x\wedge y$:

$$(x\wedge y)(i,j)=(x\otimes y)(i,j)-(y\otimes x)(i,j)=x(i)y(j)-x(j)y(i).$$

Given any subset $W\subset[n]\times[n]$, we denote by W^s its symmetric reflection in $[n]\times[n]$ with respect to the main diagonal $\Delta=\{(i,i):\ i=1,\ \dots,\ n\}$:

$$W^s=\{(j,i):\ (i,j)\in W\}.$$

Let $W\subset[n]\times[n]$ satisfy

$$W\cup W^s=[n]\times[n]; \tag{1}$$

$$W\cap W^s=\Delta. \tag{2}$$

LEMMA 1. *Given any $W\subset[n]\times[n]$ satisfying (1) and (2), the space $\wedge^2\mathbb{X}$ is isomorphic to the space $\mathbb{X}(W\setminus\Delta)$ of all real functions on $W\setminus\Delta$.*

Proof. Any function on $W\setminus\Delta$ can be extended via antisymmetry to $[n]\times[n]$ by the unique way. The received antisymmetric function is supposed to be zero on Δ. $\square$

REMARK. This simple fact is no doubt well known, but we could not find it in the literature.

LEMMA 2. *Given any $W\subset[n]\times[n]$ satisfying (1) and (2), the size of the set $W\setminus\Delta$,* $\mathrm{Card}(W\setminus\Delta)$*, is equal to* $\binom{n}{2}$.

The proof of Lemma 2 is quite obvious.

Lemma 2 implies that for any W satisfying (1) and (2) the following spaces are isomorphic:

$$\wedge^2\mathbb{R}^n\cong\mathbb{X}(W\setminus\Delta)\cong\mathbb{R}^{\binom{n}{2}}.$$

It is easy to see that we can define $2^{\binom{n}{2}}$ different sets $W\subset[n]\times[n]$, satisfying (1) and (2). In this way, we get $2^{\binom{n}{2}}$ different constructions for the space $\wedge^2\mathbb{X}\cong\mathbb{X}(W\setminus\Delta)$.

3. Binary relations on $[n]$

Binary relations on $[n]$ are defined by the subsets of $[n]\times[n]$ (see [4]). Given an arbitrary $W\subset[n]\times[n]$, we write $i\overset{W}{\prec}j$ to denote $(i,j)\in W$.

As usual, we say that a binary relation W is:

— *reflexive* if $i\overset{W}{\prec}i$ for any $i\in[n]$; equivalently, if $\Delta\subset W\cap W^s$;

— *antisymmetric* if $i\overset{W}{\prec}j,\ j\overset{W}{\prec}i$ imply $i=j$ for any $i,j\in[n]$; equivalently, if $W\cap W^s=\Delta$;

— *transitive* if $i \overset{W}{\prec} j$ and $j \overset{W}{\prec} k$ imply $i \overset{W}{\prec} k$ for any $i,j,k \in [n]$; equivalently, if $(i,j) \in W$ and $(j,k) \in W$ imply $(i,k) \in W$;

— *connected* if, for any $i,j \in [n]$, we have either $i \overset{W}{\prec} j$ or $j \overset{W}{\prec} i$; equivalently, if $W \cup W^s = [n] \times [n]$.

A binary relation $\overset{W}{\prec}$ is said to be a *linear order*, if it is reflexive, antisymmetric, transitive and connected (see [5]).

LEMMA 3. *Any set $W \subset [n] \times [n]$ satisfying (1) and (2) determines a connected antisymmetric reflexive binary relation on $[n]$. If in addition W is transitive, then it determines a linear order on $[n]$.*

Conversely, any connected antisymmetric reflexive binary relation on $[n]$ is generated by a set $W \subset [n] \times [n]$ satisfying (1) and (2), and any linear order on $[n]$ is generated by a transitive set $W \subset [n] \times [n]$ satisfying (1) and (2).

Proof. $\Rightarrow$ The first part of the proof follows from the reasoning preceding the lemma.

$\Leftarrow$ Given a binary relation $\prec$ on $[n]$, we define:

$$W = \{(i,j) \in [n] \times [n] : i \prec j\};$$

$$W^s = \{(i,j) \in [n] \times [n] : j \prec i\}.$$

Then the necessary properties of W and W^s follows from the corresponding properties of $\prec$. $\square$

The set $M = \{(i,j) \in [n] \times [n] :\ i \leqslant j\}$, which defines the natural linear order on $[n]$, is used in the classical theory of 2-TP matrices (see [1]).

4. Bases in $\wedge^2\mathbb{R}^n$

Given an arbitrary basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$, we consider the set of all possible exterior products of the form $\{e_i \wedge e_j\}$, where $1 \leqslant i < j \leqslant n$ to be the canonical basis of the space $\wedge^2\mathbb{R}^n$ (see [2], [3]). However, there exist other bases of $\wedge^2\mathbb{R}^n$ consisting of exterior products of the initial basic vectors. Namely, we can construct $2^{\binom{n}{2}}$ different bases by choosing an arbitrary element from every pair $e_i \wedge e_j$ and $e_j \wedge e_i$ $(i \neq j)$.

LEMMA 4. *Every $W \subset [n] \times [n]$ satisfying (1) and (2) uniquely defines a basis in $\wedge^2\mathbb{R}^n$, consisting of the exterior products of $e_1, \ldots, e_n$. The converse is also true: every basis in $\wedge^2\mathbb{R}^n$ consisting of some exterior products of $e_1, \ldots, e_n$ uniquely defines a set $W \subset [n] \times [n]$, satisfying (1) and (2).*

Proof. $\Rightarrow$ Given a set $W \subset [n] \times [n]$ satisfying (1) and (2), we examine the system $\Lambda = \{e_i \wedge e_j\}_{(i,j) \in W \setminus \Delta}$. Show that Λ is a basis in $\wedge^2\mathbb{X}$. For any $e_i \wedge e_j \in \Lambda$ and for any $(k,l) \in W \setminus \Delta$ we have

$$(e_i \wedge e_j)(k,l) = \begin{cases} 1 \text{ if } (i,j) = (k,l); \\ 0 \quad \text{otherwise.} \end{cases}$$

This shows that the system Λ is linearly independent. Since $\wedge^2\mathbb{X} = \mathbb{X}(W \setminus \Delta)$ is $\binom{n}{2}$-dimensional and Λ contains exactly $\binom{n}{2}$ elements, the system Λ also spans the whole space $\wedge^2\mathbb{X}$.

$\Leftarrow$ Given a basis Λ of the space $\wedge^2\mathbb{X}$ consisting of some exterior products of $e_1, \ldots, e_n$, we define the set W:

$$W = \{(i,j) \in [n] \times [n] : \ e_i \wedge e_j \in \Lambda\} \cup \Delta.$$

Show that W satisfies (1). Take a pair $(i_0, j_0) \in W \cap W^s$. In this case we have $(i_0, j_0) \in W$ and $(j_0, i_0) \in W$. If $i_0 \neq j_0$, then $e_{i_0} \wedge e_{j_0} \in \Lambda$ and $e_{j_0} \wedge e_{i_0} \in \Lambda$. It follows that $e_{i_0} \wedge e_{j_0}$ and $e_{j_0} \wedge e_{i_0}$ are linearly independent. This contradicts the equality $e_{i_0} \wedge e_{j_0} = -(e_{j_0} \wedge e_{i_0})$. So we have $i_0 = j_0$ for any pair $(i_0, j_0) \in W \cap W^s$.

We now verify condition (2). Assume that there exists a pair (i_0, j_0), $i_0 \neq j_0$, in $([n] \times [n]) \setminus (W \cup W^s)$. Then we have $(j_0, i_0) \in ([n] \times [n]) \setminus (W \cup W^s)$. It follows that neither $e_{i_0} \wedge e_{j_0}$ no $e_{j_0} \wedge e_{i_0}$ is in Λ. Add $e_{i_0} \wedge e_{j_0}$ to the system Λ. It is easy to see that the obtained system remains linearly independent. This contradicts the condition that Λ is a maximal linearly independent system in $\wedge^2\mathbb{X}$. $\square$

A basis $\{e_i \wedge e_j\}_{(i,j) \in W \setminus \Delta}$ defined by the set W is called a W*–basis*. We enumerate the elements of a W–basis in the lexicographic order.

EXAMPLE 1. Let $M = \{(i,j) \in [n] \times [n] : \ i \leqslant j\}$. Then $M \setminus \Delta = \{(i,j) \in [n] \times [n] : i < j\}$, and the corresponding basis is $\{e_i \wedge e_j\}_{i<j}$, i.e., the canonical basis of the space $\wedge^2\mathbb{R}^n$ (see [1], [3]).

5. Exterior square of a linear operator in $\mathbb{R}^n$

The exterior square $\wedge^2 A$ of the operator $A : \mathbb{X} \to \mathbb{X}$ acts on the space $\wedge^2\mathbb{X}$ according to the rule:

$$(\wedge^2 A)(x \wedge y) = Ax \wedge Ay.$$

Recall the following properties of $\wedge^2 A$ (see [1], p. 64).

1. $\wedge^2(AB) = (\wedge^2 A)(\wedge^2 B)$ for any linear operators $A, B : \mathbb{X} \to \mathbb{X}$.

2. $(\wedge^2 A)^{-1} = \wedge^2(A^{-1})$ for any invertible linear operator $A : \mathbb{X} \to \mathbb{X}$.

Below we study spectral properties of the operator A, assuming that its exterior square $\wedge^2 A$ leaves invariant a cone in $\wedge^2\mathbb{X}$. For this condition to hold, it is enough to have the matrix of $\wedge^2 A$ positive in some basis in $\wedge^2\mathbb{X}$.

Let an operator A be defined by a matrix $\mathbf{A} = \{a_{ij}\}_{i,j=1}^n$ in the basis $\{e_1, \ldots, e_n\}$. To examine the matrix of $\wedge^2 A$ in a W–basis defined by a set W satisfying (1) and (2) we recall the following definitions.

A determinant $A\begin{pmatrix} i & j \\ k & l \end{pmatrix}$, formed by the rows indexed by the integers i and j and the columns indexed by k and l $(i, \ j, \ k, \ l \in [n])$ of the matrix $\mathbf{A}$, is called a *generalized minor of the second order*.

We call the matrix consisting of all generalized minors $A\begin{pmatrix} i & j \\ k & l \end{pmatrix}$, where $(i,j),\ (k,l) \in (W \setminus \Delta)$, the *second W–matrix* of the initial matrix $\mathbf{A}$ and denote it by $\mathbf{A}_W^{(2)}$. The generalized minors are listed in the lexicographic order.

EXAMPLE 2. Let $W = M = \{(i,j) \in [n] \times [n] :\ i \leqslant j\}$. Then the corresponding W–matrix is a matrix consisting of all minors $A\begin{pmatrix} i & j \\ k & l \end{pmatrix}$ with $i < j,\ k < l$, i.e., *the second compound matrix.*

We now demonstrate the connection between $\mathbf{A}_W^{(2)}$ and the matrix of $\wedge^2 A$.

THEOREM 1. *Let the operator A be defined by a matrix $\mathbf{A} = \{a_{ij}\}_{i,j=1}^n$ in the basis $e_1, \ldots, e_n$. Then, for any $W \subset [n] \times [n]$ satisfying (1) and (2), the matrix of the exterior square $\wedge^2 A$ of the operator A in the W–basis $\{e_i \wedge e_j\}_{(i,j) \in W \setminus \Delta}$ coincides with the second W–matrix $\mathbf{A}_W^{(2)}$.*

Proof. Since $A(e_k) = \sum\limits_{i=1}^n a_{ik} e_i$ for $k = 1,\ \ldots,\ n$, we have

$$(\wedge^2 A)(e_i \wedge e_j) = Ae_i \wedge Ae_j = \left(\sum_{k=1}^n a_{ki} e_k\right) \wedge \left(\sum_{l=1}^n a_{lj} e_l\right) = \sum_{k,l=1}^n a_{ki} a_{lj} (e_k \wedge e_l) =$$

$$= \sum_{(k,l) \in (W \setminus \Delta)} a_{ki} a_{lj} (e_k \wedge e_l) + \sum_{k=l=1}^n a_{ki} a_{lj} (e_k \wedge e_l) + \sum_{(k,l) \in (W^s \setminus \Delta)} a_{ki} a_{lj} (e_k \wedge e_l) =$$

$$= \sum_{(k,l) \in (W \setminus \Delta)} a_{ki} a_{lj} (e_k \wedge e_l) + 0 - \sum_{(k,l) \in (W^s \setminus \Delta)} a_{ki} a_{lj} (e_l \wedge e_k).$$

Interchange the indices l and k in the third sum:

$$\sum_{(k,l) \in (W \setminus \Delta)} a_{ki} a_{lj} (e_k \wedge e_l) - \sum_{(k,l) \in (W \setminus \Delta)} a_{li} a_{kj} (e_k \wedge e_l) =$$

$$= \sum_{(k,l) \in (W \setminus \Delta)} (a_{ki} a_{lj} - a_{li} a_{kj})(e_k \wedge e_l) = \sum_{(k,l) \in (W \setminus \Delta)} A\begin{pmatrix} k & l \\ i & j \end{pmatrix} (e_k \wedge e_l),$$

where $A\begin{pmatrix} k & l \\ i & j \end{pmatrix}$ are the elements of the corresponding column of the matrix $\mathbf{A}_W^{(2)}$. So the matrix of $\wedge^2 A$ in the basis $\{e_i \wedge e_j\}_{(i,j) \in W \setminus \Delta}$ coincides with $\mathbf{A}_W^{(2)}$. $\square$

It follows from Theorem 1 that the matrix of $\wedge^2 A$ in the basis $\{e_i \wedge e_j\}_{i<j}$ coincides with $\mathbf{A}^{(2)}$, i.e., the second compound matrix of $\mathbf{A}$.

THEOREM 2. *Let $W \subset [n] \times [n]$ satisfy (1) and (2). Let $\{\lambda_i\}_{i=1}^n$ be the set of all eigenvalues of the matrix $\mathbf{A}$ repeated according to their multiplicity. Then all possible* *products of the type* $\{\lambda_i\lambda_j\}$, *where* $1 \leqslant i < j \leqslant n$, *form the set of all eigenvalues of the second* W*–matrix* $\mathbf{A}_W^{(2)}$ *repeated according to their multiplicity.*

Proof. Recall that all possible products of the type $\{\lambda_i\lambda_j\}$, where $1 \leqslant i < j \leqslant n$, form the set of all eigenvalues of $\wedge^2 A$, repeated according to their multiplicity (see [3]). Then apply Theorem 1. $\square$

Note, that in the case $W = M$ Theorem 2 turns into the Kronecker theorem (see [1, p. 65, Theorem 23]) about the eigenvalues of $\mathbf{A}^{(2)}$. The proof of the Kronecker theorem that does not make use of exterior products is given in [1].

6. Nonnegative and J-sign-symmetric matrices

The proof of Theorem A is based on the well-known result of Perron and Frobenius (see [6]).

THEOREM 3. (Perron) *Let the matrix* $\mathbf{A}$ *of a linear operator* $A : \mathbb{R}^n \to \mathbb{R}^n$ *be (entrywise) positive. Then the spectral radius* $\rho(A) > 0$ *is a simple positive eigenvalue of the operator* A. *Moreover,* $\rho(A)$ *is srictly bigger than the absolute value of any other eigenvalue of* A, *and the eigenvector* x_1 *corresponding to* $\lambda_1 = \rho(A)$ *is (entrywise) positive.*

It is easy to see, that the Perron theorem also holds for any matrix similar to a positive matrix. Here a natural question arises: how to determine if an arbitrary matrix is similar to some positive matrix? We now prove a criterion of similarity, which will be used later.

THEOREM 4. *The matrix* $\mathbf{A}$ *is SJS if and only if* $\mathbf{A} = \mathbf{D}\widetilde{\mathbf{A}}\mathbf{D}^{-1}$ *for some positive matrix* $\widetilde{\mathbf{A}}$ *and diagonal matrix* $\mathbf{D}$.

Proof. $\Rightarrow$ Define the diagonal matrix $\mathbf{D}$:

$$d_{ii} = \begin{cases} -1 & \text{if } i \in J; \\ \\ 1 & \text{otherwise.} \end{cases}$$

Then $\widetilde{\mathbf{A}} = \mathbf{D}^{-1}\mathbf{A}\mathbf{D}$ is positive.
$\Leftarrow$ Define $J \subseteq [n]$ as follows:

$$J = \{i \in [n] : \ \mathrm{sign}(d_{ii}) = -1\}.$$

Then $\mathbf{A}$ can be seen to be strictly J-sign-symmetric. $\square$

COROLLARY 1. *Let the matrix* $\mathbf{A}$ *of a linear operator* $A : \mathbb{R}^n \to \mathbb{R}^n$ *be SJS. Then the spectral radius* $\rho(A) > 0$ *is a simple positive eigenvalue of the operator* A, *strictly bigger than the absolute value of any other eigenvalue of* A.

Note that the number of all different types of $n \times n$ SJS matrices is equal to 2^{n-1}, while the number of all different types of $\binom{n}{2} \times \binom{n}{2}$ SJS matrices is equal to $2^{\binom{n}{2}-1}$.

The class of positive matrices is a subclass of irreducible nonnegative matrices. The following result of Frobenius is widely known:

THEOREM 5. (Frobenius) *Let the matrix* $\mathbf{A}$ *of a linear operator* A *be nonnegative and irreducible. Then the spectral radius* $\rho(A) > 0$ *is a simple positive eigenvalue of the operator* A. *The eigenvector* x_1 *corresponding to the eigenvalue* $\lambda_1 = \rho(A)$ *is positive. If* h *is a number of the eigenvalues of the operator* A *whose absolute values are equal to* $\rho(A)$, *then all of them are simple and they coincide with the* h*th roots of* $(\rho(A))^h$. *Moreover, the spectrum of* A *is invariant under rotations by* $\frac{2\pi}{h}$ *about the origin.*

The number h of the eigenvalues whose absolute values are equal to $\rho(A)$ is called *the index of imprimitivity* of the irreducible operator A. The operator A is called *primitive* if $h(A) = 1$, and *imprimitive* if $h(A) > 1$.

THEOREM 6. *The matrix* $\mathbf{A}$ *is JS if and only if* $\mathbf{A} = \mathbf{D}\widetilde{\mathbf{A}}\mathbf{D}^{-1}$ *for some nonnegative matrix* $\widetilde{\mathbf{A}}$ *and diagonal matrix* $\mathbf{D}$. *Moreover, if* $\mathbf{A}$ *is irreducible, then* $\widetilde{\mathbf{A}}$ *is also irreducible.*

Proof. The proof is analogical to the proof of Theorem 4. $\square$

COROLLARY 2. *Let the matrix* $\mathbf{A}$ *of a linear operator* A *be irreducible JS. Then the spectral radius* $\rho(A) > 0$ *is a simple positive eigenvalue of the operator* A. *If* h *is a number of the eigenvalues of the operator* A *whose absolute values are equal to* $\rho(A)$, *then all of them are simple and they coincide with the* h*th roots of* $(\rho(A))^h$. *Moreover, the spectrum of* A *is invariant under rotations by* $\frac{2\pi}{h}$ *about the origin.*

Note, that *if the matrix* $\mathbf{A}$ *is irreducible JS, then the set* J *is uniquely determined (up to the set* J^c *).*

The following sufficient criteria of primitivity was proved in [7] (see [7], p. 49, Corollary 1.1): *if a matrix* $\mathbf{A} = \{a_{ij}\}_{i,j=1}^n$ *is irreducible, and* $\sum_{i=1}^n a_{ii} > 0$, *then* $\mathbf{A}$ *is primitive.* This implies

LEMMA 5. *Let the matrix* $\mathbf{A} = \{a_{ij}\}_{i,j=1}^n$ *of a linear operator* A *be JS. Let at least one element* a_{ii} *be nonzero. Then* $\rho(A) > 0$ *and if* A *is irreducible then it is primitive.*

Proof. Since $\mathbf{A}$ is JS we have $a_{ii} \geqslant 0$ for $i = 1, \ldots, n$. Since at least one of $a_{ii} \neq 0$, we have the following estimate

$$\rho(A) \geqslant \frac{1}{n}\sum_{i=1}^{n}\lambda_i = \frac{1}{n}\sum_{i=1}^{n}a_{ii} > 0,$$

where $\{\lambda_i\}_{i=1}^n$ is the set of all eigenvalues of the operator A, repeated according to multiplicity. $\square$

Let us recall also the following result of Frobenius (see, for example, [6]).

THEOREM 7. (Frobenius) *Let the matrix* $\mathbf{A}$ *of a linear operator* A *be nonnegative and reducible. Then there is a* $n \times n$ *permutation matrix* $\mathbf{P}$ *such that*

$$\mathbf{P}\mathbf{A}\mathbf{P}^{-1} = \widehat{\mathbf{A}},$$

where $\widehat{\mathbf{A}}$ is a block-triangular form with the finite number $l \leqslant n$ of square irreducible (or zero) blocs $\mathbf{A}_j$ $(j = 1, \dots, l)$ on the principal diagonal and zero entries above the principal diagonal:

$$\widehat{\mathbf{A}} = \begin{pmatrix} \mathbf{A}_1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & \mathbf{A}_2 & \dots & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \mathbf{A}_r & 0 & 0 & \dots & 0 \\ \mathbf{B}_{r+1\,1} & \mathbf{B}_{r+1\,2} & \dots & \mathbf{B}_{r+1\,r} & \mathbf{A}_{r+1} & 0 & \dots & 0 \\ \mathbf{B}_{r+2\,1} & \mathbf{B}_{r+2\,2} & \dots & \mathbf{B}_{r+2\,r} & \mathbf{B}_{r+2\,r+1} & \mathbf{A}_{r+2} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \mathbf{B}_{l\,1} & \mathbf{B}_{l\,2} & \dots & \mathbf{B}_{l\,r} & \mathbf{B}_{l\,r+1} & \mathbf{B}_{l\,r+2} & \dots & \mathbf{A}_l \end{pmatrix}. \tag{3}$$

$\widehat{\mathbf{A}}$ is uniquely defined (up to a permutation of the blocks).

The spectral radius $\rho(A)$ is an eigenvalue of the operator A with the corresponding nonnegative eigenvector x_1. Moreover, the following equalities hold:

$$\sigma_p(A) = \bigcup_{j=1}^{l} \sigma_p(A_j), \quad \rho(A) = \max_{j=1,\dots,l} \{\rho(A_j)\},$$

where $\sigma_p(A_j)$ are the sets of all eigenvalues and $\rho(A_j)$ are the spectral radii of the irreducible blocks $\mathbf{A}_j$ $(j = 1, \dots, l)$.

If the matrix $\mathbf{A}$ is reducible JS, then we have the representation $\mathbf{A} = \mathbf{D}\mathbf{P}\widehat{\mathbf{A}}\mathbf{P}^{-1}\mathbf{D}^{-1}$, where $\widehat{\mathbf{A}}$ is the block-diagonal form of a nonnegative reducible matrix $\widetilde{\mathbf{A}}$. Note, that the algebraic multiplicity of any eigenvalue λ with $|\lambda| = \rho(A)$ does not exceed the algebraic multiplicity of $\rho(A)$.

7. Proof of Theorem 8

Enumerate the eigenvalues of the operator A decreasing order of their absolute values (taking into account their multiplicities):

$$|\lambda_1| \geqslant |\lambda_2| \geqslant |\lambda_3| \geqslant \cdots \geqslant |\lambda_n|.$$

Applying Corollary 1 to the SJS matrix $\mathbf{A}$, we get $\lambda_1 = \rho(A) > 0$ is a simple positive eigenvalue of $\mathbf{A}$. Applying Corollary 1 to the matrix $\mathbf{A}^{(2)}$, we get $\rho(\mathbf{A}^{(2)}) > 0$ is a simple positive eigenvalue of $\mathbf{A}^{(2)}$.

It follows from Theorem 2 that the matrix $\mathbf{A}^{(2)}$ has no eigenvalues other than the products of the form $\lambda_i\lambda_j$, where $i < j$. Therefore $\rho(\mathbf{A}^{(2)}) > 0$ is a product $\lambda_i\lambda_j$ for some indices $i, j,\ i < j$. Since the eigenvalues are enumerated in decreasing order, and since there is only one eigenvalue on the spectral circle $|\lambda| = \rho(\mathbf{A})$, we get $\rho(\mathbf{A}^{(2)}) = \lambda_1\lambda_2$. So $\lambda_2 = \frac{\rho(\mathbf{A}^{(2)})}{\lambda_1} > 0$. $\square$

8. Connection between $\mathbf{A}_W^{(2)}$ and $\mathbf{A}^{(2)}$

In Section 10 we will study the case when the matrix $\mathbf{A}$ is 2-TJS, i.e., $\mathbf{A}$ is similar to some nonnegative matrix, and its second compound matrix $\mathbf{A}^{(2)}$ is also similar to some nonnegative matrix. Note that these two conditions do not mean that $\mathbf{A}$ is similar to a 2-TP matrix and do not guarantee the reality of the peripheral spectrum of the matrix $\mathbf{A}$. This can be seen by invoking the above conception of a W–basis and a W–matrix. The following theorem describes the link between the matrices $\mathbf{A}_W^{(2)}$ and $\mathbf{A}^{(2)}$.

THEOREM 9. *Let the second compound matrix $\mathbf{A}^{(2)}$ of the matrix $\mathbf{A}$ be JS. Then there exists a set $W \subset [n]\times[n]$ satisfying (1) and (2) such that the corresponding W–matrix $\mathbf{A}_W^{(2)}$ is nonnegative. Moreover, if $\mathbf{A}^{(2)}$ is irreducible, then $\mathbf{A}_W^{(2)}$ is also irreducible.*

The converse is also true. Suppose for some set $W \subset [n]\times[n]$ satisfying (1) and (2), the matrix $\mathbf{A}_W^{(2)}$ is nonnegative. Then the second compound matrix $\mathbf{A}^{(2)}$ is JS. Moreover, if $\mathbf{A}_W^{(2)}$ is irreducible, then $\mathbf{A}^{(2)}$ is also irreducible.

Proof. $\Leftarrow$ Given a set $W \subset [n]\times[n]$ satisfying (1) and (2) such that the corresponding W–matrix $\mathbf{A}_W^{(2)}$ is nonnegative, we show that $\mathbf{A}^{(2)}$ is JS. Define the set $J_2 \subseteq [\binom{n}{2}]$:

$$J_2 = \{\alpha(i,j) : \ (i,j) \in (M\cap W)\setminus\Delta\},$$

where $\alpha(i,j) = \sum_{k=1}^{i-1}(n-k)+j-i$ is the number of the pairs (i,j) in the lexicographic order. Notice that $J_2^c = [\binom{n}{2}]\setminus J_2$. We get

$$J_2^c = \{\alpha(i,j) : \ (i,j) \in (M\cap W^s)\setminus\Delta\}.$$

Then

$$\left[\binom{n}{2}\right]\times\left[\binom{n}{2}\right] = (J_2\times J_2)\cup(J_2\times J_2^c)\cup(J_2^c\times J_2)\cup(J_2^c\times J_2^c).$$

Since $M = (M\cap W)\cup(M\cap W^s)$, we get the corresponding partition of $M\times M$:

$$M\times M = ((M\cap W)\times(M\cap W))\cup((M\cap W)\times(M\cap W^s))\cup$$
$$\cup((M\cap W^s)\times(M\cap W))\cup((M\cap W^s)\times(M\cap W^s)).$$

Examine an arbitrary minor $A\begin{pmatrix} i & j \\ k & l\end{pmatrix}$, where $i<j$, $k<l$. We have the following four cases.

Case 1. If $(i,j),(k,l)\in J_2$, then $(i,j),(k,l)\in(M\cap W)$, and $A\begin{pmatrix} i & j \\ k & l\end{pmatrix}$ is an element of $\mathbf{A}_W^{(2)}$ and hence is nonnegative.

Case 2. If $(i,j),(k,l)\in J_2^c$, then $(i,j),(k,l)\in(M\cap W^s)$ and $(j,i),(l,k)\in(M\cap W)$. The equality $A\begin{pmatrix} i & j \\ k & l\end{pmatrix} = A\begin{pmatrix} j & i \\ l & k\end{pmatrix}$ implies that $A\begin{pmatrix} j & i \\ l & k\end{pmatrix}$ is an element of $\mathbf{A}_W^{(2)}$ and is also nonnegative.

Case 3. If $(i,j) \in J_2$ and $(k,l) \in J_2^c$, then $(i,j) \in M \cap W$ and $(k,l) \in M \cap W^s$. The equality $A\begin{pmatrix} i & j \\ k & l \end{pmatrix} = -A\begin{pmatrix} i & j \\ l & k \end{pmatrix}$ implies that $A\begin{pmatrix} i & j \\ k & l \end{pmatrix}$ is nonpositive.

Case 4. This case $(i,j) \in M \cap W^s$, and $(k,l) \in M \cap W$ is analogous to Case 3. Here $A\begin{pmatrix} i & j \\ k & l \end{pmatrix}$ is again nonpositive.

The remaining proof of irreducibility of $\mathbf{A}_W^{(2)}$ is obvious.

$\Rightarrow$ Now let $\mathbf{A}^{(2)}$ be JS. Then we can find a set $J_2 \subseteq [\binom{n}{2}]$, such that

$$a_{ij} \geqslant 0 \quad \text{on} \quad (J_2 \times J_2) \cup (J_2^c \times J_2^c);$$

and

$$a_{ij} \leqslant \quad \text{on} \quad (J_2 \times J_2) \cup (J_2^c \times J_2^c).$$

Define a set W:

$$(i,j) \in W \Leftrightarrow \text{ either } i < j \text{ and } \alpha(i,j) \in J_2 \text{ or } i > j \text{ and } \alpha(j,i) \in J_2^c. \tag{4}$$

It is easy to see that W satisfies (1) and (2). The nonnegativity and irreducibility of $\mathbf{A}_W^{(2)}$ are proved analogously to the proof of the first part. $\square$

9. Permutations and isomorphisms of the space $\mathbb{X}$

It is well known (see Theorem B), that the two eigenvalues of a matrix $\mathbf{A}$ with largest absolute values are real and nonnegative whenever $\mathbf{A}$ is 2-TP. However, it is not true for a 2-TJS matrix $\mathbf{A}$. In Section 10 we will give some sufficient conditions for the reality of the peripheral spectrum of a 2-TJS matrix.

Let us study the case when W is transitive.

LEMMA 6. *Every transitive W satisfying (1) and (2) is uniquely defined by a permutation $\sigma_n = (\sigma(1), \ldots, \sigma(n))$. The converse is also true: every permutation σ_n of $[n]$ is uniquely defined by a transitive W satisfying (1) and (2).*

Proof. $\Rightarrow$ Given a permutation $\sigma_n = (\sigma(1), \ldots, \sigma(n))$, we define W:

$$W = \{(i,j) \in [n] \times [n] : \ \sigma_n^{-1}(i) \leqslant \sigma_n^{-1}(j)\}.$$

Properties (1) and (2) are obvious. To check transitivity, we let $(i,j),(j,k) \in W$ for some $i,j,k \in [n]$. Then we have $\sigma_n^{-1}(i) \leqslant \sigma_n^{-1}(j)$ and $\sigma_n^{-1}(j) \leqslant \sigma_n^{-1}(k)$. Since σ_n^{-1} maps $(\sigma(1), \ldots, \sigma(n))$ to $[n]$, these inequalities imply $\sigma_n^{-1}(i) \leqslant \sigma_n^{-1}(k)$ and the inclusion $(i,k) \in W$ holds.

$\Leftarrow$ Given a transitive W satisfying (1) and (2), we define σ_n by induction:

1) $\sigma_1(1) := 1$.

2) $\sigma_2(1) := 2$, $\sigma_2(2) := 1$, if $(2,1) \in W$ and $\sigma_2(1) := 1$, $\sigma_2(2) := 2$ otherwise.

3) Given σ_{j-1}, we define

$$l := \max\{k:\ 1 \leqslant k \leqslant j-1;\ (\sigma_{j-1}(k), j) \in W\}.$$

If $(\sigma_{j-1}(k), j) \in W^s$ for all $k = 1, \dots, j-1$, let $l := 0$. Define

$$\sigma_j(i) := \begin{cases} \sigma_{j-1}(i), & i = 1, \dots, l; \\ j, & i = l+1; \\ \sigma_{j-1}(i-1), & i = l+2, \dots, j. \end{cases}$$

Show that the resulting permutation σ_n defines the same set W. Let

$$V := \{(i,j) \in [n] \times [n]:\ \sigma_n^{-1}(i) \leqslant \sigma_n^{-1}(j)\}.$$

Show that V coincides with W. Let $(i,j) \in V$. In this case the inequality $\sigma_n^{-1}(i) \leqslant \sigma_n^{-1}(j)$ implies $i \leqslant j$ in $\sigma_n([n])$. Let $k_1, \dots, k_m$ be all indices between i and j in $\sigma_n([n])$. Write $\sigma_n([n])$ in the following form:

$$\sigma_n([n]) = \sigma_n(1), \dots, i, k_1, \dots, k_m, j, \dots, \sigma_n(n).$$

It follows from the construction of σ_n that all the pairs (i,k_1), (k_2,k_3), $\dots$, (k_{m-1},k_m), (k_m, j) belong to W. Since W is transitive, the inclusion $(i,k_2) \in W$ follows from the inclusions $(i,k_1) \in W$, $(k_1,k_2) \in W$. Repeating this reasoning m times, we get the inclusion $(i,j) \in W$. Therefore the inclusion $V \subseteq W$ holds. Show that $W \subseteq V$. Suppose the contrary: $\sigma_n^{-1}(i_0) > \sigma_n^{-1}(j_0)$ for some $(i_0, j_0) \in W \setminus \Delta$. Then $\sigma_n^{-1}(j_0) < \sigma_n^{-1}(i_0)$ implies $j_0 < i_0$ in $\sigma_n([n])$, and it follows from the above reasoning that $(j_0, i_0) \in W \setminus \Delta$. This contradicts condition (2). $\square$

Define a permutation operator Q_{σ_n}:

$$Q_{\sigma_n}(e_i) = e_{\sigma_n(i)},\ \ i = 1, \dots, n.$$

THEOREM 10. *Let the matrix* $\mathbf{A}$ *of a linear operator* $A : \mathbb{R}^n \to \mathbb{R}^n$ *be nonnegative, and let its second compound matrix* $\mathbf{A}^{(2)}$ *be JS. Let* $W \subset [n] \times [n]$*, defined by (4), be transitive. Then there exists a permutation operator* Q_{σ_n} *such that the matrix* $\mathbf{P} = \mathbf{Q}_{\sigma_n}^T \mathbf{A} \mathbf{Q}_{\sigma_n}$ *is 2-TP. Moreover, if* $\mathbf{A}$ *and* $\mathbf{A}^{(2)}$ *are irreducible, the* $\mathbf{P}$ *and* $\mathbf{P}^{(2)}$ *are also irreducible.*

Proof. Define σ_n as in the proof of Lemma 6. Notice that $p_{ij} = a_{\sigma_n(i)\sigma_n(j)}$. The matrix $\mathbf{P} = \mathbf{Q}_\theta^T \mathbf{A} \mathbf{Q}_\theta$ is obviously nonnegative. Prove that $\mathbf{P}^{(2)}$ is nonnegative. Examine an arbitrary minor $P\begin{pmatrix} i & j \\ k & l \end{pmatrix}$, where $i < j$, $k < l$. It is equal to the generalized minor $A\begin{pmatrix} \sigma_n(i) & \sigma_n(j) \\ \sigma_n(k) & \sigma_n(l) \end{pmatrix}$.

It follows from the construction of σ_n that $(\sigma_n(i), \sigma_n(j)) \in W$ if and only if $\sigma_n^{-1}\sigma_n(i) \leqslant \sigma_n^{-1}\sigma_n(j)$. So the inequalities $i < j$, $k < l$ imply $(\sigma_n(i), \sigma_n(j)), (\sigma_n(k), \sigma_n(l))$

$\in W$. Hence the minor $A\begin{pmatrix}\sigma_n(i) & \sigma_n(j)\\ \sigma_n(k) & \sigma_n(l)\end{pmatrix}$ is an element of the W–matrix $\mathbf{A}_W^{(2)}$. So the matrix $\mathbf{P}^{(2)}$ coincides (up to a permutation of coordinates) with $\mathbf{A}_W^{(2)}$. Applying Theorem 9 to $\mathbf{A}_W^{(2)}$, we get that $\mathbf{A}_W^{(2)}$ is nonnegative and irreducible. $\square$

Note that Theorem 10 may not hold if W is not transitive.

10. Approximation of a 2-TJS matrix by 2-STJS matrices

Let us prove the generalization of Theorem C using Theorem 10.

Given a 2-TJS matrix $\mathbf{A}$, we find two sets $J \subseteq [n]$ and $J_2 \subseteq [\binom{n}{2}]$ from Definition 2 for the matrices $\mathbf{A}$ and $\mathbf{A}^{(2)}$, respectively.

Given the sets J and J_2, we construct a set $W(J,J_2) \subseteq [n]\times[n]$: a pair of indices $(i,j) \in W(J,J_2)$ if and only if one of the following four cases occurs:

(a) $i<j$, $i,j \in J$ or $i,j \in J^c$, and $\alpha(i,j) \in J_2$;

(b) $i<j$, $i \in J$, $j \in J^c$ or $j \in J$, $i \in J^c$, and $\alpha(i,j) \in J_2^c$;

(c) $i>j$, $i,j \in J$ or $i,j \in J^c$, and $\alpha(j,i) \in J_2^c$;

(d) $i>j$, $i \in J$, $j \in J^c$ or $j \in J$, $i \in J^c$, and $\alpha(j,i) \in J_2$.

Note that since J and J_2 are not uniquely determined, the set $W(J,J_2)$ is also not uniquely determined.

Let us prove the following statement.

THEOREM 11. *Let $\mathbf{A}$ be a 2-TJS matrix. Let at least one of the possible $W(J,J_2)$ be transitive. Then there exists a sequence $\{\mathbf{A}_n\}$ of 2-STJS matrices which converges to $\mathbf{A}$.*

Proof. Since $\mathbf{A}$ is JS, we can apply Theorem 6:

$$\mathbf{A} = \mathbf{D}\widetilde{\mathbf{A}}\mathbf{D}^{-1}, \tag{5}$$

where $\widetilde{\mathbf{A}}$ is a nonnegative matrix. Examine the second compound matrix $\mathbf{A}^{(2)}$. It follows from Properties 1 and 2 of $\wedge^2 A$ that the matrix $\mathbf{A}^{(2)}$ can be represented in the form:

$$\mathbf{A}^{(2)} = \mathbf{D}^{(2)}\widetilde{\mathbf{A}}^{(2)}(\mathbf{D}^{-1})^{(2)}.$$

The equality $(\mathbf{D}^{-1})^{(2)} = (\mathbf{D}^{(2)})^{-1}$ implies

$$\mathbf{A}^{(2)} = \mathbf{D}^{(2)}\widetilde{\mathbf{A}}^{(2)}(\mathbf{D}^{(2)})^{-1}.$$

Hence $\widetilde{\mathbf{A}}^{(2)}$ can be written as

$$\widetilde{\mathbf{A}}^{(2)} = (\mathbf{D}^{(2)})^{-1}\mathbf{A}^{(2)}\mathbf{D}^{(2)}. \tag{6}$$

Since both matrices $\mathbf{D}^{(2)}$ and $(\mathbf{D}^{(2)})^{-1}$ are diagonal and the matrix $\mathbf{A}^{(2)}$ is JS, the matrix $\widetilde{\mathbf{A}}^{(2)}$ is also JS. Given a JS matrix $\widetilde{\mathbf{A}}^{(2)}$, we construct W, according to (4). Let us show that the obtained set W coincides with $W(J,J_2)$. Applying Theorem 6 to $\mathbf{A}^{(2)}$, we get:

$$\mathbf{A}^{(2)} = \widehat{\mathbf{D}}\widehat{\mathbf{A}}^{(2)}\widehat{\mathbf{D}}^{-1},$$

where $\widehat{\mathbf{A}}^{(2)}$ is a nonnegative $\binom{n}{2} \times \binom{n}{2}$ matrix, $\widehat{\mathbf{D}}$ is a diagonal matrix. The following equality follows from (6):

$$\widetilde{\mathbf{A}}^{(2)} = (\mathbf{D}^{(2)})^{-1}\widehat{\mathbf{D}}\widehat{\mathbf{A}}^{(2)}\widehat{\mathbf{D}}^{-1}\mathbf{D}^{(2)}. \tag{7}$$

Write equality (7) in the following form:

$$\widetilde{\mathbf{A}}^{(2)} = \widetilde{\mathbf{D}}\widehat{\mathbf{A}}^{(2)}\widetilde{\mathbf{D}}^{-1},$$

where $\widetilde{\mathbf{D}} = (\mathbf{D}^{(2)})^{-1}\widehat{\mathbf{D}}$. Since $\mathbf{D}^{(2)}$ is a diagonal matrix with diagonal elements equal to ± 1, we have $(\mathbf{D}^{(2)})^{-1} = \mathbf{D}^{(2)}$ and $\widetilde{\mathbf{D}} = \mathbf{D}^{(2)}\widehat{\mathbf{D}}$.

For the JS matrix $\widetilde{\mathbf{A}}^{(2)}$ we define the set $\widetilde{J_2}$ as in the proof of Theorem 6:

$$\widetilde{J_2} = \left\{ i \in \left[\binom{n}{2}\right] \; : \; \operatorname{sign}(\widetilde{d_{ii}}) = -1 \right\}.$$

The equality $\widetilde{d}_{\alpha\alpha} = d^{(2)}_{\alpha\alpha}\widehat{d}_{\alpha\alpha}$ for the elements of $\widetilde{\mathbf{D}}$ holds for all $\alpha = 1, \ \ldots, \ \binom{n}{2}$. The elements $d^{(2)}_{\alpha\alpha}$ of the matrix $\mathbf{D}^{(2)}$ are defined by the set J:

$$d^{(2)}_{\alpha\alpha} := \begin{cases} -1, \text{ if for } (i,j), \text{ such that } \alpha = \alpha(i,j) \text{ we have } i \in J, j \in J^c \text{ or } i \in J^c, j \in J; \\ 1, \;\; \text{if for } (i,j), \text{ such that } \alpha = \alpha(i,j) \text{ we have } i \in J, j \in J \text{ or } i \in J^c, j \in J^c. \end{cases}$$

The elements $\widehat{d}_{\alpha\alpha}$ of $\widehat{\mathbf{D}}$ are defined by the set J_2:

$$\widehat{d}_{\alpha\alpha} := \begin{cases} -1, \text{ if } \alpha \in J_2; \\ 1, \;\; \text{if } \alpha \in J_2^c. \end{cases}$$

Hence $\alpha \in \widetilde{J_2}$ if and only if one of the following two cases occurs:

(a) for (i,j) such that $\alpha = \alpha(i,j)$ we have $i \in J, j \in J$ or $i \in J^c, j \in J^c$, and $\alpha \in J_2$;

(b) for (i,j) such that $\alpha = \alpha(i,j)$ we have $i \in J, j \in J^c$ or $i \in J^c, j \in J$, and $\alpha \in J_2^c$.

Now (4) shows that the set W constructed from $\widetilde{J_2}$ coincides with $W(J,J_2)$.

Since $W(J,J_2)$ is transitive, so is W, and we apply Theorem 10 to the nonnegative matrix $\widetilde{\mathbf{A}}$ with a JS second compound matrix $\widetilde{\mathbf{A}}^{(2)}$. We get that for some permutation σ_n the matrix $\mathbf{P} = \mathbf{Q}^T_{\sigma_n}\widetilde{\mathbf{A}}\mathbf{Q}_{\sigma_n}$ is 2-TP. Applying Theorem C, we find a sequence of 2-STP matrices $\{\mathbf{P}_n\}_{n=1}^{\infty}$, which converges to $\mathbf{P}$. We construct the sequence $\{\mathbf{A}_n\}$ via the rule $\mathbf{A}_n = \mathbf{D}\mathbf{Q}_{\sigma_n}\widetilde{\mathbf{A}_n}\mathbf{Q}^T_{\sigma_n}\mathbf{D}^{-1}$, where $\mathbf{D}$ is a diagonal matrix from (5). It follows from Theorem 4 that the matrices $\mathbf{A}_n$ are 2-STJS for any $n = 1, 2, \ldots$. Finally, it is easy to see that the sequence $\{\mathbf{A}_n\}$ converges to the matrix $\mathbf{A}$. $\square$

The proof of Theorem 12 follows from Theorem 11 and from the continuity of eigenvalues.

Note that if $W(J,J_2)$ is not transitive, then the approximation of a 2-TJS matrix by 2-STJS matrices is not always possible, and the statement of Theorem 12 may not hold.

11. Proofs

Proof of Theorem 13. Enumerate the eigenvalues of the operator A, repeated according to their multiplicity, in decreasing order of their absolute values:

$$|\lambda_1| \geqslant |\lambda_2| \geqslant \cdots \geqslant |\lambda_n|.$$

Let us examine the first case when $W(J,J_2)$ is transitive. The positivity of λ_1 and the nonnegativity of λ_2 is proved analogously to the proof of Theorem 8. Applying Corollary 2 to A, we get that $\rho(A)$ is a simple eigenvalue of A.

Now let us examine the second case when all the possible $W(J,J_2)$ are not transitive. As usual, $h(A)$ denotes the index of imprimitivity of A. Assume that $h(A) = 2q$, where q is a positive integer. Applying Corollary 2 to A we obtain that A has a simple positive eigenvalue $\lambda_1 = \rho(A) > 0$, all the eigenvalues of the operator A equal in absolute value to $\rho(A)$ are simple and they can be written as $\lambda_j = \rho(A)e^{\frac{\pi(j-1)i}{q}}$ $(j = 1,\ldots,2q)$.

Let $h(A) = 2$. Then there are two eigenvalues $\rho(A) > 0$ and $-\rho(A)$ on the spectral circle $|\lambda| = \rho(A)$. Hence there is only one negative eigenvalue $-\rho^2(A)$ on the spectral circle $|\lambda| = \rho(\wedge^2 A)$ of the operator $\wedge^2 A$. This fact contradicts Theorem 7.

Theorem 2 implies that all the eigenvalues equal in absolute value to $\rho(\wedge^2 A)$ can be written as $\lambda_j\lambda_m = \rho^2(A)e^{\frac{\pi(j-1)i}{q}}e^{\frac{\pi(m-1)i}{q}}$, where $1 \leqslant j < m \leqslant 2q$. Thus there are exactly $\binom{2q}{2}$ eigenvalues (taking into account their multiplicities) on the spectral circle $|\lambda| = \rho(\wedge^2 A)$. The equality

$$\rho^2(A) = \rho^2(A)e^{\frac{\pi i}{q}}e^{\frac{\pi(2q-1)i}{q}} = \rho^2(A)e^{\frac{2\pi i}{q}}e^{\frac{\pi(2q-2)i}{q}} = \ldots = \rho^2(A)e^{\frac{\pi(q-1)i}{q}}e^{\frac{\pi(q+1)i}{q}}$$

shows that the algebraic multiplicity of $\rho(\wedge^2 A) = \rho^2(A)$ is equal to $q-1$.

Applying Theorems 6 and 7 to $\wedge^2 A$ we obtain, that the algebraic multiplicity of any eigenvalue λ of $\wedge^2 A$ with $|\lambda| = \rho(\wedge^2 A)$ does not exceed the algebraic multiplicity of $\rho(\wedge^2 A)$. Since all eigenvalues on $|\lambda| = \rho(\wedge^2 A)$ coincide with all the $2q$th roots of $(\rho(A))^{2q}$, we have $2q$ different eigenvalues with the greatest multiplicity $q-1$. Thus the common number of eigenvalues on $|\lambda| = \rho(\wedge^2 A)$ taking into account their multiplicities is not greater than $2q(q-1)$. We came to the contradiction because $2q(q-1) < \binom{2q}{2}$. $\square$

Now let us assume the irreducibility of $\mathbf{A}^{(2)}$.

Proof of Theorem 14. Enumerate the eigenvalues of the operator A, repeated according to their multiplicity, in decreasing order of their absolute values:

$$|\lambda_1| \geqslant |\lambda_2| \geqslant \cdots \geqslant |\lambda_n|.$$

Let us examine the first case when $W(J,J_2)$ is transitive. The equality $h(A)=1$ follows from Theorem 12. The positivity of λ_1 and λ_2 is proved analogously to the proof of Theorem 8. Applying Corollary 2 to A and $\wedge^2 A$, we get that $\rho(A)$ and $\rho(\wedge^2 A)$ are simple eigenvalues of A and $\wedge^2 A$ respectively. Then the equality $\lambda_2 = \frac{\rho(\wedge^2 A)}{\rho(A)}$ implies that λ_2 is a simple eigenvalue of A. If $h(A) = h(\wedge^2 A) = 1$, then λ_2 is obviously different from the other eigenvalues. If $h(\wedge^2 A) > 1$, the equality $\lambda_j = \frac{\rho(\wedge^2 A) e^{\frac{2\pi(j-1)i}{h(\wedge^2 A)}}}{\rho(A)}$, where $j = 2, \ldots, h(\wedge^2 A)+1$ follows from Theorem 2 and Corollary 2.

Now let us examine the second case when $W(J,J_2)$ is not transitive. We prove that $h(A) = h(\wedge^2 A) = 3$ by contradiction, excluding all the possible values $h(A)$, except for $h(A) = 3$.

Applying Theorem 6, we get

$$\mathbf{A} = \mathbf{D}\widetilde{\mathbf{A}}\mathbf{D}^{-1},$$

where $\widetilde{\mathbf{A}}$ is a nonnegative irreducible matrix, $\mathbf{D}$ is a diagonal matrix. Then

$$\mathbf{A}^{(2)} = \mathbf{D}^{(2)}\widetilde{\mathbf{A}}^{(2)}(\mathbf{D}^{(2)})^{-1}.$$

The above equality implies that $\widetilde{\mathbf{A}}^{(2)}$ is irreducible JS. Applying Theorem 9 to $\widetilde{\mathbf{A}}^{(2)}$, we get that the matrix $\widetilde{\mathbf{A}}^{(2)}_W$ where $W = W(J,J_2)$ is nonnegative and irreducible.

Suppose $h(A) = 1$. Applying Theorem 5 to the matrix $\widetilde{\mathbf{A}}$, we get that the operator A has the first positive simple eigenvalue $\lambda_1 = \rho(A) > 0$, with the corresponding positive eigenvector x_1. Applying the Frobenius theorem to the matrix $\widetilde{\mathbf{A}}^{(2)}_W$, which is also nonnegative and irreducible, we get that $\rho(\wedge^2 A)$ is a simple positive eigenvalue of $\wedge^2 A$, with the corresponding positive eigenvector φ.

Since λ_1 is different in absolute value from the other eigenvalues and since $\rho(\wedge^2 A)$ is simple, Theorem 2 shows that $\rho(\wedge^2 A) = \lambda_1\lambda_m$ for some unique value $m > 1$. Without loss of generality, we can assume that $m = 2$, i.e., $\rho(\wedge^2 A) = \lambda_1\lambda_2$. Then $\varphi = x_1 \wedge x_2$, where x_1 is the positive eigenvector corresponding to λ_1 and x_2 is the eigenvector corresponding to λ_2. Let us examine the coordinates of the vector φ in the corresponding W–basis. Since W is not transitive, there exists at least one triple of indices $i,j,k \in [n]$ for which the inclusions $(i,j),\ (j,k) \in W$, $(i,k) \in W^s$ hold. In this case the coordinates of $\varphi = x_1 \wedge x_2$ in the corresponding W–basis satisfy the following inequalities:

$$\varphi_{\alpha(i,j)} = x_i^1 x_j^2 - x_j^1 x_i^2 > 0;$$

$$\varphi_{\alpha(j,k)} = x_j^1 x_k^2 - x_k^1 x_j^2 > 0;$$

$$\varphi_{\alpha(k,i)} = x_k^1 x_i^2 - x_i^1 x_k^2 > 0.$$

(Here x_i^l, x_j^l, x_k^l are the coordinates of the vectors x_l, $l = 1,2$.) Adding the first two expressions multiplied by $x_k^1 > 0$ and $x_i^1 > 0$ respectively, we get:

$$x_j^1(x_i^1 x_k^2 - x_k^1 x_i^2) > 0;$$

$$x_k^1 x_i^2 - x_i^1 x_k^2 > 0.$$

This system has no solutions. So the case of $h(A) = 1$ is excluded.

Let $h(A) = 2$. Then there are two eigenvalues $\rho(A) > 0$ and $-\rho(A)$ on the spectral circle $|\lambda| = \rho(A)$ of the operator A. Hence there is only one negative eigenvalue $-\rho^2(A)$ on the spectral circle $|\lambda| = \rho(\wedge^2 A)$ of the operator $\wedge^2 A$. This fact contradicts Corollary 2.

It remains to exclude the case of $h(A) > 3$. Since all eigenvalues of the operator A on the spectral circle $|\lambda| = \rho(A)$ can be written in the form $\lambda_j = \rho(A) e^{\frac{2\pi(j-1)i}{h(A)}}$ $(j = 1, \ldots, h(A))$, Theorem 2 implies:

$$\lambda_2 \lambda_{h(A)} = \lambda_3 \lambda_{h(A)-1} = \cdots = \lambda_k \lambda_{h(A)-(k-2)} = \cdots = \rho^2(A).$$

Hence the eigenvalue $\rho(\wedge^2 A) = \rho^2(A)$ of the operator $\wedge^2 A$ is not simple. This fact also contradicts Corollary 2.

Finally prove that $h(\wedge^2 A) = 3$ when $h(A) = 3$. Indeed, in this case there are exactly three eigenvalues $\lambda_1 = \rho(A)$, $\lambda_2 = \rho(A) e^{\frac{2\pi i}{3}}$, $\lambda_3 = \rho(A) e^{\frac{4\pi i}{3}}$ on the spectral circle $|\lambda| = \rho(A)$, and there are also exactly three eigenvalues $\lambda_1 \lambda_2 = \rho^2(A) e^{\frac{2\pi i}{3}}$, $\lambda_1 \lambda_3 = \rho^2(A) e^{\frac{4\pi i}{3}}$ and $\lambda_2 \lambda_3 = \rho(A) e^{\frac{2\pi i}{3}} \rho(A) e^{\frac{4\pi i}{3}} = \rho^2(A)$ on the spectral circle $|\lambda| = \rho(\wedge^2 A)$. $\square$

COROLLARY 3. *If the matrix* $\mathbf{A}$ *of a linear operator* $A : \mathbb{R}^n \to \mathbb{R}^n$ *is 2-STJS, then the set* $W(J, J_2)$ *is transitive.*

Let us give the examples illustrating both cases of Theorem 14.

EXAMPLE 3. Let the operator $A : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by the matrix

$$\mathbf{A} = \begin{pmatrix} 8.5 & 0 & 6.1 \\ -5.6 & 3.2 & -7.4 \\ 6 & -2.8 & 6.6 \end{pmatrix}.$$

This matrix is irreducible JS with $J = \{1, 3\}$.

In this case the second compound matrix is the following:

$$\mathbf{A}^{(2)} = \begin{pmatrix} 27.2 & -28.74 & -19.52 \\ -23.8 & 19.5 & 17.08 \\ -3.52 & 7.44 & 0.4 \end{pmatrix}.$$

The matrix $\mathbf{A}^{(2)}$ is also irreducible JS with $J_2 = \{2, 3\}$.

Examine the set $W(J, J_2)$. We have

$(1,2) \in W(J, J_2)$, since $1 < 2$, $1 \in J$, $2 \in J^c$, and $\alpha(1,2) = 1 \in J_2^c$;

$(1,3) \in W(J, J_2)$, since $1 < 3$, $1, 3 \in J$, and $\alpha(1,3) = 2 \in J_2$;

$(3,2) \in W(J, J_2)$, since $3 > 2$, $3 \in J$, $2 \in J^c$, and $\alpha(2,3) = 3 \in J_2$.

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Illustration 1. The set $W(J,J_2)$.

Applying Lemma 6, we get that $W(J,J_2)$ defines the linear order $1 \prec 3 \prec 2$ on $[3]$. The operator A satisfies the conditions of Theorem 14, case (1). The two largest eigenvalues of A are $\lambda_1 = \rho(A) = 15.102$ and $\lambda_2 = 3.53642$; all other eigenvalues have smaller absolute values.

EXAMPLE 4. Let the operator $A : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

This matrix is obviously nonnegative and irreducible.

In this case the second compound matrix is the following:

$$\mathbf{A}^{(2)} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}.$$

The matrix $\mathbf{A}^{(2)}$ is irreducible JS with $J_2 = \{1,\ 3\}$. Examine the set W, corresponding to the set of indices $J_2 = \{1,\ 3\}$. It consists of the pairs $(1,2)$, $(2,3)$ and $(3,1)$ (see Illustration 2).

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Illustration 2. The set W.

The set W defines the non-transitive binary relation $1 \prec 2,\ 2 \prec 3,\ 3 \prec 1$ on the set of the indices $[3]$. The operator A satisfies the conditions of Theorem 14, case (2). Then $\lambda = \rho(A) = 1$, and there are exactly three eigenvalues 1, $e^{\frac{2\pi i}{3}}$ and $e^{\frac{4\pi i}{3}}$ on the spectral circle $|\lambda| = 1$, all of which are simple and coincide with 3th roots of unity.

The proof of Theorem 15 follows from Lemma 5.

Proof of Theorem 16. Applying Theorems 6 and 7 we obtain block representation (3) of the matrix $\mathbf{A}$. We consider only those blocks $\mathbf{A}_j$ with $\rho(A_j) = \rho(A)$. The number of such blocks is equal to the algebraic multiplicity m of $\rho(A)$. Every square submatrix $\mathbf{A}_j$ $(j = 1, \ldots, m)$ is obviously irreducible 2-TJS. Applying Theorem 13 to every $\mathbf{A}_j$, we obtain that there is an odd number $k_j \geqslant 1$ of eigenvalues on the spectral circle $|\lambda| = \rho(A_j)$. Each eigenvalue is simple and they coincide with the k_j-th roots of $(\rho(A))^{k_j}$. The equality

$$\sigma_p(A) = \bigcup_j \sigma_p(A_j)$$

completes the proof. $\square$

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