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MONOTONICITY OF GENERALIZED FURUTA TYPE FUNCTIONS

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Abstract. The monotonicity of generalized Furuta type operator function $F_{s_0}(r,s)=C^{\frac{-r}{2}}(C^{\frac{r}{2}}(A^{\frac{t}{2}}B^pA^{\frac{t}{2}})^sC^{\frac{-r}{2}})^{\frac{(p+t)s_0+r}{(p+t)s+r}}C^{\frac{-r}{2}}$ is discussed via the equivalent relations between operator inequalities. Let $-1\leqslant t<0,\ p\geqslant 1\ (p+t\neq 0)$, $C\geqslant A\geqslant B\geqslant 0$ with $A>0$. It is shown that, for each s_0 such that $\frac{t}{p+t}<s_0$, the function $F_{s_0}(r,s)$ is decreasing for both $r\geqslant -t$ and $s\geqslant \max\{1,s_0\}$. Moreover, some examples are given which imply that, for each $s_0\geqslant 1$ and $r\geqslant -t$, the monotone interval $[s_0,\infty)$ of s in $F_{s_0}(r,s)$ is unique in the interval $[-\frac{r}{p+t},\infty)$.

1. Introduction

Throughout this paper, an operator T means a bounded linear operator on a Hilbert space. The classical Loewner-Heinz inequality (L-H) is stated below.

THEOREM 1.1. (Loewner-Heinz inequality (L-H), [23]) *Let* $p\in[0,1]$, *then* $A\geqslant B\geqslant 0$ *ensures*

$$A^p\geqslant B^p.$$

In general, (L-H) is not true for $p>1$ [23, page 3]. In order to overcome the restraint $p\in[0,1]$ in (L-H), Furuta developed a kind of order preserving operator inequality [4, Theorem 1].

THEOREM 1.2. (Furuta inequality, [4]) *Let* $r\geqslant 0,\ p>0$. *Then* $A\geqslant B\geqslant 0$ *ensures*

$$\left(A^{\frac{r}{2}}A^pA^{\frac{r}{2}}\right)^{\frac{\min\{1,p\}+r}{p+r}}\geqslant\left(A^{\frac{r}{2}}B^pA^{\frac{r}{2}}\right)^{\frac{\min\{1,p\}+r}{p+r}}.$$

$$\left(B^{\frac{r}{2}}B^pB^{\frac{r}{2}}\right)^{\frac{\min\{1,p\}+r}{p+r}}\leqslant\left(B^{\frac{r}{2}}A^pB^{\frac{r}{2}}\right)^{\frac{\min\{1,p\}+r}{p+r}}.$$

Tanahashi [11] proved the optimality of the outer exponent $\min\{1,p\}+r$ in Theorem 1.2.

In [22], the complete form of Furuta inequality was introduced to establish the order structure on Aluthge transform of nonnormal operators.

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THEOREM 1.3. (Complete form, [22]) *Let* $\delta > 0$, $r \geqslant 0$, $p > 0$, $p > p_0 > 0$ *and* $s(\delta) = \min\{p, 2p_0 + \min\{\delta, r\}\}$. *Then* $A \geqslant 0$ *and* $B \geqslant 0$ *such that* $A^{\delta} \geqslant B^{\delta}$ *ensures*

$$(A^{\frac{r}{2}} B^{p_0} A^{\frac{r}{2}})^{\frac{s(\delta)+r}{p_0+r}} \geqslant (A^{\frac{r}{2}} B^{p} A^{\frac{r}{2}})^{\frac{s(\delta)+r}{p+r}}.$$

$$(B^{\frac{r}{2}} A^{p_0} B^{\frac{r}{2}})^{\frac{s(\delta)+r}{p_0+r}} \leqslant (B^{\frac{r}{2}} A^{p} B^{\frac{r}{2}})^{\frac{s(\delta)+r}{p+r}}.$$

We call Theorem 1.3 the complete form of Furuta inequality Theorem 1.2, because the case $p_0 = \delta = 1$ of it implies the essential part ($p > 1$) of Theorem 1.2 by (L-H) for $\frac{1+r}{s(1)+r} \in (0,1]$.

Inspired by Ando-Hiai log majorization, Uchiyama showed a kind of generalized Furuta type inequalities.

THEOREM 1.4. ([13]) *Let* $t \in [-1, 0]$ *and* $p \geqslant 1$. *Then* $C \geqslant A \geqslant B \geqslant 0$ *with* $A > 0$ *ensures the function*

$$F(r,s) = C^{\frac{-r}{2}} \left(C^{\frac{r}{2}} (A^{\frac{t}{2}} B^{p} A^{\frac{t}{2}})^{s} C^{\frac{r}{2}}\right)^{\frac{1+t+r}{(p+t)s+r}} C^{\frac{-r}{2}}$$

is decreasing for both $r \geqslant -t$ *and* $s \geqslant 1$. *In particular, the inequality*

$$C^{1+t+r} \geqslant \left(C^{\frac{r}{2}} (A^{\frac{t}{2}} B^{p} A^{\frac{t}{2}})^{s} C^{\frac{r}{2}}\right)^{\frac{1+t+r}{(p+t)s+r}} \tag{1.1}$$

holds for $r \geqslant -t$ *and* $s \geqslant 1$.

Furuta [5] proved the case $C = A$ of Theorem 1.4 which interpolates the essential part of Theorem 1.2 (as extremal case $t = 0$ in (1.1)) and Ando-Hiai inequality (A-H) [1] (as extremal case $t = -1$ and $r = s$ in (1.1)). See [3, 19] for alternate proofs of Theorem 1.4.

It is known that there are many applications of Furuta type inequalities, we cite [2], [10], [14].

This paper is to consider the generalized Furuta type function

$$F_{s_0}(r,s) = C^{\frac{-r}{2}} (C^{\frac{r}{2}} (A^{\frac{t}{2}} B^{p} A^{\frac{t}{2}})^{s} C^{\frac{r}{2}})^{\frac{(p+t)s_0+r}{(p+t)s+r}} C^{\frac{-r}{2}}.$$

Let $-1 \leqslant t < 0$, $p \geqslant 1$ $(p+t \neq 0)$, $C \geqslant A \geqslant B \geqslant 0$ with $A > 0$. It is shown in section 2 that, for each s_0 such that $\frac{t}{p+t} < s_0$, the function $F_{s_0}(r,s)$ is decreasing for both $r \geqslant -t$ and $s \geqslant \max\{1, s_0\}$ (see Theorem 2.1). In section 3, some examples (Theorem 3.1 and Theorem 3.3) on Furuta type inequalities are given. In particular, it is proved that, for each $s_0 \geqslant 1$ and $r \geqslant -t$, the monotone interval $[s_0, \infty)$ of s in $F_{s_0}(r,s)$ is unique in the interval $[-\frac{r}{p+t}, \infty)$.

2. Monotonicity of $F_{s_0}(r,s)$

Denote $D := (A^{\frac{t}{2}}B^pA^{\frac{t}{2}})^{\frac{1}{p+t}}$.

THEOREM 2.1. (Main result) *Let* $-1 \leqslant t < 0$, $p \geqslant 1$ $(p+t \neq 0)$, $C \geqslant A \geqslant B \geqslant 0$ *with* $A > 0$.

(1) *For each* r *such that* $r \geqslant -t$ *and* s_0 *such that* $s_0 > \frac{-r}{p+t}$, *the function*

$$F_{s_0}(s) = \left(C^{\frac{r}{2}}(A^{\frac{t}{2}}B^pA^{\frac{t}{2}})^sC^{\frac{r}{2}}\right)^{\frac{(p+t)s_0+r}{(p+t)s+r}}$$

is decreasing for $s \geqslant \max\{1, s_0\}$.

(2) *For each* s *such that* $s \geqslant 1$ *and* s_0 *such that* $s_0 < s$, *the function*

$$G_{s_0}(r) = (D^{\frac{(p+t)s}{2}}C^rD^{\frac{(p+t)s}{2}})^{\frac{(p+t)(s-s_0)}{(p+t)s+r}}$$

is increasing for $r \geqslant \max\{-t, -(p+t)s_0\}$.

(3) *For each* s_0 *such that* $\frac{t}{p+t} < s_0$, *the function*

$$F_{s_0}(r,s) = C^{-\frac{r}{2}}\left(C^{\frac{r}{2}}D^{(p+t)s}C^{\frac{r}{2}}\right)^{\frac{(p+t)s_0+r}{(p+t)s+r}}C^{-\frac{r}{2}}$$

is decreasing for both $r \geqslant -t$ *and* $s \geqslant \max\{1, s_0\}$.

We remark that the special case $s_0 = \frac{1+t}{p+t}$ of Theorem 2.1 (3) is just Uchiyama's result Theorem 1.4 (GF).

In order to give a proof, we prepare some results in advance.

For $A \geqslant 0$, A^0 means the projection $P_{(\ker A)^{\perp}}$.

THEOREM 2.2. ([9]) *Let* $r > 0$, $0 \leqslant p_0 < p$, $A \geqslant 0$ *and* $B \geqslant 0$.

(1) *If* $\ker(AB^{\frac{p_0}{2}}) \subseteq \ker B$, *then for each* r, p_0 *and* p, *the following inequalities are equivalent to each other.*

$$\left(B^{\frac{p}{2}}A^rB^{\frac{p}{2}}\right)^{\frac{p-p_0}{r+p}} \geqslant \left(B^{\frac{p}{2}}B^rB^{\frac{p}{2}}\right)^{\frac{p-p_0}{r+p}}. \tag{2.1}$$

$$(A^{\frac{r}{2}}B^{p_0}A^{\frac{r}{2}})^{\frac{p_0+r}{p_0+r}} \geqslant (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{p_0+r}{p+r}}. \tag{2.2}$$

In particular, (2.1) *implies* (2.2) *without the condition* $\ker(AB^{\frac{p_0}{2}}) \subseteq \ker B$.

(2) *For each* r, p_0 *and* p, *the following inequalities are equivalent to each other.*

$$\left(A^{\frac{p}{2}}B^rA^{\frac{p}{2}}\right)^{\frac{p-p_0}{r+p}} \leqslant \left(A^{\frac{p}{2}}A^rA^{\frac{p}{2}}\right)^{\frac{p-p_0}{r+p}}. \tag{2.3}$$

$$(B^{\frac{r}{2}}A^{p_0}B^{\frac{r}{2}})^{\frac{p_0+r}{p_0+r}} \leqslant (B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{p_0+r}{p+r}}. \tag{2.4}$$

The case $p_0 = 0$ of Theorem 2.2 is an extension of [8, Theorem 1], and (2.2) ensures (2.1) is not true without the condition $\ker(AB^{\frac{p_0}{2}}) \subseteq \ker B$ [9, Remark 1].

THEOREM 2.3. ([18]) *Let* $r > 0$, $0 < p_0 < p$, $A \geqslant 0$ *and* $B \geqslant 0$.

(1) *If* $\ker(AB^{\frac{p_0}{2}}) \subseteq \ker B$, *then for each* r, p_0 *and* p, *the following inequalities are equivalent to each other.*

$$\left(B^{\frac{p_0}{2}}A^rB^{\frac{p_0}{2}}\right)^{\frac{p-p_0}{r+p_0}} \geqslant \left(B^{\frac{p_0}{2}}B^rB^{\frac{p_0}{2}}\right)^{\frac{p-p_0}{r+p_0}}. \tag{2.5}$$

$$(A^{\frac{r}{2}}B^{p_0}A^{\frac{r}{2}})^{\frac{p+r}{p_0+r}} \geqslant (A^{\frac{r}{2}}B^{p}A^{\frac{r}{2}})^{\frac{p+r}{p+r}}. \tag{2.6}$$

In particular, (2.5) *implies* (2.6) *without the condition* $\ker(AB^{\frac{p_0}{2}}) \subseteq \ker B$.

(2) *If* $\ker(BA^{\frac{p_0}{2}}) \subseteq \ker A$, *then for each* r, p_0 *and* p, *the following inequalities are equivalent to each other.*

$$\left(A^{\frac{p_0}{2}}B^rA^{\frac{p_0}{2}}\right)^{\frac{p-p_0}{r+p_0}} \leqslant \left(A^{\frac{p_0}{2}}A^rA^{\frac{p_0}{2}}\right)^{\frac{p-p_0}{r+p_0}}. \tag{2.7}$$

$$(B^{\frac{r}{2}}A^{p_0}B^{\frac{r}{2}})^{\frac{p+r}{p_0+r}} \leqslant (B^{\frac{r}{2}}A^{p}B^{\frac{r}{2}})^{\frac{p+r}{p+r}}. \tag{2.8}$$

In particular, (2.7) *implies* (2.8) *without the condition* $\ker(BA^{\frac{p_0}{2}}) \subseteq \ker A$.

Theorem 2.3 can be regarded as a parallel result to Theorem 2.2, and (2.6) ensures (2.5) is not true without the condition $\ker(AB^{\frac{p_0}{2}}) \subseteq \ker B$ [18].

THEOREM 2.4. ([17]) *Let* $\alpha > 0$, $\beta_0 > 0$, $A \geqslant 0$, $B \geqslant 0$. *For* δ *such that* $-\beta_0 < \delta \leqslant \alpha$, *if*

$$(B^{\frac{\beta_0}{2}}A^{\alpha}B^{\frac{\beta_0}{2}})^{\frac{\delta+\beta_0}{\alpha+\beta_0}} \geqslant (resp. \leqslant) B^{\delta+\beta_0}, \tag{2.9}$$

then

$$(B^{\frac{\beta}{2}}A^{\alpha}B^{\frac{\beta}{2}})^{\frac{\delta+\beta}{\alpha+\beta}} \geqslant (resp. \leqslant) B^{\delta+\beta} \tag{2.10}$$

where $\beta \geqslant \beta_0$. *Moreover, for each* $\delta' > -\alpha$, *the function*

$$f(\beta) = (A^{\frac{\alpha}{2}}B^{\beta}A^{\frac{\alpha}{2}})^{\frac{\delta'+\alpha}{\beta+\alpha}}$$

is decreasing (resp. increasing) for $\beta \geqslant \max\{\beta_0, \delta'\}$.

The case $\delta = 0$ of Theorem 2.4 is just Yanagida [16, Proposition 4].

It should be pointed out that, if $\delta = 0$ and $0 < \beta < \beta_0$, the assertion that (2.9) ensures (2.10) is not true [21, Theorem 2.8].

LEMMA 2.5. *Let* $-1 \leqslant t < 0$, $p \geqslant 1$ $(p+t \neq 0)$ *and* $s \geqslant 1$. *Then* $A > 0$ *and* $C \geqslant A \geqslant B \geqslant 0$ *ensures the function*

$$f(s) = \left(C^{-\frac{t}{2}}(A^{\frac{t}{2}}B^{p}A^{\frac{t}{2}})^{s}C^{-\frac{t}{2}}\right)^{\frac{1}{(p+t)s-t}}$$

is decreasing for $s \geqslant 1$. *In particular,*

$$C \geqslant \left(C^{-\frac{t}{2}}(A^{\frac{t}{2}}B^{p}A^{\frac{t}{2}})^{s}C^{-\frac{t}{2}}\right)^{\frac{1}{(p+t)s-t}} \tag{2.11}$$

Lemma 2.5 is the steps (I)-(II) in [19, Proof of Theorem 1.2].

LEMMA 2.6. *Let* $r > 0$, $A \geqslant 0$ *and* $B \geqslant 0$. *Then the following assertion (1) implies (2).*

(1) *There exists an increasing function* $d(t) : (0,\infty) \to (0,\infty)$ *such that, for each* $t_0 > 0$, *if* $t_0 < t \leqslant t_0 + d(t_0)$ *then*

$$(A^{\frac{r}{2}}B^{t_0}A^{\frac{r}{2}})^{\frac{t+r}{t_0+r}} \geqslant (resp. \leqslant)(A^{\frac{r}{2}}B^{t}A^{\frac{r}{2}})^{\frac{t+r}{t+r}}.$$

(2) *There exists an increasing function* $d(p) : (0,\infty) \to (0,\infty)$ *such that, for each* $p_0 > 0$, *if* $p_0 < p$ *then*

$$(A^{\frac{r}{2}}B^{p_0}A^{\frac{r}{2}})^{\frac{\min\{p,p_0+d(p_0)\}+r}{p_0+r}} \geqslant (resp. \leqslant)(A^{\frac{r}{2}}B^{p}A^{\frac{r}{2}})^{\frac{\min\{p,p_0+d(p_0)\}+r}{p+r}}.$$

Lemma 2.6 is an improvement of Step 2 in [22, Proof of Theorem 1.3].

Proof. It is sufficient to prove the case $\geqslant$ for the case $\leqslant$ can be proved in a similar manner. We need to show that the function d in (1) satisfies the conditions of (2).

For each $p_0 > 0$ and $p_0 < p$, if $p \leqslant p_0 + d(p_0)$, then (2) follows by (1) immediately. Suppose $p_n < p \leqslant p_{n+1} = p_n + d(p_n)$ for some positive integer n and $p_1 = p_0 + d(p_0)$. By (1), for $k = 0, 1, \cdots, n-1$, we have

$$\begin{aligned}(A^{\frac{r}{2}}B^{p_k}A^{\frac{r}{2}})^{\frac{p_{k+1}+r}{p_k+r}} &\geqslant (A^{\frac{r}{2}}B^{p_{k+1}}A^{\frac{r}{2}})^{\frac{p_{k+1}+r}{p_{k+1}+r}},\\ (A^{\frac{r}{2}}B^{p_n}A^{\frac{r}{2}})^{\frac{p+r}{p_n+r}} &\geqslant (A^{\frac{r}{2}}B^{p}A^{\frac{r}{2}})^{\frac{p+r}{p+r}}.\end{aligned}$$

Noting that $\frac{p_1+r}{p_{k+1}+r} \in [0,1]$ and $\frac{p_1+r}{p+r} \in [0,1]$, these together with (L-H) deduce that

$$\begin{aligned}(A^{\frac{r}{2}}B^{p_0}A^{\frac{r}{2}})^{\frac{p_1+r}{p_0+r}} &\geqslant (A^{\frac{r}{2}}B^{p_1}A^{\frac{r}{2}})^{\frac{p_1+r}{p_1+r}}\\ &\geqslant \cdots \geqslant (A^{\frac{r}{2}}B^{p_n}A^{\frac{r}{2}})^{\frac{p_1+r}{p_n+r}} \geqslant (A^{\frac{r}{2}}B^{p}A^{\frac{r}{2}})^{\frac{p_1+r}{p+r}}.\end{aligned}$$

Therefore the function d in (1) satisfies the conditions of (2). □

LEMMA 2.7. *Let* $-1 \leqslant t < 0$, $p \geqslant 1$ $(p+t \neq 0)$, $r \geqslant -t$ *and* $C \geqslant A \geqslant B \geqslant 0$ *with* $A > 0$.

(1) *For each* $s_0 \geqslant 1$ *and* $s_0 < s \leqslant 2s_0$*, the inequalities below holds and they are equivalent to each other.*

$$(D^{\frac{(p+t)s_0}{2}} C^r D^{\frac{(p+t)s_0}{2}})^{\frac{(p+t)(s-s_0)}{(p+t)s_0+r}} \geqslant (D^{\frac{(p+t)s_0}{2}} D^r D^{\frac{(p+t)s_0}{2}})^{\frac{(p+t)(s-s_0)}{(p+t)s_0+r}}. \tag{2.12}$$

$$\left(C^{\frac{r}{2}} D^{(p+t)s_0} C^{\frac{r}{2}}\right)^{\frac{(p+t)s+r}{(p+t)s_0+r}} \geqslant \left(C^{\frac{r}{2}} D^{(p+t)s} C^{\frac{r}{2}}\right)^{\frac{(p+t)s+r}{(p+t)s+r}}. \tag{2.13}$$

(2) *Let* $\delta = \min\{(p+t)s, 2(p+t)s_0\}$*, then*

$$\left(C^{\frac{r}{2}} D^{(p+t)s_0} C^{\frac{r}{2}}\right)^{\frac{\delta+r}{(p+t)s_0+r}} \geqslant \left(C^{\frac{r}{2}} D^{(p+t)s} C^{\frac{r}{2}}\right)^{\frac{\delta+r}{(p+t)s+r}}. \tag{2.14}$$

Proof. (1) It is enough to prove (2.12) by Theorem 2.3. By (2.11) and Theorem 2.4 for $s_0 \geqslant 1$ and $r \geqslant -t$, we have

$$C^{1+t-t} \geqslant \left(C^{\frac{-t}{2}} D^{(p+t)s_0} C^{\frac{-t}{2}}\right)^{\frac{1+t-t}{(p+t)s_0-t}},$$
$$C^{1+t+r} \geqslant \left(C^{\frac{r}{2}} D^{(p+t)s_0} C^{\frac{r}{2}}\right)^{\frac{1+t+r}{(p+t)s_0+r}},$$

$$C^r \geqslant \left(C^{\frac{r}{2}} D^{(p+t)s_0} C^{\frac{r}{2}}\right)^{\frac{r}{(p+t)s_0+r}}.$$

This together with the case $p_0 = 0$ of Theorem 2.2 (or [8, Theorem 1]) implies

$$(D^{\frac{(p+t)s_0}{2}} C^r D^{\frac{(p+t)s_0}{2}})^{\frac{(p+t)s_0}{(p+t)s_0+r}} \geqslant (D^{\frac{(p+t)s_0}{2}} D^r D^{\frac{(p+t)s_0}{2}})^{\frac{(p+t)s_0}{(p+t)s_0+r}}.$$

So (2.12) holds by (L-H) for $\frac{s-s_0}{s_0} \in (0,1]$.

(2) follows by (2.13) and Lemma 2.6 easily. □

Proof of Theorem 2.1. It is obvious that (1) is a direct result of Lemma 2.7 (2) and (L-H).

(2) (2.11) in Lemma 2.5 means

$$C^{1+t-t} \geqslant (C^{-\frac{t}{2}} (A^{\frac{t}{2}} B^p A^{\frac{t}{2}})^s C^{-\frac{t}{2}})^{\frac{1+t-t}{(p+t)s-t}}.$$

This together with Theorem 2.4 implies that, for each s_0 such that $s_0 < s$, the function

$$G_{s_0}(r) = (D^{\frac{(p+t)s}{2}} C^r D^{\frac{(p+t)s}{2}})^{\frac{(p+t)(s-s_0)}{(p+t)s+r}}$$

is increasing for $r \geqslant \max\{-t, -(p+t)s_0\}$.

(3) Since $r \geqslant -t$, $s_0 > \frac{t}{p+t} \geqslant \frac{-r}{p+t}$ holds and (1) implies the monotonicity of s in the function $F_{s_0}(r,s)$. On the other hand, assume that B is invertible without loss of generality, then

$$\begin{aligned} F_{s_0}(r,s) =& C^{-\frac{r}{2}} \left(C^{\frac{r}{2}} D^{(p+t)s} C^{\frac{r}{2}}\right)^{\frac{(p+t)s_0+r}{(p+t)s+r}} C^{-\frac{r}{2}} \\ =& D^{\frac{(p+t)s}{2}} (D^{\frac{(p+t)s}{2}} C^r D^{\frac{(p+t)s}{2}})^{\frac{-(p+t)(s-s_0)}{(p+t)s+r}} D^{\frac{(p+t)s}{2}}. \end{aligned}$$

So $F_{s_0}(r,s)$ is decreasing for $r \geqslant \max\{-t, -(p+t)s_0\} = -t$ by (2). □

3. Examples

Furuta [6] showed some concrete counterexamples on Theorem 1.4. Now we gave some counterexamples on Theorem 2.1.

THEOREM 3.1. (Main result) *For each* $-1 \leqslant t < 0$, $p \geqslant 1$ $(p+t \neq 0)$, $r > 0$ *and* $s_0 \geqslant \frac{1+t}{p+t}$. *If* s_1 *and* s_2 *satisfy* $-\frac{r}{p+t} < s_1 < s_2 < s_0$, *then there exist two operators* A *and* B *such that*

$$A \geqslant B > 0,\ \ F_{s_0}(s_1) \ngeqslant F_{s_0}(s_2)(C = A).$$

The case $s_0 \geqslant 1$ of Theorem 3.1 means that the monotone interval $[s_0, \infty)$ in Theorem 2.1 (1) is unique in the interval $[-\frac{r}{p+t}, \infty)$.

To give a proof, we need the following result.

THEOREM 3.2. ([20]) *Denote* $f_\delta(p) := \left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}}\right)^{\frac{\delta+r}{p+r}}$.

(1) *For each* $r > 0$ *and* $\delta > -r$, *if* p_1 *and* p_2 *satisfy* $-r < p_1 < p_2 < \delta$, *then there exist two operators* A *and* B *such that*

$$A \geqslant B > 0,\ \ f_\delta(p_1) \ngeqslant f_\delta(p_2).$$

(2) *For each* $r > 0$ *and* $\delta > -r$, *the monotone interval* $[\max\{\delta, 0\}, \infty)$ *of* p *in* $f_\delta(p)$ *under the order* $\log A \geqslant \log B$ *is unique in the interval* $[-r, \infty)$.

Proof of Theorem 3.1. The proof is inspired by Yamazaki's technique [15]. In the case that $-1 < t < 0$ and $s_0 \geqslant \frac{1+t}{p+t}$, and the case that $t = -1$ and $s_0 > \frac{1+t}{p+t} = 0$, denote $r_1 = \frac{r}{(p+t)s_0}$, $\delta_1 = 1$, $p_1 = \frac{s_1}{s_0}$ and $p_2 = \frac{s_2}{s_0}$. Then $r_1 > 0$, $\delta_1 > -r_1$ and $-r_1 < p_1 < p_2 < \delta_1$ by $-\frac{r}{p+t} < s_1 < s_2 < s_0$. (1) of Theorem 3.2 implies that there exist operators $A_1 > 0$ and $B_1 > 0$ satisfy

$$A_1 \geqslant B_1,\ \ (A_1^{\frac{r_1}{2}} B_1^{p_1} A_1^{\frac{r_1}{2}})^{\frac{1+r_1}{p_1+r_1}} \ngeqslant (A_1^{\frac{r_1}{2}} B_1^{p_2} A^{\frac{r_1}{2}})^{\frac{1+r_1}{p_1+r_1}}.$$

Denote $A = A_1^{\frac{1}{(p+t)s_0}}$, $B = (A_1^{\frac{-t}{2(p+t)s_0}} B_1^{\frac{1}{s_0}} A_1^{\frac{-t}{2(p+t)s_0}})^{\frac{1}{p}}$, then $A \geqslant B$ by Theorem 1.2 and (L-H) for $s_0 \geqslant \frac{1+t}{p+t}$ and $\frac{1}{p} = \frac{1}{p+t-t} \leqslant \frac{\min\{1, \frac{1}{s_0}\} + \frac{-t}{(p+t)s_0}}{\frac{1}{s_0} + \frac{-t}{(p+t)s_0}}$. Meanwhile, it is easy to check that, if $C = A$, then

$$F_{s_0}(s_i) = (A_1^{\frac{r_1}{2}} B_1^{p_i} A_1^{\frac{r_1}{2}})^{\frac{1+r_1}{p_i+r_1}}$$

where $i = 1, 2$. Therefore, Theorem 3.1 follows.

In the case that $t = -1$ and $s_0 = \frac{1+t}{p+t} = 0$, denote $r_1 = r$, $q_1 = 0$, $p_1 = (p-1)s_1$ and $p_2 = (p-1)s_2$. Then $r_1 > 0$, $q_1 > -r_1$ and $-r_1 < p_1 < p_2 < q_1$ by $-\frac{r}{p-1} < s_1 < s_2 < s_0$. By Theorem 3.2 (2), there exist operators $A_1 > 0$ and $B_1 > 0$ satisfy

$$\log A_1 \geqslant \log B_1,\ \ (A_1^{\frac{r_1}{2}} B_1^{p_1} A_1^{\frac{r_1}{2}})^{\frac{r_1}{p_1+r_1}} \ngeqslant (A_1^{\frac{r_1}{2}} B_1^{p_2} A^{\frac{r_1}{2}})^{\frac{r_1}{p_2+r_1}}.$$

Denote $A = A_1$, $B = (A_1^{\frac{1}{2}} B_1^{p-1} A_1^{\frac{1}{2}})^{\frac{1}{p}}$, then $A \geqslant B$ by $\log A_1 \geqslant \log B_1$ and the Furuta inequality under chaotic order ([7, page 139]) for $\frac{1}{p} = \frac{1}{p-1+1}$. Therefore, Theorem 3.1 holds. □

THEOREM 3.3. (Main result) *Let* $-1 \leqslant t < 0$, $p > 1$, $r \geqslant -t$, $s > s_0 > 0$ *and* $\delta' = \min\{(p+t)s, 2(p+t)s_0 + \min\{1+t, r\}\}$. *For each* $\alpha > 1$, *if* $(p+t)s \leqslant 2(p+t)s_0 + \min\{1+t, r\}$, *then there exist operators* $A > 0$ *and* $B > 0$ *satisfy* $A \geqslant B$ *and*

$$\left(A^{\frac{r}{2}}(A^{\frac{t}{2}} B^p A^{\frac{t}{2}})^{s_0} A^{\frac{r}{2}}\right)^{\frac{(\delta'+r)\alpha}{(p+t)s_0+r}} \ngeqslant \left(A^{\frac{r}{2}}(A^{\frac{t}{2}} B^p A^{\frac{t}{2}})^{s} A^{\frac{r}{2}}\right)^{\frac{(\delta'+r)\alpha}{(p+t)s+r}}.$$

Theorem 3.3 implies that the outer exponent $\delta + r$ in (2.14) is optimal when $s_0 < s \leqslant 2s_0$. We prepare some results to prove Theorem 3.3.

THEOREM 3.4. *Let* $r > 0$, $p > 0$, $s(0) = \min\{p, 2p_0\}$, $A > 0$ *and* $B > 0$. *Then* $\log A \geqslant \log B$ *ensures*

$$(A^{\frac{r}{2}} B^{p_0} A^{\frac{r}{2}})^{\frac{s(0)+r}{p_0+r}} \geqslant (A^{\frac{r}{2}} B^{p} A^{\frac{r}{2}})^{\frac{s(0)+r}{p+r}}.$$

Proof. We use Uchiyama's method [12] (see also [7, page 139]). Denote $A_n = 1 + \frac{\log A}{n}$ and $B_n = 1 + \frac{\log B}{n}$. Then for sufficiently large n, by Theorem 1.3 we have $A_n \geqslant B_n$ and

$$(A_n^{\frac{nr}{2}} B_n^{np_0} A_n^{\frac{nr}{2}})^{\frac{s_n(1)+nr}{np_0+nr}} \geqslant (A_n^{\frac{nr}{2}} B_n^{np} A_n^{\frac{nr}{2}})^{\frac{s_n(1)+nr}{np+nr}}$$

where $s_n(1) = \min\{np, 2np_0 + \min\{1, nr\}\}$. Letting $n \to \infty$, The assertion holds by $A_n^n \to A$, $B_n^n \to B$ and $\frac{s_n(1)}{n} \to s(0)$. □

Theorem 3.4 can be regarded as the case $q = 0$ of Theorem 1.3.

THEOREM 3.5. *For each* $\alpha > 1$, $r > 0$ *and* $p > p_0 > 0$, *there exist operators* $A > 0$ *and* $B > 0$ *satisfy*

$$\log A \geqslant \log B, \ \ (A^{\frac{r}{2}} B^{p_0} A^{\frac{r}{2}})^{\frac{(s(0)+r)\alpha}{p_0+r}} \ngeqslant (A^{\frac{r}{2}} B^{p} A^{\frac{r}{2}})^{\frac{(s(0)+r)\alpha}{p+r}}.$$

This result implies that the outer exponent $s(0) + r$ in Theorem 3.4 is optimal.

Proof. If $2p_0 \geqslant p$, then $2p_0 + \min\{q, r\} \geqslant p$ for $q > 0$. By [22, Theorem 3.6], there exist operators $A > 0$ and $B > 0$ satisfy

$$A^q \geqslant B^q, \ (A^{\frac{r}{2}} B^{p_0} A^{\frac{r}{2}})^{\frac{(p+r)\alpha}{p_0+r}} \ngeqslant (A^{\frac{r}{2}} B^{p} A^{\frac{r}{2}})^{\alpha}.$$

So Theorem 3.5 holds because $A^q \geqslant B^q$ implies $\log A \geqslant \log B$.

If $2p_0 < p$, take a sufficiently small q such that $0 < q < \min\{r, p - 2p_0, (2p_0 + r)(\alpha - 1)\}$ and $\alpha_q = \frac{(2p_0+r)\alpha}{2p_0+q+r} > 1$. By [22, Theorem 3.6 (2)], there exist $A > 0$ and $B > 0$ satisfy $A^q \geqslant B^q$ and

$$(A^{\frac{r}{2}} B^{p_0} A^{\frac{r}{2}})^{\frac{(2p_0+q+r)\alpha_q}{p_0+r}} \ngeqslant (A^{\frac{r}{2}} B^{p} A^{\frac{r}{2}})^{\frac{(2p_0+q+r)\alpha_q}{p+r}}.$$

Hence Theorem 3.5 follows. □

Proof of Theorem 3.3. If $(p+t)s \leqslant 2(p+t)s_0 + \min\{1+t,r\}$ and $-1<t<0$, denote $r_1 = \frac{r}{1+t}$, $p_1 = \frac{(p+t)s_0}{1+t}$, $p_2 = \frac{(p+t)s}{1+t}$ and $\delta_1 = \frac{\delta'}{1+t}$. Then $r_1>0$, $p_2>p_1>0$ and $\delta_1 = \min\{p_2, 2p_1+\min\{1,r_1\}\} = p_2$. By [22, Theorem 3.6 (1)], there exist operators $A_1>0$ and $B_1>0$ satisfy

$$A_1 \geqslant B_1, \ \ (A_1^{\frac{r_1}{2}} B_1^{p_1} A_1^{\frac{r_1}{2}})^{\frac{(p_2+r_1)\alpha}{p_1+r_1}} \ngeqslant (A_1^{\frac{r_1}{2}} B_1^{p_2} A_1^{\frac{r_1}{2}})^{\alpha}. \tag{3.1}$$

Take $A = A_1^{\frac{1}{1+t}}$, $B = (A_1^{-\frac{t}{2(1+t)}} B_1^{\frac{p+t}{1+t}} A_1^{-\frac{t}{2(1+t)}})^{\frac{1}{p}}$, then $A \geqslant B$ by Theorem 1.2 for $\frac{p+t}{1+t} \geqslant 1$ and $\frac{1}{p} = \frac{1+\frac{-t}{1+t}}{\frac{p+t}{1+t}+\frac{-t}{1+t}}$. Meanwhile, it is easy to check that

$$\left(A^{\frac{r}{2}} D^{(p+t)s_i} A^{\frac{r}{2}}\right)^{\frac{\delta'+r}{(p+t)s_i+r}} = (A_1^{\frac{r_1}{2}} B_1^{p_i} A_1^{\frac{r_1}{2}})^{\frac{\delta_1+r_1}{p_i+r_1}}$$

where $i=1,2$, $s_1=s_0$ and $s_2=s$. Therefore, Theorem 3.3 follows by (3.1).

If $(p+t)s \leqslant 2(p+t)s_0+\min\{1+t,r\}$ and $t=-1$, then $2(p-1)s_0 \geqslant (p-1)s$ and $r \geqslant 1$. Denote $r_1 = r$, $p_1 = (p-1)s_0$, $p_2 = (p-1)s$ and $\delta_1 = \delta'$. Then $r_1>0$, $p_2>p_1>0$ and $\delta_1 = \min\{p_2, 2p_1\} = p_2$. By Theorem 3.5, there exist operators $A_1>0$ and $B_1>0$ satisfy

$$\log A_1 \geqslant \log B_1, \ \ (A_1^{\frac{r_1}{2}} B_1^{p_1} A_1^{\frac{r_1}{2}})^{\frac{(p_2+r_1)\alpha}{p_1+r_1}} \ngeqslant (A_1^{\frac{r_1}{2}} B_1^{p_2} A^{\frac{r_1}{2}})^{\alpha}. \tag{3.2}$$

Take $A=A_1$, $B = (A_1^{\frac{1}{2}} B_1^{p-1} A_1^{\frac{1}{2}})^{\frac{1}{p}}$, then $A \geqslant B$ by $\log A_1 \geqslant \log B_1$ and the Furuta inequality under chaotic order ([7, page 139]) for $\frac{1}{p} = \frac{1}{p-1+1}$. Therefore, Theorem 3.3 holds by (3.2). □

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