Operators and Matrices
Volume 6, Number 4 (2012), 819–831
doi:10.7153/oam-06-53

LINEAR MAPS STRONGLY PRESERVING MOORE–PENROSE INVERTIBILITY

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(*Communicated by Peter Šemrl*)

Abstract. Let A and B be C^*-algebras. We investigate linear maps from A to B strongly preserving Moore-Penrose invertibility, where A is unital, and either it is linearly spanned by its projections, or has large socle, or has real rank zero (in this last case the map T is assumed to be bounded).

1. Introduction

Let A be a (complex) Banach algebra. An element a in A is (*von Neumann*) *regular* if it has a generalized inverse, that is, if there exists b in A such that $a = aba$ (b is an *inner inverse* of a) and $b = bab$ (b is an *outer inverse* of a). Observe that the first equality $a = aba$ is a necessary and sufficient condition for a to be regular, and that, if a has generalized inverse b, then $p = ab$ and $q = ba$ are idempotents in A with $aA = pA$ and $Aa = Aq$.

The generalized inverse of a regular element a is not unique. For an element a in A let us consider the left and right multiplication operators $L_a : x \mapsto ax$ and $R_a : x \mapsto xa$, respectively. If a is regular, then so are L_a and R_a, and thus their ranges $aA = L_a(A)$ and $Aa = R_a(A)$ are both closed. The *conorm* (or the *reduced minimum modulus*) of an element a in a Banach algebra A, is defined as the reduced minimum modulus of the left multiplication operator by a,

$$\gamma(a) := \gamma(L_a) = \begin{cases} \inf\{\|ax\| : \operatorname{dist}(x, \ker(L_a)) \geqslant 1\} & \text{if } a \neq 0 \\ \infty & \text{if } a = 0. \end{cases}$$

If b is a generalized inverse of a, with $a \neq 0$, then

$$\|b\|^{-1} \leqslant \gamma(a) \leqslant \|ba\| \, \|ab\| \, \|b\|^{-1}$$

Mathematics subject classification (2010): 47B49, 46L05, 47B48.
Keywords and phrases: Linear preservers, C^*-algebras, Moore-Penrose invertibility, Jordan *-homomorphism.

The authors are partially supported by I+D MEC projects no. MTM 2011-23843, MTM 2010-17687 and Junta de Andalucía grants FQM 0199, FQM 0194 and FQM 3737. The second author is supported by a Plan Propio contract from the Universidad de Almería. They wish to thank A. Peralta, A. Kaidi and P. García-Sánchez for their kindly hospitality and help during the preparation of this manuscript, and the referee for their comments and suggestions.

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(see [11, Theorem 2]).

Regular elements in unital C^*-algebras have been studied by Harte and Mbekhta in [10] and [11]. The main results in those papers state that an element a in a C^*-algebra A is regular if and only if aA is closed, equivalently $\gamma(a) > 0$, and that

$$\gamma(a)^2 = \gamma(a^*a) = \inf\{\lambda : \lambda \in \sigma(a^*a) \setminus \{0\}\} = \gamma(a^*)^2.$$

Given a and b in A, b is said to be a *Moore-Penrose inverse* of a if b is a generalized inverse of a and the associated idempotents ab and ba are selfadjoint. It is known that every regular element a in A has a unique Moore-Penrose inverse that will be denoted by $a^\dagger$, and that

$$\gamma(a) = \|a^\dagger\|^{-1}.$$

Denote by $A^\dagger$ the set of regular elements in the C^*-algebra A.

Let A and B be C^*-algebras. We say that a map $T : A \to B$ *strongly preserves Moore-Penrose invertibility* if $T(a^\dagger) = T(a)^\dagger$, for all $a \in A^\dagger$. If A and B are both unital, with identity elements $\mathbf{1}_A$ and $\mathbf{1}_B$ respectively, the mapping T is called *unital* if $T(\mathbf{1}_A) = \mathbf{1}_B$.

In [21] Mbekhta started the study of the so called *Hua type theorems* and *strongly preserver problems* between Banach algebras. In the context of C^*-algebras, he proved that a surjective unital bounded linear map from a real rank zero C^*-algebra to a prime C^*-algebra strongly preserves Moore-Penrose invertibility if and only if it is either a *-homomorphism or a *-antihomomorphism, and he conjectures that the same holds without any assumption on the C^*-algebras and when T is not assumed to be unital. (Linear maps strongly preserving Moore-Penrose invertibility for matrix algebras over some fields were previously considered by Zhang, Cao and Bu, [24].)

Recall that a linear (additive) map $T : A \to B$ between Banach algebras is a *Jordan homomorphism* if $T(a^2) = T(a)^2$, for all $a \in A$, or equivalently, $T(ab+ba) = T(a)T(b)+T(b)T(a)$ for every a,b in A. A bijective Jordan homomorphism is called *Jordan isomorphism*. Clearly every homomorphism and every antihomomorphism is a Jordan homomorphism. It is well known that if $T : A \to B$ is a Jordan homomorphism then

$$T(abc+cba) = T(a)T(b)T(c)+T(c)T(b)T(a), \tag{1}$$

for all $a,b,c \in A$.

If A and B are C^*-algebras, then T is called *selfadjoint* if $T(a^*) = T(a)^*$, for every $a \in A$. Selfadjoint Jordan homomorphism are named *Jordan *-homomorphisms*. Notice that every Jordan *-homomorphism strongly preserves Moore-Penrose invertibility (see Remark 8).

The study of linear maps strongly preserving Moore-Penrose invertibility is connected with regularity linear preserver problems in C^*-algebras. Recall that a linear map $T : A \to B$ between C^*-algebras *preserves regularity* if $T(a) \in B^\dagger$ whenever $a \in A^\dagger$, and that T *preserves regularity in both directions* when $T(a) \in B^\dagger$ if and only if $a \in A^\dagger$.

In [3] the authors showed that every surjective linear map $T : A \to B$ preserving regularity in both directions factorizes as a Jordan isomorphism through the generalized

Calkin algebras, in the case that A and B are prime C^*-algebras with non zero socle and A has real rank zero. Also if A has real rank zero and B has zero socle, it is proved that T preserves regularity if and only if it is a Jordan homomorphism.

Linear maps between C^*-algebras (strongly) preserving the conorm are considered in [4]. In that paper the authors proved that every unital (respectively, surjective) linear map $T: A \to B$, between unital C^*-algebras such that $\gamma(a) = \gamma(T(a))$, for every $a \in A$, is an isometric Jordan *-isomorphism (respectively, multiplied by a unitary element of B). Notice that every linear map strongly preserving the conorm preserves regularity in both directions, and that every bounded linear map $T: A \to B$ strongly preserving Moore-Penrose invertibility satisfies

$$\gamma(a) \leqslant ||T||\gamma(T(a)) \qquad (a \in A).$$

The present manuscript focusses on the study of strongly Moore-Penrose invertibility preservers in C^*-algebras with a rich structure of projections. Section 2 gathers some technical preliminary results. In Section 3 we consider C^*-algebras in which every element is a linear combination of projections. We prove that a linear map strongly preserving Moore-Penrose invertibility $T: A \to B$ between C^*-algebras, is a Jordan *-homomorphism multiplied by a regular element of B commuting with $T(A)$, whenever A is unital and linearly spanned by its projections, or when A is unital and has real rank zero and T is bounded.

Section 4 deals with C^*-algebras having large socle. We show that if $T: A \to B$ is a linear map strongly preserving Moore-Penrose invertibility between C^*-algebras, and A is unital with non zero socle, then T restricted to the socle of A is a Jordan *-homomorphism multiplied by a regular element commuting with the range of T. Also, we prove that every bijective linear map strongly preserving Moore-Penrose invertibility from a unital C^*-algebra with essential socle is a Jordan *-isomorphism multiplied by an involutory element.

Notice that our mappings are never assumed to be unital, and even the codomains do not necessarily have an identity element.

2. Preliminaries

In [21], Mbekhta showed that every unital linear continuous mapping $T: A \to B$ between unital C^*-algebras that strongly preserves Moore-Penrose invertibility is a Jordan homomorphism and preserves projections. In particular, T sends mutually orthogonal projections into mutually orthogonal projections. In the following proposition we generalize this result by showing that every linear mapping between C^*-algebras (not necessarily unital), strongly preserving Moore-Penrose invertibility, preserves orthogonality of regular elements. Recall that two elements a, b in a C^*-algebra A are said to be *orthogonal*, denoted by $a \perp b$, if $ab^* = b^*a = 0$.

PROPOSITION 1. *Let A and B be C^*-algebras, and let $T: A \to B$ be a linear map strongly preserving Moore-Penrose invertibility. Then $a \perp b$ implies $T(a) \perp T(b)$ for all $a, b \in A^\dagger$.*

Proof. Let $a,b \in A^\dagger$ with $a \perp b$. For every $\alpha \in \mathbb{Q} \setminus \{0\}$ it is easy to see that $(a+\alpha b)^\dagger = a^\dagger + \alpha^{-1} b^\dagger$. By assumption,

$$(T(a)+\alpha T(b))(T(a)^\dagger + \alpha^{-1} T(b)^\dagger)(T(a)+\alpha T(b)) = T(a)+\alpha T(b),$$

which yields

$$\begin{aligned}\alpha^{-1} T(a)T(b)^\dagger T(a) &+ (T(a)T(b)^\dagger T(b) + T(b)T(b)^\dagger T(a)) \\ &+ \alpha(T(b)T(a)^\dagger T(a) + T(a)T(a)^\dagger T(b)) \\ &+ \alpha^2 T(b)T(a)^\dagger T(b) = 0,\end{aligned}$$

for every $\alpha \in \mathbb{Q} \setminus \{0\}$. Hence

$$T(a)T(b)^\dagger T(b) + T(b)T(b)^\dagger T(a) = 0.$$

By multiplying the last equation on the right, and on the left, respectively, by $T(b)^\dagger$ it follows that

$$T(a)T(b)^\dagger = -T(b)T(b)^\dagger T(a)T(b)^\dagger, \tag{2}$$

and

$$T(b)^\dagger T(a) = -T(b)^\dagger T(a)T(b)^\dagger T(b). \tag{3}$$

As

$$(T(a)^\dagger + \alpha^{-1} T(b)^\dagger)(T(a)+\alpha T(b))(T(a)^\dagger + \alpha^{-1} T(b)^\dagger) = T(a)^\dagger + \alpha^{-1} T(b)^\dagger$$

for every $\alpha \in \mathbb{Q} \setminus \{0\}$, we get analogously

$$T(b)^\dagger T(a)T(b)^\dagger = 0. \tag{4}$$

From Equations (2), (3) and (4) we deduce that $T(a)T(b)^\dagger = 0$ and $T(b)^\dagger T(a) = 0$. Equivalently, $T(a)T(b)^* = 0$ and $T(b)^* T(a) = 0$, that is, $T(a) \perp T(b)$. □

Let A and B be C^*-algebras. In what follows, let us assume that A is unital with identity element $\mathbf{1}$. It is clear that the zero map strongly preserves Moore-Penrose invertibility. In the next proposition we show that this is the only map strongly preserving Moore-Penrose invertibility that annihilates the identity element.

PROPOSITION 2. *Let A and B be C^*-algebras, and let $T: A \to B$ be a linear map strongly preserving Moore-Penrose invertibility. Then either $T(\mathbf{1}) \neq 0$ or $T = 0$.*

Proof. If b and $\mathbf{1}+b$ are invertible elements in A, then as consequence of Hua's identity (see [13])

$$\mathbf{1} = (\mathbf{1}+b)^{-1} + (\mathbf{1}+b^{-1})^{-1}.$$

Since T strongly preserves Moore-Penrose invertibility,

$$T(\mathbf{1}) = T((\mathbf{1}+b)^{-1}) + T((\mathbf{1}+b^{-1})^{-1}) = (T(\mathbf{1})+T(b))^\dagger + (T(\mathbf{1})+T(b)^\dagger)^\dagger.$$

If we assume that $T(\mathbf{1}) = 0$, then we get

$$T(b)^{\dagger} + T(b) = 0,$$

for every invertible element $b \in A$ with $\mathbf{1} + b$ invertible. Thus, let a be an invertible element in A, and $\alpha \in \mathbb{Q} \setminus \{0\}$ be such that $|\alpha| < ||a||^{-1}$. It is clear that αa and $\mathbf{1} + \alpha a$ are invertible, and therefore $T(a)^{\dagger} = -\alpha^2 T(a)$. By the uniqueness of the Moore-Penrose inverse, it follows that $T(a) = 0$. Thus T is the zero map. $\square$

Notice that if $T : A \to B$ is a non zero linear map strongly preserving Moore-Penrose invertibility, and $T(\mathbf{1})$ commutes with $T(A)$, then $B' = T(\mathbf{1})^2 B T(\mathbf{1})^2$ is a C^*-algebra with identity $T(\mathbf{1})^2$ ($T(\mathbf{1})^2 \neq 0$ in view of the preceding proposition), the map $S = T(\mathbf{1})^2 T$ from A to B' strongly preserves Moore-Penrose invertibility, and $S(\mathbf{1})$ is invertible. A closer look at the arguments employed in Theorem 3.5, Lemma 3.7 and Proposition 3.10 in [5], where the authors only required the Hua's identity and the inner relation of the generalized inverse on invertible elements, reveals that the same reasoning works with Moore-Penrose invertibility. Obviously B' has an identity element even if B is not unital.

PROPOSITION 3. *Let A and B be C^*-algebras, and T be a linear map such that $T(\mathbf{1})$ commutes with the range of T. If T strongly preserves Moore-Penrose invertibility, then $T(\mathbf{1})T$ is a Jordan homomorphism.*

We finish this section with a technical lemma which together with Proposition 3 will be the key tool for the next sections. It describes the behaviour of a linear map strongly preserving Moore-Penrose invertibility with respect to the projections.

LEMMA 4. *Let A and B be C^*-algebras. Let $T : A \to B$ be a linear map strongly preserving Moore-Penrose invertibility. For every projection $p \in A$:*

(a) $T(p)T(\mathbf{1})^* = T(\mathbf{1})T(p)^*$ *and* $T(\mathbf{1})^* T(p) = T(p)^* T(\mathbf{1})$,

(b) $T(p) = T(p)T(\mathbf{1})^2 = T(\mathbf{1})^2 T(p)$,

(c) $T(p)T(\mathbf{1}) = T(\mathbf{1})T(p) = (T(p)T(\mathbf{1}))^*$.

Proof. For the sake of simplicity, write $h = T(\mathbf{1})$. Let p be a non zero projection in A. As $p \perp (\mathbf{1} - p)$, and by Proposition 1, T preserves orthogonality of regular elements, then $T(p) \perp (h - T(p))$, that is, $T(p)h^* = T(p)T(p)^*$ and $h^* T(p) = T(p)^* T(p)$. In particular, $T(p)h^* = hT(p)^*$ and $h^* T(p) = T(p)^* h$. Since $h = h^{\dagger}$, it is clear that $h^3 = h$, and $h^2 = (h^2)^*$. Hence

$$\begin{aligned} T(p)^* T(p) h^2 &= h^* T(p) h^2 = T(p)^* h h^2 = T(p)^* h \\ &= h^* T(p) = T(p)^* T(p). \end{aligned}$$

Again, $T(p) = T(p)^{\dagger}$ gives $T(p)^3 = T(p)$ and thus $T(p)^* T(p)(h^2 - T(p)^2) = 0$. By the cancellation law, $T(p)h^2 = T(p)^3 = T(p)$. In the same way, $T(p) = h^2 T(p)$,

$T(p)^* = h^2T(p)^*$ and $T(p)^*h^2 = T(p)^*$. Also,

$$\begin{aligned} T(p)h &= h^2T(p)h = h^*h^*T(p)h = h^*T(p)^*h^2 = h^*T(p)^* \\ &= (T(p)h)^*. \end{aligned}$$

Analogously, $(hT(p))^* = hT(p)$.

It only remains to prove that $hT(p) = T(p)h$. Since $T(p) = h^2T(p) = T(p)h^2$, it suffices to show that $T(p) = hT(p)h$. Having in mind the uniqueness of the Moore-Penrose inverse, and that $T(p)^\dagger = T(p^\dagger) = T(p)$ we proceed by checking that $hT(p)h$ is the Moore-Penrose inverse of $T(p)$. As $T(p)h = h^*T(p)^*$, $hT(p) = T(p)^*h^*$, $T(p)^*T(p) = h^*T(p)$, $h^2 = (h^2)^*$ and $h^3 = h$, we get

$$\begin{aligned} T(p)(hT(p)h)T(p) &= T(p)hh^*T(p)^*T(p) = T(p)hh^*h^*T(p) \\ &= T(p)hT(p) = T(p)T(p)^*h^* = T(p)(h^*)^2 = T(p). \end{aligned}$$

From this it is clear that, $(hT(p)h)T(p)(hT(p)h) = hT(p)h$, and since

$$T(p)(hT(p)h) = (T(p)h)T(p)h = h^*T(p)^*T(p)h = h^*h^*T(p)h = T(p)h,$$

and similarly $(hT(p)h)T(p) = hT(p)$, are selfadjoint, this shows that $hT(p)h = T(p)^\dagger$, as desired. □

REMARK 5. Note that, as every additive map $T : A \to B$ between Banach algebras is $\mathbb{Q}$-linear, the results in this section also hold if we change the linearity with additivity.

3. C*-algebras linearly spanned by their projections and real rank zero C*-algebras

In many C*-algebras every element can be expressed as a finite linear combination of projections: properly infinite C*-algebras, von Neumann algebras of type II_1, unital simple C*-algebras of real rank zero with no tracial states, unital simple AF C*-algebras with finitely many extremal states, UHF C*-algebras, Bunce-Deddens algebras, irrotational rotation algebras...(See for instance [17], [18], [19], [22], [15] and the references therein.) The following theorem describes linear maps strongly preserving Moore-Penrose invertibility from C*-algebras linearly spanned by their projections. In particular, by [22, Corollary 2.3], it applies to the algebra of all bounded linear operators on a complex infinite dimensional Hilbert space (compare with Theorem 3.3 $(i) \Rightarrow (ii)$ in [21], where T is assumed to be bounded, unital, and bijective).

THEOREM 6. *Let $T : A \to B$ be a linear map strongly preserving Moore-Penrose invertibility between C^*-algebras, where A is unital. Assume that every element of A is a finite linear combination of projections. Then $T(\mathbf{1})T$ is a Jordan *-homomorphism and $T(\mathbf{1})$ commutes with the range of T.*

Proof. From A being linearly spanned by its projections, by Lemma 4 it is clear that $T(x)T(\mathbf{1}) = (T(x^*)T(\mathbf{1}))^* = T(\mathbf{1})T(x)$, for every $x \in A$. The conclusions can be obtained directly by applying Proposition 3. □

Recall that a C^*-algebra A is of *real rank zero* if the set of all real linear combinations of orthogonal projections is dense in the set of all hermitian elements of A (see [7]). Notice that every von Neumann algebra, and, in particular, the algebra of all bounded linear operators on a complex Hilbert space H is of real rank zero.

THEOREM 7. *Let A and B be C^*-algebras, and $T : A \to B$ be a bounded linear map strongly preserving Moore-Penrose invertibility. Suppose that A is unital of real rank zero. Then:*

1. *$T(\mathbf{1})$ commutes with the range of T,*
2. *$T(\mathbf{1})T$ is a Jordan *-homomorphism.*

Proof. As T is continuous and A has real rank zero, the theorem can be proved as the previous one by applying Lemma 4 and Proposition 3. □

REMARK 8. Every Jordan *-homomorphism between C*-algebras strongly preserves Moore-Penrose invertibility. Indeed if $a \in A^\dagger$, and $b = a^\dagger$, from Equation (1) it is clear that $T(a) = T(a)T(b)T(a)$ and $T(b) = T(b)T(a)T(b)$. Thus it remains to show that $T(b)T(a)$ and $T(a)T(b)$ are selfadjoint. As $a = b^*a^*a = aa^*b^*$, in particular $2a = b^*a^*a + aa^*b^*$, and since T is a Jordan *-homomorphism, it is clear that

$$2T(a) = T(b)^*T(a)^*T(a) + T(a)T(a)^*T(b)^*.$$

By multiplying on the left by $T(a)^*$, we get that

$$T(a)^*T(a) = T(a)^*T(a)T(a)^*T(b)^*,$$

or equivalently $T(a)^*T(a)(T(b)T(a) - T(a)^*T(b)^*) = 0$, which implies that $T(a) = T(a)T(a)^*T(b)^*$, and hence $T(b)T(a) = T(b)T(a)T(a)^*T(b)^*$ is selfadjoint.

Moreover if $T : A \to B$ is a Jordan *-homomorphism between C*-algebras and u is a regular element in B such that $u = u^\dagger$, and u commutes with the range if T, it is clear that uT also strongly preserves Moore-Penrose invertibility.

As we have mentioned in the Introduction, in [21, Theorem 3.2] Mbekhta shows that a *surjective unital* continuous linear mapping from a real rank zero unital C*-algebra onto a prime unital C*-algebra is either a *-homomorphism or a *-antihomomorphism if and only if it strongly preserves Moore-Penrose invertibility. In the following result, we characterize bounded linear maps (not necessarily surjective nor unital) strongly preserving Moore-Penrose invertibility on a real rank zero unital C*-algebra.

COROLLARY 9. *Let A and B be C^*-algebras, and let $T : A \to B$ be a bounded linear map. Suppose that A is unital of real rank zero. The following are equivalent:*

1. *T strongly preserves Moore-Penrose invertibility,*
2. *$T(\mathbf{1})^\dagger = T(\mathbf{1})$, $T = ST(\mathbf{1}) = T(\mathbf{1})S$ for a Jordan *-homomorphism S.*

Proof. The implication $(1) \Rightarrow (2)$ is a direct consequence of Theorem 7, and the converse follows from Remark 8. □

4. C*-algebras of large socle

Let A be a C*-algebra. An element x of A is *finite* (*compact*) in A, if the wedge operator $x \wedge x : A \to A$, given by $x \wedge x(a) = xax$, is a finite rank (compact) operator on A. It is known that the ideal $\mathscr{F}(A)$ of finite rank elements in A coincides with the *socle* of A, $\operatorname{soc}(A)$, that is, the sum of all minimal right (equivalently left) ideals of A, and that $\mathscr{K}(A) = \overline{\operatorname{soc}(A)}$ is the ideal of compact elements in A. Every element in the socle of a C*-algebra is a linear combination of minimal projections (a projection p in a C*-algebra A is said to be *minimal* if $pAp = \mathbb{C}p$). We refer to [1, 2, 23], for the basic references on the socle.

It is known that every element in the socle of a C^*-algebra A is regular and that $A^\dagger + \operatorname{soc}(A) \subset A^\dagger$ (see for instance [16, Theorem 6.3]). This fact together with Proposition 1 allow us to employ the techniques on orthogonality preserving maps on C^*-algebras with large socle in order to determine the structure of strongly Moore-Penrose invertibility linear preservers. The following lemma is inspired in [9] (see also [8]). Recall that every C^*-algebra can be endowed with a Jordan product $a \circ b := \frac{1}{2}(ab + ba)$, and a Jordan triple product defined as $\{abc\} := \frac{1}{2}(ab^*c + cb^*a)$.

For the rest of the section A and B are C*-algebras, and $T : A \to B$ is a linear map strongly preserving Moore-Penrose invertibility. We assume that A is unital with non zero socle.

LEMMA 10. *For every $a \in A$ and $x \in \operatorname{soc}(A)$, the following identities hold:*

(a) $2T(a \circ x)T(\mathbf{1})^* = T(a)T(x^*)^* + T(x)T(a^*)^*$ *and*
$2T(\mathbf{1})^*T(a \circ x) = T(x^*)^*T(a) + T(a^*)^*T(x)$,

(b) $T(x)T(\mathbf{1})^*T(a) = T(x)T(a^*)^*T(\mathbf{1})$ *and*
$T(a)T(\mathbf{1})^*T(x) = T(\mathbf{1})T(a^*)^*T(x)$,

(c) $T(x)T(\mathbf{1})T(a) = T(x)T(a^*)^*T(\mathbf{1})^*$ *and*
$T(a)T(\mathbf{1})T(x) = T(\mathbf{1})^*T(a^*)^*T(x)$,

(d) $\{T(x)T(a)T(x)\} = T(\{xax\})T(\mathbf{1})^*T(\mathbf{1})$,

Proof. As above denote $T(\mathbf{1})$ by h. In view of Lemma 4, since every element of the socle is a linear combination of minimal projections, it follows directly that, $T(x)h^* = hT(x^*)^*$, $h^*T(x) = T(x^*)^*h$, $T(x)h = hT(x) = h^*T(x^*)^* = T(x^*)^*h^*$, and $T(x) = T(x)h^2$ for every $x \in \operatorname{soc}(A)$.

Let p,q be minimal projections in A. Since qp and $(\mathbf{1}-q)(\mathbf{1}-p) = \mathbf{1}-p-q+qp$ are mutually orthogonal regular elements, by Proposition 1, $T(qp) \perp T(\mathbf{1}-q-p+qp)$. Therefore

$$T(qp)h^* - T(qp)T(q)^* - T(qp)T(p)^* + T(qp)T(qp)^* = 0.$$

As $q(\mathbf{1}-p) \perp (\mathbf{1}-q)p$, we also have $T(q-qp) \perp T(p-qp)$, that is

$$T(q)T(p)^* - T(q)T(qp)^* - T(qp)T(p)^* + T(qp)T(qp)^* = 0.$$

Taking into account these equations and $\mathrm{soc}(A)$ being linearly spanned by the minimal projections, we can prove

$$T(yx+xy)h^* = T(y)T(x^*)^* + T(x)T(y^*)^*, \tag{5}$$

for all $x,y \in \mathrm{soc}(A)$ (compare with the proof of Theorem 14 in [8]). Besides, given a minimal projection p in A and an invertible element b in A, p and $(\mathbf{1}-p)b(\mathbf{1}-p) = b-bp-pb+pbp$ are mutually orthogonal regular elements. Thus $T(p)^*T(b) = T(p)^*T(bp+pb-pbp)$ and $T(b)T(p)^* = T(bp+pb-pbp)T(p)^*$. Equation (5) yields

$$\begin{aligned} T(bp+pb)h^* &= T((bp+pb)p + p(bp+pb) - 2pbp)h^* \\ &= T(bp+pb)T(p)^* + T(p)T(b^*p+pb^*)^* \\ &\quad - T(pbp)T(p)^* - T(p)T(pb^*p)^* \\ &= T(bp+pb-pbp)T(p)^* + T(p)T(b^*p+pb^*-pb^*p)^* \\ &= T(b)T(p)^* + T(p)T(b^*)^*. \end{aligned}$$

As $T(p)h^* = hT(p)^*$, given $a \in A$ and $\alpha \in \mathbb{C}$ such that $a-\alpha$ is invertible, the last equation gives $T(ap+pa)h^* = T(a)T(p)^* + T(p)T(a^*)^*$, and by the linearity of T

$$T(ax+xa)h^* = T(a)T(x^*)^* + T(x)T(a^*)^* \qquad (a \in A, x \in \mathrm{soc}(A)). \tag{6}$$

The other equality of (a) can be proved analogously.

Again for a minimal projection p in A, and an invertible element $b \in A$, from $p \perp (\mathbf{1}-p)b(\mathbf{1}-p)$, we obtain

$$\begin{aligned} T(p)h^*T(b) &= T(p)T(p)^*T(b) = T(p)T(p)^*T(bp+pb-pbp) \\ &= T(p)h^*T(bp+pb-pbp) = T(p)T((bp+pb-pbp)^*)^*h \\ &= T(p)T(b^*)^*h, \end{aligned}$$

and

$$\begin{aligned} T(p)hT(b) &= h^*T(p)^*T(b) = h^*T(p)^*T(bp+pb-pbp) \\ &= T(p)hT(bp+pb-pbp) = T(p)T((bp+pb-pbp)^*)^*h^* \\ &= T(p)T(b^*)^*h^*. \end{aligned}$$

This proves that

$$T(x)h^*T(a) = T(x)T(a^*)^*h, \tag{7}$$

and

$$T(x)hT(a) = T(x)T(a^*)^*h^*, \tag{8}$$

for all $x \in \mathrm{soc}(A)$ and $a \in A$. The other relations of (b) and (c) can be deduced in an obvious way.

In order to prove equality (d), let $x \in \mathrm{soc}(A)$ and $a \in A$. By the definition of the triple product in a C^*-algebra and the statements just proved we get

$$\begin{aligned}
T(\{xax\})h^*h &= 2T((x\circ a^*)\circ x)h^*h - T(x^2\circ a^*)h^*h \\
&= \left(T(x\circ a^*)T(x^*)^* + T(x)T(x^*\circ a)^*\right)h \\
&\quad -\frac{1}{2}\left(T(x^2)T(a)^* + T(a^*)T((x^2)^*)^*\right)h \\
&= T(x\circ a^*)h^*T(x) + T(x)h^*T(x\circ a^*) \\
&\quad -\frac{1}{2}\left(T(x^2)h^*T(a^*) + T(a^*)T((x^2)^*)^*h\right) \\
&= \frac{1}{2}\left((T(x)T(a)^* + T(a^*)T(x^*)^*)T(x)\right) \\
&\quad +\frac{1}{2}\left(T(x)(T(x^*)^*T(a^*) + T(a)^*T(x))\right) \\
&\quad -\frac{1}{2}\left((T(x)T(x^*)^*T(a^*) + T(a^*)T(x^*)^*T(x)\right) \\
&= \{T(x)\,T(a)\,T(x)\}. \qquad \square
\end{aligned}$$

REMARK 11. From the preceding lemma, it is clear that

$$T(\mathbf{1})T(x) = (T(\mathbf{1})T(x^*))^*,$$

and

$$T(\mathbf{1})T(x^2) = T(\mathbf{1})T(x^2)(T(\mathbf{1})^*)^2 = T(\mathbf{1})T(x)T(x^*)^*T(\mathbf{1})^* = (T(\mathbf{1})T(x))^2,$$

for every element x in the socle of A. This shows that the mapping $x \mapsto T(\mathbf{1})T(x)$ is a Jordan *-homomorphism from $\mathrm{soc}(A)$ to B.

It is well known that every element of the socle is a finite sum of rank-one elements. Recall that a non zero element $u \in A$ is said to be of *rank-one* if u belongs to some minimal left ideal of A, that is, if $u = ue$ for some minimal idempotent e of A. A non zero element $u \in A$ has rank-one if and only if $uAu = \mathbb{C}u$, and this is equivalent to the condition $|\sigma(xu)\setminus\{0\}| \leqslant 1$, for all $x \in A$ (also equivalent to $|\sigma(ux)\setminus\{0\}| \leqslant 1$, for all $x \in A$), where $\sigma(x)$ denotes the spectrum of x. Let us denote the set of rank-one elements in A by $\mathscr{F}_1(A)$.

Recall that an ideal I of a Banach algebra A is called *essential* if it has non zero intersection with every non zero ideal of A. If A is semisimple (in particular, if A is a C*-algebra) this is equivalent to the condition $aI = 0$ implies $a = 0$.

THEOREM 12. *Assume that T does not annihilate rank-one elements.*

1. *If $T(a)T(\mathbf{1})-T(\mathbf{1})T(a)\in T(A)$ for every $a\in A$, then $T^{-1}(T(a)T(\mathbf{1})-T(\mathbf{1})T(a))\operatorname{soc}(A)=\{0\}$, for every $a\in A$.*

2. *If $T(a)T(\mathbf{1})-(T(a)T(\mathbf{1}))^*\in T(A)$ for every selfadjoint element $a\in A$, then $T^{-1}(T(a)T(\mathbf{1})-T(\mathbf{1})^*T(a^*)^*)\operatorname{soc}(A)=\{0\}$, for every $a\in A$.*

In particular, if $\operatorname{soc}(A)$ is essential, then $T(\mathbf{1})T$ is a Jordan *-homomorphism, and $T(\mathbf{1})$ commutes with the range of T.

Proof. Again write $h=T(\mathbf{1})$. Let $x\in\operatorname{soc}(A)$ and $a\in A$. From (d) of Lemma 10, by multiplying on the right by hh^*

$$\begin{aligned}T(\{xax\}) &= T(x)T(a)^*T(x)hh^* = T(x)T(a)^*hT(x)h^*\\ &= T(x)T(a)^*h^2T(x^*)^* = T(x)T(a)^*T(x^*)^*.\end{aligned}$$

Moreover, since $T(\{xax\})h^*h=hh^*T(\{xax\})$, we also get (by multiplying on the left by h^*h)

$$T(\{xax\})=h^*T(x)hT(a)^*T(x)=T(x^*)^*h^2T(a)^*T(x)=T(x^*)^*T(a)^*T(x).$$

Therefore,

$$\begin{aligned}\{T(x)\,(T(a)h)\,T(x)\} &= T(x)h^*T(a)^*T(x)=hT(x^*)^*T(a)^*T(x)\\ &= hT(\{xax\})=T(\{xax\})h=T(x)T(a)^*T(x^*)^*h\\ &= T(x)T(a)^*h^*T(x)=\{T(x)\,(hT(a))\,T(x)\}.\end{aligned}$$

If $T(a)h-hT(a)\in T(A)$, there exists $b\in A$ such that $T(b)=T(a)h-hT(a)$. The last identities show that

$$0=\{T(x)\,T(b)\,T(x)\}=T(\{xbx\})h^*h,$$

and hence $T(\{xbx\})=0$. In particular $T(\{ubu\})=0$ for every $u\in\mathscr{F}_1(A)$. As T does not annihilate rank-one elements, and for every $u\in\mathscr{F}_1(A)$, $ubu=0$ or ubu has rank-one, it follows that $ubu=0$ for all $u\in\mathscr{F}_1(A)$. This implies that $bu=0$ for every $u\in\mathscr{F}_1(A)$ (see for instance the proof of Theorem 1.1 in [6]). Hence $bSoc(A)=\{0\}$, that is,

$$T^{-1}(T(a)h-hT(a))\operatorname{soc}(A)=\{0\}. \tag{9}$$

From Lemma 10 (c), it follows that

$$\begin{aligned}\{T(x)\,(T(a)h)\,T(x)\} &= T(x)h^*T(a)^*T(x)=T(x)T(a^*)hT(x)\\ &= \{T(x)\,(h^*T(a^*)^*)\,T(x)\}.\end{aligned}$$

Whenever $T(z)h-(T(z)h)^*\in T(A)$, for every selfadjoint element $z\in A$, it is clear that $T(a)h-h^*T(a^*)^*$ lies in $T(A)$, for every $a\in A$. Then, as above, we can prove

$$T^{-1}(T(a)h-h^*T(a^*)^*)\operatorname{soc}(A)=\{0\}. \tag{10}$$

If $\mathrm{soc}(A)$ is essential, by Equation (9), h commutes with $T(A)$, and by Proposition 3, $S = hT$ is a Jordan homomorphism. Besides, Equation (10) gives $T(a)h = h^*T(a^*)^* = (T(a^*)h)^*$ for all $a \in A$, which shows that S is selfadjoint. □

Notice that if $T : A \to B$ is a bijective linear map strongly preserving Moore-Penrose invertibility, and $\mathrm{soc}(A)$ is essential, since

$$\{T(x)\,(T(a)T(\mathbf{1}_A)^2)\,T(x)\} = \{T(x)\,T(a)\,T(x)\} = \{T(x)\,(T(\mathbf{1}_A)^2T(a))\,T(x)\},$$

we can obtain that $T(a)T(\mathbf{1}_A)^2 = T(a) = T(\mathbf{1}_A)^2T(a)$, for every $a \in A$, and hence B is unital with identity element $\mathbf{1}_B = T(\mathbf{1}_A)^2$. The following corollary can be derived now as an easy consequence.

COROLLARY 13. *Let A and B be C^*-algebras. Suppose that A is unital with essential socle. Let $T : A \to B$ be a bijective linear map. The following are equivalent:*

1. *T strongly preserves Moore-Penrose invertibility,*
2. *$T(\mathbf{1}_A)^2 = \mathbf{1}_B$, $T = T(\mathbf{1}_A)S = ST(\mathbf{1}_A)$ for a Jordan *-isomorphism S.*

We conclude this section by considering the case of linear mappings from prime C^*-algebras with non zero socle. Recall that every prime C^*-algebra A with non zero socle is primitive (see [20]) and hence its socle is a simple algebra which is contained in every non zero (Jordan) ideal of A (see [14, IV §9] and [12, Theorem 1.1]). As we have noted in Remark 11, if $T : A \to B$ is a linear map strongly preserving Moore-Penrose invertibility, then $T(\mathbf{1})T|_{\mathrm{soc}(A)} : \mathrm{soc}(A) \to B$ is a Jordan *-homomorphism and hence $\mathrm{Ker(T)} \cap \mathrm{soc(A)}$ is a Jordan ideal of A. Therefore, if A is prime, either $\mathrm{Ker(T)} \cap \mathrm{soc(A)} = \{0\}$ or $T(\mathrm{soc}(A)) = \{0\}$.

Having in mind these considerations and the proof of Theorem 12, we get the following result.

COROLLARY 14. *Let A and B be C^*-algebras, and let $T : A \to B$ be a surjective linear map strongly preserving Moore-Penrose invertibility. Suppose that A is prime, unital, with non zero socle. If $T(\mathrm{soc}(A)) \neq \{0\}$, then $T(\mathbf{1})T$ is a Jordan *-homomorphism, and $T(\mathbf{1})$ commutes with the range of T.*

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(Received July 5, 2011)

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