Operators and Matrices
Volume 6, Number 4 (2012), 869–875
doi:10.7153/oam-06-56

ON YUAN–GAO'S "COMPLETE FORM" OF FURUTA INEQUALITY

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(*Communicated by F. Hansen*)

Abstract. Recently Yuan and Gao gave a "complete form" of Furuta inequality. We present its extension by an expression of operator mean: If $A \geqslant B \geqslant 0$ with $A > 0$, $p \geqslant p_0 \geqslant 0$ and $r, r_0 > 0$, then

$$A^{-r_0} \natural_{\frac{\delta+r_0}{p_0+r_0}} B^{p_0} \geqslant B^{\delta} \geqslant A^{-r} \sharp_{\frac{\delta+r}{p+r}} B^p$$

for $p_0 \leqslant \delta \leqslant \min\{p, 2p_0 + \min\{1, r_0\}\}$. Furthermore we also obtain a grand Furuta type inequality related to our extension.

1. Introduction

Throughout this note a capital letter means a bounded linear operator acting on a Hilbert space.

In 1987, Furuta [5] established the so-called Furuta inequality, see [2, 3, 6, 7, 13, 16].

Furuta inequality. *If $A \geqslant B \geqslant 0$, then for each $r \geqslant 0$,*

$$(A^{\frac{r}{2}} A^p A^{\frac{r}{2}})^{\frac{1}{q}} \geqslant (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}$$

and

$$(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1}{q}} \geqslant (B^{\frac{r}{2}} B^p B^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for p and q such that $p \geqslant 0$ and $q \geqslant 1$ with

$$(1+r)q \geqslant p+r.$$

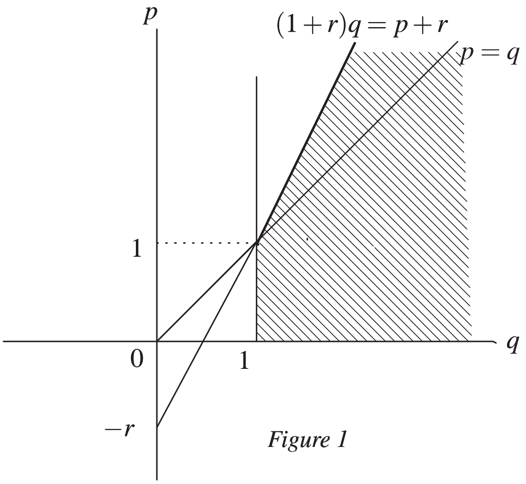

Figure 1

Mathematics subject classification (2010): Primary 47A63, 47A64.

Keywords and phrases: Positive operators, operator mean, Furuta inequality and grand Furuta inequality.

The most important fact on Furuta inequality is that it is an extension of Löwner-Heinz inequality (LH), i.e.,

$$A \geqslant B \geqslant 0 \quad \Longrightarrow \quad A^{\alpha} \geqslant B^{\alpha} \quad (\alpha \in [0,1]).$$

Related to (LH), Kubo-Ando theory says that α-geometric mean $\sharp_\alpha$ just corresponds to (LH; α). That is, it is defined by

$$A \sharp_\alpha B = A^{1/2}(A^{-1/2}BA^{-1/2})^{\alpha}A^{1/2}$$

for positive operators A and B. As stated in [13], when $A > 0$ and $B \geqslant 0$, Furuta inequality can be arranged in terms of α-geometric mean as follows: If $A \geqslant B \geqslant 0$ with $A > 0$, then

$$A \geqslant B \geqslant A^{-r} \sharp_{\frac{1+r}{p+r}} B^p \quad \text{for } p \geqslant 1 \text{ and } r \geqslant 0. \tag{FI}$$

Furthermore Furuta [9] obtained the following as an interpolation between Furuta inequality and Ando-Hiai one [1].

The grand Furuta inequality. *If $A \geqslant B \geqslant 0$ with $A > 0$, then for each $t \in [0,1]$,*

$$A^{1-t+r} \geqslant \{A^{\frac{r}{2}}(A^{-\frac{t}{2}}B^pA^{-\frac{t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} \tag{GFI}$$

holds for all $s \geqslant 1$, $p \geqslant 1$ and $r \geqslant t$.

For the grand Furuta inequality see [4, 10, 14, 15, 17].

Now in order to provide an elementary and alternative proof of Furuta inequality, Furuta proved the following inequality.

THEOREM 1.A. ([8]) *Let $A \geqslant B \geqslant 0$, $1 \geqslant r \geqslant 0$ and $p > p_0 > 0$. If $2p_0 + r \geqslant p$, then*

$$(A^{r/2}B^{p_0}A^{r/2})^{\frac{p+r}{p_0+r}} \geqslant A^{r/2}B^pA^{r/2}.$$

Yuan and Gao [18] provided a "complete form" of Theorem 1.A.

THEOREM 1.B. ([18]) *Let $A \geqslant B \geqslant 0$, $r > 0$, $p > p_0 > 0$ and $\delta = \min\{p, 2p_0 + \min\{1,r\}\}$. Then*

$$(A^{r/2}B^{p_0}A^{r/2})^{\frac{\delta+r}{p_0+r}} \geqslant (A^{r/2}B^pA^{r/2})^{\frac{\delta+r}{p+r}}. \tag{1.1}$$

In this note, we shall give an extension of Theorem 1.B and related results by an expression of operator mean. As a matter of fact, (1.1) is expressed as

$$A^{-r} \natural_{\frac{\delta+r}{p_0+r}} B^{p_0} \geqslant A^{-r} \sharp_{\frac{\delta+r}{p+r}} B^p,$$

where $A \natural_\alpha B = A^{1/2}(A^{-1/2}BA^{-1/2})^{\alpha}A^{1/2}$ $(\alpha \notin [0,1])$.

For this, we present a generalization as follows: Let $A \geqslant B \geqslant 0$ with $A > 0$, $p \geqslant p_0 > 0$ and $r, r_0 > 0$. Then

$$A^{-r_0} \natural_{\frac{\delta+r_0}{p_0+r_0}} B^{p_0} \geqslant B^{\delta} \geqslant A^{-r} \sharp_{\frac{\delta+r}{p+r}} B^p$$

for $p_0 \leqslant \delta \leqslant \min\{p, 2p_0 + \min\{1, r_0\}\}$. If we put $r_0 = r$, then we have Theorem 1.B obviously. Furthermore we also obtain a grand Furuta type inequality related to our extension.

2. The main theorem

In this section, we shall give an extension of Theorem 1.B. First of all, we cite useful formulae on $A \natural_{\alpha} B$ for convenience. They are easily checked by the direct computations and frequently used in the below.

LEMMA 2.A. *The following formulae hold for all real numbers s and t:*

1. $A \natural_s B = B \natural_{1-s} A$,
2. $A \natural_s B = B(B^{-1} \natural_{s-1} A^{-1})B$, *and*
3. $A \natural_{st} B = A \natural_s (A \natural_t B)$.

Under this preparation, we extend Theorem 1.A as follows:

THEOREM 2.1. *Let $A \geqslant B \geqslant 0$ with $A > 0$, $p_0 \geqslant 0$ and $r_0 > 0$. Then*

$$A^{-r_0} \natural_{\frac{\delta+r_0}{p_0+r_0}} B^{p_0} \geqslant B^{\delta}$$

for $p_0 \leqslant \delta \leqslant 2p_0 + \min\{1, r_0\}$.

Proof. We may assume that B is invertible. By Furuta inequality, $A \geqslant B > 0$ ensures

$$B^{-p_0} \sharp_{\frac{\delta-p_0}{p_0+r_0}} A^{r_0} = B^{-p_0} \sharp_{\frac{\delta-2p_0+p_0}{r_0+p_0}} A^{r_0} \geqslant B^{\delta-2p_0}$$

for $-p_0 \leqslant \delta - 2p_0 \leqslant \min\{1, r_0\}$. Then we have

$$A^{-r_0} \natural_{\frac{\delta+r_0}{p_0+r_0}} B^{p_0} = B^{p_0}(B^{-p_0} \sharp_{\frac{\delta-p_0}{p_0+r_0}} A^{r_0})B^{p_0} \geqslant B^{p_0} B^{\delta-2p_0} B^{p_0} = B^{\delta}.$$

Hence the proof is complete. $\square$

Recently, the following result was shown in [11, Theorem 2.1].

THEOREM 2.B. ([11]) *For $A, B > 0$, $p > 0$ and $r > 0$, if $A^{-r} \sharp_{\frac{r}{p+r}} B^p \leqslant 1$, then*

$$A^{-r} \sharp_{\frac{\delta+r}{p+r}} B^p \leqslant A^{-t} \sharp_{\frac{\delta+t}{s+t}} B^s$$

for $0 \leqslant s \leqslant p$, $0 \leqslant t \leqslant r$ and $-t \leqslant \delta \leqslant s$.

By Theorem 2.1 and Theorem 2.B, we obtain an extension of Theorem 1.B.

THEOREM 2.2. *Let* $A \geqslant B \geqslant 0$ *with* $A > 0$, $p \geqslant p_0 \geqslant 0$ *and* $r, r_0 > 0$. *Then*

$$A^{-r_0} \natural_{\frac{\delta+r_0}{p_0+r_0}} B^{p_0} \geqslant B^{\delta} \geqslant A^{-r} \sharp_{\frac{\delta+r}{p+r}} B^p$$

for $p_0 \leqslant \delta \leqslant \min\{p, 2p_0 + \min\{1, r_0\}\}$.

Proof. The former inequality is just Theorem 2.1, so that we have only to prove the latter one. We may assume that B is invertible. By Furuta inequality, $A \geqslant B > 0$ ensures $A^r \geqslant (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{r}{p+r}}$, that is, $A^{-r} \sharp_{\frac{r}{p+r}} B^p \leqslant 1$ for $p > 0$ and $r > 0$. Then we have

$$B^{\delta} = 1 \sharp_{\frac{\delta+0}{p+0}} B^p \geqslant A^{-r} \sharp_{\frac{\delta+r}{p+r}} B^p \tag{2.1}$$

for $0 \leqslant \delta \leqslant p$ and $r > 0$ by applying Theorem 2.B. $\square$

3. A grand Furuta type inequality

Next we shall show a grand Furuta type inequality related to Theorem 2.2. By putting $\beta = (p-t)s + t$ and $\gamma = r - t$, we can arrange (GFI) in terms of α-geometric mean as follows [4]: If $A \geqslant B \geqslant 0$ with $A > 0$, then for each $t \in [0,1]$ and $p \geqslant 1$ with $p \neq t$,

$$A \geqslant B \geqslant A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta+\gamma}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \quad \text{for } \beta \geqslant p \text{ and } \gamma \geqslant 0. \tag{3.1}$$

The following Theorem 3.1 is a grand Furuta type extension of Theorem 2.2.

THEOREM 3.1. *Let* $A \geqslant B \geqslant 0$ *with* $A > 0$, $p \geqslant 1$, $t \in [0,1]$, $p \neq t$, $\gamma, \gamma_0 \geqslant 0$ *and* $\beta \geqslant \beta_0 \geqslant p$. *Then for* $\beta_0 \leqslant \delta \leqslant \min\{\beta, 2\beta_0 - t\}$,

$$\begin{aligned} A^{-\gamma_0} \natural_{\frac{\delta+\gamma_0}{\beta_0+\gamma_0}} (A^t \natural_{\frac{\beta_0-t}{p-t}} B^p) &\geqslant (A^t \natural_{\frac{\beta_0-t}{p-t}} B^p)^{\frac{\delta}{\beta_0}} \geqslant A^t \natural_{\frac{\delta-t}{p-t}} B^p \\ &\geqslant (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{\delta}{\beta}} \geqslant A^{-\gamma} \sharp_{\frac{\delta+\gamma}{\beta+\gamma}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p). \end{aligned}$$

To prove Theorem 3.1, we use the following lemma shown in [4] (cf. [12]).

LEMMA 3.A. ([4]) *Let* $A \geqslant B \geqslant 0$ *with* $A > 0$. *Then*

$$A \geqslant B \geqslant (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{1}{\beta}}$$

holds for $t \in [0,1]$, $\beta \geqslant p \geqslant 1$ *and* $p \neq t$.

Proof of Theorem 3.1. By Lemma 3.A, for $t \in [0,1]$ and $\beta, \beta_0 \geqslant p \geqslant 1$, we have

$$A \geqslant (A^t \natural_{\frac{\beta_0-t}{p-t}} B^p)^{\frac{1}{\beta_0}} \tag{3.2}$$

and

$$A \geqslant (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{1}{\beta}}. \tag{3.3}$$

Applying Theorem 2.1 to (3.2), we obtain

$$A^{-\gamma_0} \natural_{\frac{\delta+\gamma_0}{\beta_0+\gamma_0}} (A^t \natural_{\frac{\beta_0-t}{p-t}} B^p) \geqslant (A^t \natural_{\frac{\beta_0-t}{p-t}} B^p)^{\frac{\delta}{\beta_0}} \tag{3.4}$$

for $\beta_0 \leqslant \delta \leqslant 2\beta_0 + \min\{1, \gamma_0\}$, $\gamma_0 \geqslant 0$. Applying (2.1) in the proof of Theorem 2.2 to (3.3), we obtain

$$(A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{\delta}{\beta}} \geqslant A^{-\gamma} \sharp_{\frac{\delta+\gamma}{\beta+\gamma}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \tag{3.5}$$

for $0 \leqslant \delta \leqslant \beta$, $\gamma \geqslant 0$.

Put $C = (A^t \natural_{\frac{\beta_0-t}{p-t}} B^p)^{\frac{1}{\beta_0}}$ and $D = (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{1}{\beta}}$. Then by (3.2),

$$\begin{aligned} A^t \natural_{\frac{\delta-t}{p-t}} B^p &= A^t \natural_{\frac{\delta-t}{\beta_0-t}} (A^t \natural_{\frac{\beta_0-t}{p-t}} B^p) = A^t \natural_{\frac{\delta-t}{\beta_0-t}} C^{\beta_0} \\ &= C^{\beta_0} (C^{-\beta_0} \sharp_{\frac{\delta-\beta_0}{\beta_0-t}} A^{-t}) C^{\beta_0} \\ &\leqslant C^{\beta_0} (C^{-\beta_0} \sharp_{\frac{\delta-\beta_0}{\beta_0-t}} C^{-t}) C^{\beta_0} = C^{\delta} = (A^t \natural_{\frac{\beta_0-t}{p-t}} B^p)^{\frac{\delta}{\beta_0}} \end{aligned}$$

for $t \in [0,1]$, $1 \leqslant p \leqslant \beta_0 \leqslant \delta \leqslant 2\beta_0 - t$, and also by (3.3),

$$\begin{aligned} A^t \natural_{\frac{\delta-t}{p-t}} B^p &= A^t \sharp_{\frac{\delta-t}{\beta-t}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) = A^t \sharp_{\frac{\delta-t}{\beta-t}} D^{\beta} \\ &\geqslant D^t \sharp_{\frac{\delta-t}{\beta-t}} D^{\beta} = D^{\delta} = (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{\delta}{\beta}} \end{aligned}$$

for $t \in [0,1]$, $1 \leqslant p \leqslant \delta \leqslant \beta$. Therefore we have

$$(A^t \natural_{\frac{\beta_0-t}{p-t}} B^p)^{\frac{\delta}{\beta_0}} \geqslant A^t \natural_{\frac{\delta-t}{p-t}} B^p \geqslant (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{\delta}{\beta}} \tag{3.6}$$

for $t \in [0,1]$, $1 \leqslant p \leqslant \beta_0 \leqslant \delta \leqslant \beta$ and $\delta \leqslant 2\beta_0 - t$.

Hence the desired inequalities are obtained by (3.4), (3.5) and (3.6). □

By putting $\beta_0 = (p-t)s_0 + t$, $\gamma_0 = r_0 - t$, $\beta = (p-t)s + t$ and $\gamma = r - t$, we get the following corollary.

COROLLARY 3.2. *Let $A \geqslant B \geqslant 0$ with $A > 0$, $p \geqslant 1$, $t \in [0,1]$, $p \neq t$, $r, r_0 \geqslant t$ and $s \geqslant s_0 \geqslant 1$. Then for $(p-t)s_0 + t \leqslant \delta \leqslant \min\{(p-t)s + t, 2(p-t)s_0 + t\}$,*

$$A^{-r_0} \natural_{\frac{\delta-t+r_0}{(p-t)s_0+r_0}} (A^{-t/2} B^p A^{-t/2})^{s_0} \geqslant (A^{-t/2} B^p A^{-t/2})^{\frac{\delta-t}{p-t}} \geqslant A^{-r} \sharp_{\frac{\delta-t+r}{(p-t)s+r}} (A^{-t/2} B^p A^{-t/2})^{s}.$$

REMARK. We may expect that Corollary 3.2 holds for $\delta \leqslant \min\{(p-t)s+t, 2(p-t)s_0+t+\min\{1,r_0\}\}$. Unfortunately it does not hold in general.

Let $A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}, B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, t = 1, p = 2, r_0 = 1, r = 2, s_0 = 1$ and $s = 3$. Then $A \geqslant B$, and $\delta = 2(p-t)s_0+t+\min\{1,r_0\} = (p-t)s+t = 4$. Moreover we have

$$C = A^{-r_0} \natural_{\frac{\delta-t+r_0}{(p-t)s_0+r_0}} (A^{-t/2}B^pA^{-t/2})^{s_0} = A^{-1/2}B^4A^{-1/2} = \begin{pmatrix} \frac{34}{3} & 7\sqrt{\frac{3}{2}} \\ 7\sqrt{\frac{3}{2}} & \frac{13}{2} \end{pmatrix},$$

and

$$D = A^{-r} \sharp_{\frac{\delta-t+r}{(p-t)s+r}} (A^{-t/2}B^pA^{-t/2})^{s} = (A^{-1/2}B^2A^{-1/2})^3 = \begin{pmatrix} \frac{601}{54} & \frac{125}{6\sqrt{6}} \\ \frac{125}{6\sqrt{6}} & \frac{13}{2} \end{pmatrix}.$$

Hence $\det(C-D) < 0$, that is,

$$A^{-r_0} \natural_{\frac{\delta-t+r_0}{(p-t)s_0+r_0}} (A^{-t/2}B^pA^{-t/2})^{s_0} - A^{-r} \sharp_{\frac{\delta-t+r}{(p-t)s+r}} (A^{-t/2}B^pA^{-t/2})^{s} \ngeqslant 0.$$

Acknowledgement

The authors would like to express their thanks to the referee for valuable suggestions.

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(Received September 8, 2011)

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