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APPROXIMATE INNER PRODUCTS ON HILBERT C^*–MODULES; A FIXED POINT APPROACH

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Abstract. We present a fixed point method to investigate the stability and superstability of inner products on Banach modules over a C^*-algebra. Moreover, we show that under some conditions on approximate inner product, the Banach modules over a C^*-algebra has a Hilbert C^*-module structure.

1. Introduction

The study of stability problems for functional equations is strongly related to the following question of S. M. Ulam [26] concerning the stability of group homomorphisms:

Let G_1 be a group and let G_2 be a metric group with the metric d. Given $\varepsilon > 0$ does there exist a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then a homomorphism $H : G_1 \to G_2$ exists with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

D. H. Hyers[12] gave the first affirmative answer to the question of Ulam, for Banach spaces. Th. M. Rassias in [24] and Z. Gajda in [10] considered the stability problem with unbounded Cauchy differences. For more details about the results concerning such problems, the reader refer to [3, 4, 9, 11, 16, 18, 21]. Subsequently, the stability theory was extended and generalized in several ways (see e.g. [13, 23, 5, 14, 19, 25]).

In 2003 Cădariu and Radu applied the fixed point method to the investigation of the Jensen functional equation [5]. They could present a short and a simple proof (different from the "*direct method*", initiated by Hyers in to the 1941) for the generalized Hyers–Ulam stability of Jensen functional equation [5], for Cauchy functional equation [2].

Hilbert C^*-modules provide a natural generalization of Hilbert spaces arising when the field of scalars $\mathbb{C}$ is replaced by an arbitrary C^*-algebra.

DEFINITION 1.1. Suppose that M is a left A-module over the C^*-algebra A. An inner product is a mapping $\langle .,. \rangle : M \times M \to A$ satisfying the following conditions:

(i) $\langle x,x \rangle \geqslant 0$ for all $x \in M$ and $\langle x,x \rangle = 0$ iff $x = 0$;

(ii) $\langle x,y \rangle = \langle y,x \rangle^*$. for all $x, y \in M$;

(iii) $\langle .,. \rangle$ is A-linear in the first variable.

The pair $(M, \langle .,. \rangle)$ is inner product space, M is said to be a Hilbert C^*-module if M is complete with respect to the norm $\|x\| = \|\langle x,x \rangle\|^{1/2}$.

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Throughout this paper, we assume that M is a left A-module over the C^*-algebra A, $n_0 \in \mathbb{N}$ is a positive integer. Suppose that $\mathbb{T}^1 := \{z \in \mathbb{C} : |z| = 1\}$ and that $\mathbb{T}^1_{\frac{1}{n_o}} := \left\{e^{i\theta} \; ; \; 0 \leqslant \theta \leqslant \frac{2\pi}{n_o}\right\}$. It is easy to see that $\mathbb{T}^1 = \mathbb{T}^1_{\frac{1}{1}}$. Moreover, we suppose that A is a unital C^*- algebra, $U := \{a \in A : \|a\| = 1\}$ and suppose that A^+ is the set of all positive elements of A. Recall that an element a in a C^*- algebra A is positive if and only if there exists $b \in A$ such that $a = b^* \, b$.

In this paper, by using fixed point methods, we prove the Hyers-Ulam-Rassias stability of inner products associated with the following Jensen- type functional equation

$$f(x,2z) = rf\left(\frac{x+y}{r}, z\right) + rf\left(\frac{x-y}{r}, z\right).$$

where r is a fixed positive real number in $(1,\infty)$. For a given mapping $f : M \times M \to A$, we define

$$D_\mu f(x,y,z) = r\mu f\left(\frac{x+y}{r}, z\right) + r\mu f\left(\frac{x-y}{r}, z\right) - f(\mu x, 2z)$$

for all $\mu \in \mathbb{T}^1_{\frac{1}{n_o}}$ and all $x, y \in M$. Moreover, under some conditions on f, the left A-module M has a Hilbert C^*-module structure. For more details about the results concerning such problems, the reader refer to works of Chmielin'ski (see for example [1, 6, 7, 8]).

We recall the fundamental result in fixed point theory.

THEOREM 1.2. ([20, 22]) *Suppose that we are given a complete generalized metric space* (Ω, d) *and a strictly contractive function* $T : \Omega \to \Omega$ *with Lipschitz constant* L. *Then for each given* $x \in \Omega$, *either*

$$d(T^m x, T^{m+1} x) = \infty \quad \textit{for all } m \geqslant 0,$$

or other exists a natural number m_0 *such that*

- $d(T^m x, T^{m+1} x) < \infty$ *for all* $m \geqslant m_0$;
- *the sequence* $\{T^m x\}$ *is convergent to a fixed point* y^* *of* T;
- y^* *is the unique fixed point of* T *in* $\Lambda = \{y \in \Omega : d(T^{m_0} x, y) < \infty\}$;
- $d(y, y^*) \leqslant \frac{1}{1-L} d(y, Ty)$ *for all* $y \in \Lambda$.

2. Main results

Before proceeding to the main results, we recall the following theorem by R. V. Kadison and G. Pedersen.

THEOREM 2.1. ([15]) *If the element s of a C^*- algebra B has the property that $\|s\| < 1-2n^{-1}$ for some integer greater than 2, then there are n unitary elements $u_1,\cdots,u_n$ in B such that $s = n^{-1}(u_1+u_2+\cdots+u_n)$.*

THEOREM 2.2. *Let $f : M\times M \to A$ be a mapping for which there exists a function $\phi : M\times M\times M \to [0,\infty)$ such that*

$$\|D_\mu f(x,y,z)\| \leqslant \phi(x,y,2z), \tag{2.1}$$
$$\|f(ax,z)-(f(z,x)a^*)^*\| \leqslant \phi(x,0,z), \tag{2.2}$$
$$\lim_n 2^{-n}r^{-n}f(2^nz,r^nx) = \lim_n 2^{-n}r^{-n}(f(r^nx,2^nz))^*, \tag{2.3}$$
$$\lim_n 2^{-n}r^{-n}f(r^nx,2^nx) = 0 \implies x=0 \tag{2.4}$$

for all $\mu\in\mathbb{T}^1_{\frac{1}{n_o}}$, $x,y,z\in M$ and $a\in U$. If there exists a $L<1$ such that $\phi(x,y,z)\leqslant 2rL\phi(\frac{x}{r},\frac{y}{r},\frac{z}{2})$ for all $x,y,z\in M$, then there exists a unique A- bilinear mapping $T : M\times M\to A$ such that

$$\|f(x,z)-T(x,z)\| \leqslant \frac{L}{1-L}\phi(x,0,z) \tag{2.5}$$

for all $x,z\in M$. Moreover, if

$$\lim_n 2^{-n}r^{-n}f(r^nx,2^nx)\in A^+ \tag{2.6}$$

for all $x\in M$, then (M,T) is an inner product space with inner product $\langle x,y\rangle = T(x,y)$ for all $x,y\in M$.

Proof. Put $\mu=1$ and $y=0$ in (2.1), we get

$$\left\|2rf\left(\frac{x}{r},z\right)-f(x,2z)\right\| \leqslant \phi(x,0,2z)$$

for all $x,z\in M$. Hence,

$$\left\|\frac{1}{2r}f(rx,2z)-f(x,z)\right\| \leqslant \frac{1}{2r}\phi(rx,0,2z)\leqslant L\phi(x,0,z) \tag{2.7}$$

for all $x,z\in M$. Consider the set $X := \{g \mid g : M\times M\to A\}$ and introduce the generalized metric on X:

$$d(h,g) := \inf\{C\in\mathbb{R}^+ : \|g(x,z)-h(x,z)\|\leqslant C\phi(x,0,z) \text{ for all } x,z\in M\}.$$

It is easy to show that (X,d) is complete. Now we define mapping $J:X\to X$ by

$$J(h)(x,z)=\frac{1}{2r}h(rx,2z)$$

for all $x,z\in M$. We have

$$\begin{aligned} d(g,h)<C &\Rightarrow \|g(x,z)-h(x,z)\|\leqslant C\phi(x,0,z),\quad \forall x,z\in M\\ &\Rightarrow \left\|\frac{1}{2r}g(rx,2z)-\frac{1}{2r}h(rx,2z)\right\|\leqslant \frac{1}{2r}C\phi(rx,0,2z)\quad \forall x,z\in M\\ &\Rightarrow d(J(g),J(h))\leqslant LC. \end{aligned}$$

for all $g,h\in X$. Therefore, we see that

$$d(J(g),J(h))\leqslant Ld(g,h)$$

for all $g,h\in X$. It follows from (2.7) that

$$d(f,J(f))\leqslant L.$$

Then J has a unique fixed point in the set $X_1:=\{I\in X: d(f,I)<\infty\}$. Let T be the fixed point of J. We have $\lim_n d(J^n(f),T)=0$. It follows that

$$\lim_n \frac{1}{2^n r^n}f(r^n x,2^n z)=T(x,z) \tag{2.8}$$

for all $x,z\in M$. On the other hand, we have $d(f,J(f))\leqslant L$ and $J(T)=T$, then

$$d(f,T)\leqslant d(f,J(f))+d(J(f),J(T))\leqslant L+Ld(f,T).$$

So

$$d(f,T)\leqslant \frac{L}{1-L}.$$

This implies the inequality (2.5). By inequality $\phi(x,y,z)\leqslant 2rL\phi(\frac{x}{r},\frac{y}{r},\frac{z}{2})$, we have

$$\lim_j 2^{-j}r^{-n}\phi(r^j x,r^j y,2^j z)=0 \tag{2.9}$$

for all $x,y,z\in M$. It follows from (2.1), (2.8) and (2.9) that

$$\begin{aligned} &\left\|rT\left(\frac{x+y}{r},z\right)+rT\left(\frac{x-y}{r},z\right)-T(x,2z)\right\|\\ &=\lim_n \frac{1}{2^n}r^{-n}\left\|rf\left(\frac{r^n x+r^n y}{r},2^n z\right)+rf\left(\frac{r^n x-r^n y}{r},2^n z\right)-f(r^n x,2^{n+1}z)\right\|\\ &\leqslant \lim_n \frac{1}{2^n r^n}\phi(2^r x,2^r y,2^n z)=0 \end{aligned}$$

for all $x,y,z\in M$. So

$$rT\left(\frac{x+y}{r},z\right)+rT\left(\frac{x-y}{r},z\right)=T(x,2z)$$

for all $x,y,z \in M$. This shows that

$$T\left(\frac{r(x+y)}{2}, 2z\right) = rT(x,z) + rT(y,z) \tag{2.10}$$

for all $x,y,z \in M$. By putting $a = 1$ in (2.2), we get

$$\|f(x,z) - (f(z,x))^*\| \leqslant \phi(x,0,z)$$

for all $x,z \in M$. Hence

$$\left\| \frac{f(r^n x, 2^n z)}{2^n r^n} - \left(\frac{f(2^n z, r^n x)}{2^n r^n}\right)^* \right\| \leqslant \frac{\phi(r^n x, 0, 2^n z)}{2^n r^n}$$

for all $x,z \in M$. This implies that

$$T(x,z) = (T(z,x))^* \tag{2.11}$$

for all $x,z \in M$. It is easy to see that $T(0,0) = 0$ and $T(x,0) = 0$ for all $x \in M$. Also

$$rT\left(\frac{2x}{r}, z\right) = T(x,2z)$$

for all $x,z \in M$. It follows from $T(x,0) = 0$ and (2.11) that $T(0,x) = 0$ for all $x \in M$. By putting $y := x$ in (2.1), we have

$$\left\| r\mu f\left(\frac{2x}{r}, z\right) + r\mu f(0,z) - f(\mu x, 2z) \right\| \leqslant \phi(x,x,2z)$$

for all $x,z \in M$ and all $\mu \in \mathbb{T}^1_{\frac{1}{n_o}}$. So

$$\begin{aligned}
&\left\| r\mu T\left(\frac{2x}{r}, z\right) - T(\mu x, 2z) \right\| \\
&\quad = \lim_{n\to\infty} \frac{1}{2^n r^n} \left\| r\mu f\left(r^n \frac{2x}{r}, 2^n z\right) + r\mu f(0, 2^n z) - f(r^n \mu x, 2^{n+1} z) \right\| \\
&\quad \leqslant \lim_{n\to\infty} \frac{1}{2^n r^n} \phi(r^n x, r^n x, 2^{n+1} z) = 0
\end{aligned}$$

for all $x,z \in M$ and $\mu \in \mathbb{T}^1_{\frac{1}{n_o}}$. Hence

$$T(\mu x, 2z) = r\mu T\left(\frac{2x}{r}, z\right) \tag{2.12}$$

for all $x,z \in M$ and $\mu \in \mathbb{T}^1_{\frac{1}{n_o}}$. Hence, we get

$$\begin{aligned}
\|T(\mu x, 2z) - \mu T(x,2z)\| &= \left\| r\mu T\left(\frac{2x}{r}, z\right) - \mu T(x,2z) \right\| \\
&= \lim_{n\to\infty} \frac{1}{2^n r^n} \left\| rf\left(r^n \frac{2x}{r}, 2^n z\right) + rf(0, 2^n z) - f(r^n x, 2^{n+1} z) \right\| \\
&\leqslant \lim_{n\to\infty} \frac{1}{2^n r^n} \phi(r^n x, r^n x, 2^{n+1} z) = 0
\end{aligned}$$

for all $x,z \in M$ and $\mu \in \mathbb{T}^1_{\frac{1}{n_o}}$. Hence, $T(\mu x, 2z) = \mu T(x,2z)$ for all $x,z \in M$ and $\mu \in \mathbb{T}^1_{\frac{1}{n_o}}$. If λ belongs to $\mathbb{T}^1$, then there exists $\theta \in [0,2\pi]$ such that $\lambda = e^{i\theta}$. We set $\lambda_1 = e^{\frac{i\theta}{n_o}}$, thus λ_1 belongs to $\mathbb{T}^1_{\frac{1}{n_o}}$. By using (2.12), we have

$$T(\lambda x, 2z) = T(\lambda_1^{n_0} x, 2z) = \lambda_1^{n_0} T(x,2z)$$

for all $x,z \in M$. If λ belongs to $n\mathbb{T}^1 = \{nz\ ;\ z \in \mathbb{T}^1\}$, then for some $n \in \mathbb{N}$, $\lambda = n\lambda_1$ such that $\lambda_1 \in \mathbb{T}^1$, by (2.10) we have

$$\begin{aligned}
T(\lambda x, 2z) &= T(n\lambda_1 x, 2z) = T(\lambda_1(nx), 2z) \\
&= \lambda_1 T(nx, 2z) \\
&= \lambda_1 T\left(\frac{r}{2}\left(\frac{2}{r}x + \frac{2}{r}(n-1)x\right), 2z\right) \\
&= \lambda_1 \left[rT\left(\frac{2}{r}x, z\right) + rT\left(\frac{2}{r}(n-1)x, z\right)\right] \\
&= \lambda_1 [T(x,2z) + T((n-1)x, 2z] \\
&= \lambda_1 \left[T(x,2z) + rT\left(\frac{2}{r}x, z\right) + rT\left(\frac{2}{r}(n-2)x, z\right)\right] \\
&= \lambda_1 [T(x,2z) + (T(x,2z) + T((n-2)x, 2z)] \\
&\ \ \vdots \\
&= n\lambda_1 T(x,2z) = \lambda T(x,2z)
\end{aligned}$$

for all $x,z \in M$. Let $t \in (0,\infty)$ then by archimedean property of $\mathbb{C}$, there exists a positive real number n such that the point $(t,0)$ lies in the interior of circle with center at origin and radius n. Putting $t_1 := t + \sqrt{n^2 - t^2}\, i$, $t_2 := t - \sqrt{n^2 - t^2}\, i$. Then we have $t = \dfrac{t_1 + t_2}{2}$ and $t_1, t_2 \in n\mathbb{T}^1$. It follows from (2.10) that

$$\begin{aligned}
T(tx, 2z) &= T\left(\frac{t_1 + t_2}{2}x, 2z\right) = T\left(t_1\frac{x}{2} + t_2\frac{x}{2}, 2z\right) \\
&= T\left(\frac{r}{2}\left(t_1\frac{2}{r}\frac{x}{2} + t_2\frac{2}{r}\frac{x}{2}\right), 2z\right) \\
&= rT\left(t_1\frac{x}{r}, z\right) + rT\left(t_2\frac{x}{r}, z\right) \\
&= t_1 rT\left(\frac{x}{r}, z\right) + t_2 rT\left(\frac{x}{r}, z\right) \\
&= \frac{t_1 + t_2}{2}\ 2rT\left(\frac{x}{r}, z\right) \\
&= tT(x,2z)
\end{aligned}$$

for all $x,z \in M$. On the other hand, if λ belongs to $\mathbb{C}$ then there exists $\theta \in [0,2\pi]$ such that $\lambda = |\lambda| e^{i\theta}$. Then

$$T(\lambda x, 2z) = T(|\lambda| e^{i\theta} x, 2z) = |\lambda| T(e^{i\theta} x, 2z) = |\lambda| e^{i\theta} T(x,2z) = \lambda T(x,2z) \tag{2.13}$$

for all $x,z \in M$. Hence $T: M\times M \to A$ is homogeneous in the first variable. It follows from (2.10), (2.11) and (2.13) that

$$\begin{aligned} T(x+y,z) &= T\left(\frac{r}{2}\left(\frac{2}{r}x+\frac{2}{r}y\right),z\right) = rT\left(\frac{2}{r}x,\frac{z}{2}\right)+rT\left(\frac{2}{r}y,\frac{z}{2}\right) \\ &= 2T\left(x,\frac{z}{2}\right)+2T\left(y,\frac{z}{2}\right) \\ &= 2\left(T\left(\frac{z}{2},x\right)\right)^*+2\left(T\left(\frac{z}{2},y\right)\right)^* \\ &= (T(z,x))^*+(T(z,y)^* \\ &= T(x,z)+T(y,z) \end{aligned}$$

for all $x,y,z \in M$. This implies that T is additive in the first variable. It follows from (2.2) that

$$\begin{aligned} &\frac{1}{2^n r^n}\|f(r^n ax,2^n z)-(f(2^n z,r^n x)a^*)^*\| \\ &\leqslant \lim_{n\to\infty}\frac{1}{2^n r^n}\phi(r^n ax,0,2^n z)=0 \end{aligned}$$

for all $x,z\in M$ and $a\in U$. Hence, by (2.11) and (2.3)

$$T(ax,z)=(T(x,z)^*a^*)^*=aT(x,z) \tag{2.14}$$

for all $x,z\in M$ and $a\in U$. Now, let $a\in A(a\neq 0)$ and M an integer greater than $4\|a\|$. Then $\|\frac{a}{m}\|<\frac{1}{4}<1-\frac{2}{3}=\frac{1}{3}$. By Theorem (2.1), there exists three elements $a_1,a_2,a_3\in U$ such that $3\frac{a}{m}=a_1+a_2+a_3$. So by (2.14), we have

$$\begin{aligned} T(ax,z) &= T\left(\frac{M}{3}.3\frac{a}{M}x,z\right)=\frac{M}{3}T\left(3\frac{a}{M}x,z\right)=\frac{M}{3}T((a_1+a_2+a_3)x,z) \\ &= \frac{M}{3}[T(a_1x,z)+T(a_2x,z)+T(a_3x,z)] \\ &= \frac{M}{3}(a_1+a_2+a_3)T(x,z) \\ &= aT(x,z) \end{aligned}$$

for all $x,z\in M$. On the other hand by (2.4)

$$T(x,x)=0 \quad\Leftrightarrow\quad x=0$$

for all $x\in M$ and by (2.6), $T(x,x)\in A^+$ for all $x\in M$. Thus $T: M\times M\to A$ is an inner product satisfying (2.8), and (M,T) is Hilbert C^*-module. To prove the uniqueness property of T, let $T': M\times M\to A$ be an other inner product satisfies (2.8), then we have

$$\begin{aligned} \|T(x,z)-T'(x,z)\| &= \lim_{n\to\infty}\left\|\frac{f(r^n x,2^n z)}{2^n r^n}-\frac{T'(r^n x,2^n z)}{2^n r^n}\right\| \\ &\leqslant \lim_{n\to\infty}\frac{1}{2^n r^n}\left(\frac{L}{1-L}\right)\phi(r^n x,0,2^n z)=0 \end{aligned}$$

for all $x,z \in M$. This means that $T = T'$. $\square$

We continue to suppose that M is a Banach left A-module.

COROLLARY 2.3. *Let $p \in (0,1)$ and $\theta \in [0,\infty)$ be real numbers. Let $f : M \times M \to A$ be a mapping satisfying*

$$\|D_\mu f(x,y,z)\| \leqslant \theta(\|x\|^p + \|y\|^p)\ \|z\|^p,$$
$$\|f(ax,z) - (f(z,x)a^*)^*\| \leqslant \theta\|x\|^p\ \|z\|^p,$$
$$\lim_n 2^{-n}r^{-n}f(2^n z, r^n x) = \lim_n 2^{-n}r^{-n}(f(r^n x, 2^n z))^*,$$
$$\lim_m 2^{-m}r^{-m}f(r^m x, 2^m x) = 0 \Longrightarrow x = 0$$

for all $\mu \in \mathbb{T}^1_{\frac{1}{n_o}}$, all $x,y,z \in M$ and all $a \in U$. Then there exists a unique A-bilinear mapping $T : M \times M \to A$ such that

$$\|f(x,z) - T(x,z)\| \leqslant \frac{\theta}{2^{(1-p)}\ r^{(1-p)} - 1}(\|x\|^p\ \|z\|^p)$$

for all $x,z \in M$. Moreover, if

$$\lim_m 2^{-m}r^{-m}f(r^m x, 2^m x) \in A^+$$

for all $x \in M$, then (M,T) is an inner product space with inner product T.

Proof. It follows from Theorem 2.2 by putting $\phi(x,y,z) := \theta\ (\|x\|^p + \|y\|^p)\ \|z\|^p$ for all $x,y,z \in M$, and $L = 2^{p-1}r^{p-1}$. $\square$

We have the following superstability of inner products on Banach A-modules.

COROLLARY 2.4. *Let $p \in (0,\frac{1}{2})$ and $\theta \in [0,\infty)$ be real numbers. Let $f : M \times M \to A$ be a mapping satisfying*

$$\|D_\mu f(x,y,z)\| \leqslant \theta(\|x\|^p\ \|y\|^p\ \|z\|^p),$$
$$f(ax,z) = a\ (f(z,x))^*,$$
$$\lim_m 2^{-m}r^{-m}f(r^m x, 2^m x) = 0 \Longrightarrow x = 0$$

for all $\mu \in \mathbb{T}^1_{\frac{1}{n_o}}$, all $x,y,z \in M$ and all $a \in U$. Moreover, if

$$\lim_m 2^{-m}r^{-m}f(r^m x, 2^m x) \in A^+$$

for all $x \in M$, then f is an inner product on M.

Proof. It follows from Theorem 2.2 by putting $\phi(x,y,z) := \theta\ (\|x\|^p\ \|y\|^p\ \|z\|^p)$ for all $x,y,z \in M$, and $L = 2^{(p-1)}r^{(2p-1)}$. $\square$

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