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ON INVERSE PROBLEMS FOR LEFT–DEFINITE DISCRETE STURM–LIOUVILLE EQUATIONS

R. ALAHMAD AND R. WEIKARD

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Abstract. We establish an expansion theorem and investigate inverse spectral and inverse scattering problems for the discrete Sturm-Liouville problem

$$-u''(n-1)+q(n)u(n)=\lambda w(n)u(n)$$

where q is nonnegative and w may change sign. If w is positive, the ℓ^2-space weighted by w is a Hilbert space and it is customary to use that space for setting the problem. In the present situation the right-hand-side of the equation does not give rise to a positive-definite quadratic form and we use instead the left-hand side to define such a form and hence a Hilbert space (such problems are called left-definite). The difference equation gives rise to a linear relation which, upon proper restrictions, generates a self-adjoint operator. For this operator we define a Fourier transform and investigate the relationship between two operators with the same transform (the inverse spectral problem). If $q-q_0$ and $w-1$ are summable one may define the scattering process and we solve the inverse scattering problem. For coefficients decaying sufficiently fast to q_0 and 1, respectively, the concept of a resonance is introduced as a generalization of the notion of an eigenvalue and the set of iso-resonant operators, *i.e.*, operators having the same eigenvalues and resonances, is described.

1. Introduction

In this paper we will study spectral and scattering theory as well as their inverse counterparts for a discrete, left-definite Sturm-Liouville problem, *i.e.*, a problem determined by the difference equation

$$-u''(n-1)+q(n)u(n)=\lambda w(n)u(n) \tag{1.1}$$

set in a space defined using the left-hand-side of this equation (see below for precise definitions). Such an approach is particularly appropriate if w changes signs. The corresponding problem in a continuous setting has been treated recently in [1]. Note, however, that we address here also the inverse resonance problem which was not touched upon in [1].

The left-definite spectral problem was first raised by Weyl in his seminal paper [17] and treated by him in [16]. There is now a large body of literature on the problem of determining spectral properties for such systems. We mention here for instance Niessen

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and Schneider [13, 14], Krall [9, 10], Marletta and Zettl [12], Littlejohn and Wellman [11], Kong, Wu, and Zettl [8] and the references therein. Inverse spectral or scattering theory for left-definite problems were considered much less frequently, but we refer the reader to Freiling and Yurko [5, 6] and to Binding, Browne, and Watson [4].

If $w(n)$ is always different from zero, one may divide equation (1.1) by $w(n)$ and treat the resulting equation as a special case of the difference equation

$$a_{n-1}u(n-1)+b_nu(n)+c_nu(n+1)=\lambda u(n) \tag{1.2}$$

set in $\ell^2(\mathbb{R})$, see e.g., [15] or Guseinov [7][1]. We emphasize that the corresponding spectral problems are not equivalent. Moreover, the expansion theorem in Section 3 allows for zeros of w while our inverse results assume that w is never zero largely for convenience.

We denote the complex-valued sequences on $\mathbb{N}_0$ and $\mathbb{N}$ by $\mathbb{C}^{\mathbb{N}_0}$ and $\mathbb{C}^{\mathbb{N}}$, respectively. The forward difference operator maps a sequence u to the sequence u' defined by $u'(n)=u(n+1)-u(n)$. Note that then

$$u''(n-1)=u(n+1)-2u(n)+u(n-1).$$

If u is a function of several variables, a $'$ denotes the forward difference operator with respect to the last variable. The notation $[a,b]$ for intervals is used for subsets of both the real numbers and the integers.

Our main interest is studying the equation (1.1) where λ is a complex parameter and where q and w are sequences with the following properties:

1. q is defined on $\mathbb{N}_0$ and assumes non-negative real values but is not identically equal to zero and
2. w is defined on $\mathbb{N}$ and real-valued.

Subsequently we may abbreviate the operator on the left-hand side of (1.1) by L, *i.e.,*

$$(Lu)(n)=-u''(n-1)+(qu)(n), \quad n\in\mathbb{N}$$

Note that L operates from $\mathbb{C}^{\mathbb{N}_0}$ to $\mathbb{C}^{\mathbb{N}}$.

In Section 2 we will define a Hilbert space $\mathscr{H}$ and a self-adjoint operator T acting in $\mathscr{H}$ representing the difference equation $Lu=wf$. In Section 3 we introduce a generalized Fourier transform which diagonalizes T and prove a theorem of Paley-Wiener type, *i.e.*, a theorem which relates support properties of $u\in\mathscr{H}$ with growth properties of the Fourier transform. In Sections 4, 5, and 6 we respectively address the inverse spectral problem, the inverse scattering problem, and the inverse resonance problem.

[1] Under certain conditions on the coefficients a_n and c_n the operator defined by (1.2) is similar to a formally symmetric one.

2. Definition of the operator

2.1. Difference equations

We have already defined the forward difference operator which maps a sequence u to u' by the assignment $u'(n) = u(n+1) - u(n)$. The forward difference operator is linear and satisfies the product rules

$$(fg)'(n) = f(n+1)g'(n) + f'(n)g(n) = f(n)g'(n) + f'(n)g(n+1).$$

The latter of these implies immediately the following summation by parts formula

$$\sum_{n=j}^{k} f(n)g'(n) = (fg)(k+1) - (fg)(j) - \sum_{n=j+1}^{k+1} f'(n-1)g(n). \tag{2.1}$$

Two particular instances of this formula are

$$\begin{aligned}\sum_{n=0}^{N}(u'(n)\overline{v'(n)} + q(n)u(n)\overline{v(n)})& \\ = u'(N)\overline{v(N+1)} - (u'-qu)(0)\overline{v(0)} + \sum_{n=1}^{N}(Lu)(n)\overline{v(n)}& \\ = u(N+1)\overline{v'(N)} - u(0)\overline{(v'-qv)(0)} + \sum_{n=1}^{N} u(n)\overline{(Lv)(n)}.&\end{aligned} \tag{2.2}$$

We define the Wronskian of two sequences f and g by

$$[f,g](n) = f(n)g'(n) - f'(n)g(n).$$

Just as in the continuous case, $[u,v]$ is a constant if u and v are both solutions of the homogeneous equation $Ly = \lambda wy$. We also remind the reader of the existence and uniqueness theorem for solutions of initial value problems for the (possibly) non-homogeneous equation: given a $f \in \mathbb{C}^{\mathbb{N}}$ the initial value problem

$$Lu = \lambda wu + wf, \quad u(n_0) = A, u'(n_0) = B$$

has a unique solution $u \in \mathbb{C}^{\mathbb{N}_0}$ whenever $n_0 \in \mathbb{N}_0$ and $A, B \in \mathbb{C}$.

2.2. Maximal and minimal relations associated with L

Due to the fact that the sign of w is indefinite it is not convenient to phrase the spectral and scattering theory in the usual setting of a weighted ℓ^2-space, since it is not a Hilbert space. Instead the requirement that q is non-negative but not identically equal to zero allows us to define an inner product associated with the left-hand side of the equation $Lu = wf$ giving rise to the term *left-definite problem.* To do so define the set

$$\mathscr{H}_1 = \{u \in \mathbb{C}^{\mathbb{N}_0} : \sum_{n=0}^{\infty}(|u'(n)|^2 + q(n)|u(n)|^2) < \infty\}$$

and introduce the scalar product

$$\langle u,v\rangle=\sum_{n=0}^{\infty}(u'(n)\overline{v'(n)}+q(n)u(n)\overline{v(n)}).$$

The associated norm is denoted by $\|\cdot\|$. We will also use the norm in $\ell^2(\mathbb{N}_0)$ which we denote by $\|\cdot\|_2$. We claim $\mathscr{H}_1$ is a complete space. To see this we begin with the following lemma.

LEMMA 2.1. *For any $k\in\mathbb{N}_0$ there is a constant C_k such that for all $u\in\mathscr{H}_1$ and all $n\in[0,k]$ the estimate $|u(n)|\leqslant C_k\|u\|$ holds.*

Proof. The triangle inequality and Cauchy-Schwarz's inequality show that $|u(n)|\leqslant |u(m)|+|n-m|^{1/2}\|u'\|_2$. Multiplying this by $q(m)$ and summing over m we find

$$|u(n)|\sum_{m=0}^{k}q(m)\leqslant\sum_{m=0}^{k}q(m)|u(m)|+k^{1/2}\|u'\|_2\sum_{m=0}^{k}q(m)$$

if $0\leqslant n\leqslant k$. Now choose k large enough so that $\sum_{m=0}^{k}q(m)>0$. Using Cauchy-Schwarz again we get

$$|u(n)|\leqslant\Big(\sum_{m=0}^{\infty}q(m)|u(m)|^2\Big)^{1/2}\Big(\sum_{m=0}^{k}q(m)\Big)^{-1/2}+k^{1/2}\|u'\|_2$$

which gives the desired result upon a proper choice of C_k. □

LEMMA 2.2. *The space $\mathscr{H}_1$ is complete.*

Proof. Suppose u_n is a Cauchy sequence in $\mathscr{H}_1$. Lemma 2.1 shows that $u_n(k)$ converges for every $k\in\mathbb{N}_0$. Let the limit be $u(k)$. It follows that the sequences u_n' and $\sqrt{q}u_n$ converge pointwise to u' and $\sqrt{q}u$, respectively. However, these sequences converge also in $\ell^2(\mathbb{N}_0)$ and the corresponding limits are given by the pointwise limits. Hence u is in $\mathscr{H}_1$ and is indeed the limit of u_n in $\mathscr{H}_1$. □

Our goal is to investigate the equation $Lu=wf$ when (u,f) are pairs in a certain subspace T_1 of $\mathscr{H}_1\oplus\mathscr{H}_1$. Suppose now that $u,f,v\in\mathscr{H}_1$. If v is an element of

$$\ell_0=\{u\in\mathbb{C}^{\mathbb{N}_0}:u(0)=0,\ \operatorname{supp}u\text{ is finite}\}\subset\mathscr{H}_1$$

and if $Lu=wf$ the summation by parts formula (2.2) yields

$$\langle u,v\rangle=\sum_{n=1}^{\infty}w(n)f(n)\overline{v(n)}. \tag{2.3}$$

This leads us to study the functional $u:u\mapsto\sum_{n=1}^{\infty}u(n)\overline{v(n)}$ on $\mathscr{H}_1$ defined for any fixed function v in ℓ_0. Using Lemma 2.1 one shows that this functional is, in fact, continuous. Hence, by Riesz' representation theorem, there exists, for any such v, a $v^* \in \mathscr{H}_1$ so that $\sum_{n=1}^{\infty} u(n)\overline{v(n)} = \langle u, v^* \rangle$. This gives rise to an operator $\mathscr{G}_0 : \ell_0 \to \mathscr{H}_1$ such that $\mathscr{G}_0 v = v^*$ and $\langle u, \mathscr{G}_0 v \rangle = \sum_{n=1}^{\infty} u(n)\overline{v(n)}$.

Another important consequence of Lemma 2.1, in conjunction with Riesz' representation theorem, is the existence of an evaluation operator, *i.e.*, for every $k \in \mathbb{N}_0$, there is a unique element $g_0(k,\cdot) \in \mathscr{H}_1$ such that

$$u(k) = \langle u, g_0(k,\cdot) \rangle. \tag{2.4}$$

We will determine g_0 explicitly in terms of solutions of $Lu = 0$ in Lemma 2.6 below. Making use of this evaluation operator gives an explicit form to $\mathscr{G}_0$, namely

$$(\mathscr{G}_0 v)(n) = \langle \mathscr{G}_0 v, g_0(n,\cdot) \rangle = \sum_{m=1}^{\infty} v(m)\overline{g_0(n,m)}.$$

We can now define precisely the subspace T_1 mentioned above.

$$T_1 = \{(u,f) \in \mathscr{H}_1 \oplus \mathscr{H}_1 : \langle u, v \rangle = \langle f, \mathscr{G}_0(wv) \rangle \text{ for all } v \in \ell_0\}.$$

Before we proceed we recall some facts about linear relations (for more details see, *e.g.*, Bennewitz [2]). A (closed) linear subset E of $\mathscr{H}_1 \oplus \mathscr{H}_1$ is called a (closed) linear relation on $\mathscr{H}_1$. The adjoint E^* of E is defined as

$$E^* = \{(u^*, v^*) \in \mathscr{H}_1 \oplus \mathscr{H}_1 : \langle u^*, v \rangle = \langle v^*, u \rangle \text{ for all } (u,v) \in E\}.$$

E^* is always a closed linear relation. E is called symmetric if $E \subset E^*$ and self-adjoint if $E = E^*$. E^{**} is the closure of E, and $F^* \subset E^*$ if $E \subset F$.

Thus we see that T_1 is the adjoint of

$$T_c = \{(\mathscr{G}_0(wv), v) : v \in \ell_0\}.$$

One checks easily that T_c is symmetric, *i.e.* $T_c \subset T_1$. We denote $\overline{T_c} = T_1^*$, the closure of T_c, by T_0. Of course, T_0 is also a symmetric relation.

In the following we will make frequent use of the δ-sequences defined on $\mathbb{N}_0$ by the requirement that $\delta_n(m)$ equals one if $n = m$ and zero otherwise. These are in ℓ_0 when $n \in \mathbb{N}$ (but δ_0 is not in ℓ_0). We record here that

$$\langle u, \delta_n \rangle = \begin{cases} (Lu)(n) & \text{if } n \in \mathbb{N}, \\ -u'(0) + q(0)u(0) & \text{if } n = 0. \end{cases} \tag{2.5}$$

THEOREM 2.3. *The set T_1 can be characterized in the following way.*

$$T_1 = \{(u,f) \in \mathscr{H}_1 \oplus \mathscr{H}_1 : (Lu)(n) = (wf)(n) \textit{ for all } n \in \mathbb{N}\}.$$

Proof. Suppose that $(u,f) \in T_1$. Thus, when $n \in \mathbb{N}$,

$$(Lu)(n) = \langle u, \delta_n \rangle = \langle f, \mathscr{G}_0(w\delta_n) \rangle = f(n)w(n).$$

Conversely, assume that $(u,f) \in \mathscr{H}_1 \oplus \mathscr{H}_1$, $(Lu)(n) = (wf)(n)$ for $n \geqslant 1$ and that $v \in \ell_0$. We may then employ equation (2.3) to get

$$\langle u, v\rangle = \sum_{n=1}^{\infty} w(n) f(n) \overline{v(n)} = \langle f, \mathscr{G}_0(wv)\rangle$$

and this completes the proof. □

In order to study the self-adjoint restrictions of T_1 we rely on a generalization to relations of von Neumann's formula for symmetric operators, see Theorem 1.4 in Bennewitz [2]. Thus

$$T_1 = T_0 \oplus \mathscr{D}_i \oplus \mathscr{D}_{-i}$$

where

$$\mathscr{D}_\lambda = \{(u, \lambda u) \in T_1\}.$$

We also define D_λ to be the projection of $\mathscr{D}_\lambda$ on its first component, *i.e.*, $D_\lambda = \{u \in \mathscr{H}_1 : (u, \lambda u) \in T_1\}$. The following lemma gives some preliminary information which we will use in Lemma 2.5 to establish the dimension of the spaces $\mathscr{D}_{\pm i}$ and in Lemma 2.6 to determine the kernel g_0 of the evaluation operator.

LEMMA 2.4. *The following statements hold true:*

1. $D_0 = \ell_0^\perp$, *that is* D_0 *is the orthogonal complement of* ℓ_0 *in* $\mathscr{H}_1$.
2. *If* $u \in D_0$ *and* $v \in \mathscr{H}_1$ *then* $\lim_{N\to\infty} u'(N)\overline{v(N+1)} = 0$.
3. *If* $0 \neq u \in D_0$ *then* $(u' - qu)(0)\overline{u(0)} < 0$.
4. $\dim D_0 = 1$.
5. *Finitely supported functions are dense in* $\mathscr{H}_1$.

Proof. If $u \in D_0$ then $(u,0) \in T_1$ which means $\langle u, v\rangle = \langle 0, \mathscr{G}_0(wv)\rangle = 0$ for all $v \in \ell_0$. Hence $D_0 \subset \ell_0^\perp$. To prove the other inclusion let $v \in \ell_0^\perp$. Then $0 = \langle v, \delta_n\rangle = (Lv)(n)$ for all $n \in \mathbb{N}$ according to equation (2.5). This implies $(v,0) \in T_1$ and $v \in D_0$.

The summation by parts formula (2.2) shows that, when $u \in D_0$ and $v \in \mathscr{H}_1$,

$$\sum_{k=0}^{N} [u'(k)\overline{v'(k)} + q(k)u(k)\overline{v(k)}] = u'(N)\overline{v(N+1)} - (u' - qu)(0)\overline{v(0)}. \tag{2.6}$$

Since the left-hand side of this equation has a limit as N tends to infinity, it follows that $u'(N)\overline{v(N+1)}$ does, too. We claim that this limit is zero. If this were not the case, then $(u'(N)\overline{v(N+1)})^{-1}$ would be bounded by some constant C near infinity so that $1/|v(N+1)| \leqslant C|u'(N)|$ if N is sufficiently large. It would follow from this that $1/v$ is square summable near infinity. On the other hand,

$$v(N) = v(0) + \sum_{k=0}^{N-1} v'(k)$$

so that the Cauchy-Schwarz inequality implies

$$|v(N)| \leqslant |v(0)| + \|v'\|_2 \sqrt{N}$$

which prevents $1/v$ from being square summable. This proves our second assertion.

Choosing $v = u \in D_0$ in (2.6) gives $\|u\|^2 = -(u' - qu)(0)\overline{u(0)}$ and shows the third claim.

The uniqueness of solutions of initial value problems for equation $Ly = 0$ shows that $\dim D_0 \leqslant 2$. It can not even be equal to 2 since in that case we would be able to choose initial conditions for an element in D_0 which would violate the requirement $(u' - qu)(0)\overline{u(0)} < 0$. Also $D_0 = \ell_0^\perp$ can not be trivial since δ_0 is not an element of ℓ_0. Thus $\dim D_0$ must be one.

Finally, suppose $u \in \mathscr{H}_1$ is orthogonal to all finitely supported functions. In particular, then, $u \in \ell_0^\perp = D_0$. However, since δ_0 is also finitely supported, we get from (2.5) that $0 = \langle u, \delta_0 \rangle = (-u' + qu)(0)$. This forces $u = 0$. □

LEMMA 2.5. *If λ is not real, then* $\dim \mathscr{D}_\lambda = \dim D_\lambda = 1$.

Proof. By Corollary 1.5 in Bennewitz [2] it is enough to deal with $\lambda = \pm i$. Recall that $\dim D_i = \dim D_{-i} \leqslant 2$. First assume that $\dim D_i$ is zero, *i.e.*, $T_1 = T_0 = T_1^*$. When u is in D_0 then (u, δ_0) is in T_1 and hence in T_1^*. Therefore $0 = \langle u, 0 \rangle = \langle \delta_0, v \rangle = \overline{(-v' + qv)(0)}$ for all $v \in D_0$. Since this is impossible by part (3) of Lemma 2.4, we have that $\dim D_i$ must at least be one.

Now assume that $\dim D_i = 2$. If this also leads to a contradiction our proof is finished. Our assumption allows us to choose a non-zero $u \in D_i$ such that $u(0) = 0$. Then we get from formula (2.2)

$$\sum_{k=0}^{N} [|u'(k)|^2 + q(k)|u(k)|^2] = u'(N)\overline{u(N+1)} + i \sum_{k=1}^{N} w(k)|u(k)|^2.$$

Thus we see that $\operatorname{Re}(u'(N)\overline{u(N+1)})$ tends to $\|u\|^2 > 0$ as N tends to infinity. This means that $(u'(N)\overline{u(N+1)})^{-1}$ is bounded near infinity. We conclude, in the same way as in the proof of part (2) in Lemma 2.4, both that $1/u$ is square integrable near infinity and that it is not, the desired contradiction. □

The following lemma gives us some information about the kernel g_0.

LEMMA 2.6. *There are real-valued functions ψ_0 and ϕ_0 which solve the equation $Ly = 0$, such that*

1. $\phi_0(0) = -1$, $(\phi_0' - q\phi_0)(0) = 0$,
2. $\psi_0 \in D_0 \subset \mathscr{H}_1$, $(\psi_0' - q\psi_0)(0) = 1$,
3. $\phi_0'(n) \leqslant 0$ *and* $\phi_0(n) \leqslant -1$ *for all* $n \in \mathbb{N}_0$,
4. $\psi_0(0) < 0$ *and* $\lim_{N\to\infty} \psi_0'(N)\overline{u(N+1)} = 0$ *for all* $u \in \mathscr{H}_1$,

5. $\langle u, \psi_0 \rangle = -u(0)$ *for all* $u \in \mathscr{H}_1$,

6. $[\psi_0, \phi_0](n) = 1$ *for all* $n \in \mathbb{N}_0$, *and*

7. $g_0(m,k) = \phi_0(\min(m,k))\psi_0(\max(m,k))$.

Proof. The existence of ϕ_0 is guaranteed by the existence and uniqueness theorem and that of ψ_0 by Lemma 2.4 picking the appropriate element in D_0. Since ψ_0 and ϕ_0 satisfy $Ly = 0$ with real initial values they must be real. The third claim follows from (1) by induction using $q(n) \geqslant 0$ while the fourth claim has been established already in Lemma 2.4. Part (5) is a special case of part (7), namely the case $m = 0$. To establish (6) recall that the Wronskian is independent of n. Evaluating it at zero gives 1.

To prove the last claim let $f_0(m,\cdot) = \phi_0(\min(m,\cdot))\psi_0(\max(m,\cdot))$ and notice that it is in $\mathscr{H}_1$. A straightforward computation gives $(Lf_0(m,\cdot))(k) = \delta_m(k)$ for all $k \in \mathbb{N}$. This and the formula (2.2) give

$$\sum_{k=0}^{N}[u'(k)f_0'(m,k) + q(k)u(k)f_0(m,k)]$$
$$= u(N+1)f_0'(m,N) - u(0)(f_0'(m,\cdot) - qf_0(m,\cdot))(0) + \sum_{k=1}^{N} u(k)\delta_m(k).$$

The first term on the right-hand side tends to zero as N tends to infinity because of part (2) in Lemma 2.4. The second term evaluates to $u(0)\delta_m(0)$, while the last is equal to $u(m)(1-\delta_m(0))$ if $N \geqslant m$. Hence

$$\langle u, f_0(m,\cdot)\rangle = u(m)$$

which implies that $g_0 = f_0$. $\square$

In the course of this proof we have shown the following two identities which we record here for future reference.

$$(Lg_0(m,\cdot))(k) = \delta_m(k), \text{ for all } k \in \mathbb{N} \text{ and } m \in \mathbb{N}_0 \tag{2.7}$$

and

$$-g_0'(m,0) + q(0)g_0(m,0) = \delta_m(0) \text{ for all } m \in \mathbb{N}_0. \tag{2.8}$$

2.3. Construction of a self-adjoint relation

In this section, we consider restrictions T' of T_1 (or extensions of T_0) given by the boundary condition

$$f(0)\cos\alpha - (u' - qu)(0)\sin\alpha = 0 \tag{2.9}$$

where α is a given number in $(-\pi/2, \pi/2]$. More precisely, we define

$$T' = \{(u,f) \in T_1 : f(0)\cos\alpha - (u' - qu)(0)\sin\alpha = 0\}.$$

LEMMA 2.7. *T' is self-adjoint.*

Proof. We first show that T' is an extension of T_0, *i.e.*, that any element (u,f) of T_0 satisfies the boundary condition (2.9). In fact we will show that $f(0) = (u' - qu)(0) = 0$. Since $\langle u,h\rangle = \langle f,v\rangle$ whenever $(v,h) \in T_1$ we get, choosing $(v,h) = (\psi_0, 0)$, that $f(0) = 0$ and, choosing $(v,h) = (\psi_0, \delta_0)$, that $(u' - qu)(0) = 0$.

Next we see that T' is a proper subspace of T_1 since $(\psi_0, \beta\delta_0)$ is in T_1 for all $\beta \in \mathbb{C}$ but in T' only when $\beta = \tan\alpha$. This shows that there is a $\psi \in D_i$ for which

$$i\psi(0)\cos\alpha - (\psi' - q\psi)(0)\sin\alpha = 1$$

(using a proper normalization) and hence that

$$T' \ominus T_0 = \{C(\psi - \overline{\psi}, i(\psi + \overline{\psi})) : C \in \mathbb{C}\}.$$

Using this, the fact that $T' \subset T_1 = T_0^*$, and the mutual orthogonality of T_0, $\mathscr{D}_i$ and $\mathscr{D}_{-i}$ in a straightforward calculation shows next that T' is symmetric, *i.e.*, $T' \subset T'^*$.

Finally, to show the converse inclusion, note first that $T'^* \subset T_1$ since $T_0 \subset T'$. Thus, if $(u,f) \in T'^*$ we only have to show that it satisfies the boundary condition (2.9). Since $(v,g) = ((\cos\alpha)\psi_0, (\sin\alpha)\delta_0)$ is in T' we have $\langle u,g\rangle = \langle f,v\rangle$. The identity (2.5) shows that $\langle u,g\rangle = (-u' + qu)(0)\sin\alpha$. Part (5) of Lemma 2.6 implies $\langle f,v\rangle = -f(0)\cos\alpha$. Hence (u,f) satisfies the boundary condition which completes our proof. □

Now assume that we have a self-adjoint relation T' as described above. Consider the set

$$\mathscr{H}_\infty = \{h \in \mathscr{H}_1 : (0,h) \in T'\}$$

which is a closed subspace of $\mathscr{H}_1$. Since (u,h) is in T' if and only if it satisfies the boundary condition (2.9) and the equation $Lu = wh$ we obtain immediately that

$$\mathscr{H}_\infty = \{h \in \mathscr{H}_1 : h(0)\cos\alpha = 0 \text{ and } w(n)h(n) = 0 \text{ for all } n \in \mathbb{N}\}.$$

Now set $\mathscr{H} = \mathscr{H}_1 \ominus \mathscr{H}_\infty$. We claim that $\operatorname{dom}(T')$, the domain of T', is a dense subset of $\mathscr{H}$. Indeed, if $u \in \operatorname{dom}(T')$ then there exists $f \in \mathscr{H}_1$ such that $(u,f) \in T'$. Since $(0, f_\infty) \in T'$ for all $f_\infty \in \mathscr{H}_\infty$ and since T' is self-adjoint we get

$$\langle u, f_\infty\rangle = \langle f, 0\rangle = 0.$$

Hence, $\operatorname{dom}(T') \subset \mathscr{H}$. Now assume $v \in \mathscr{H}$ is orthogonal to $\operatorname{dom}(T')$. Then

$$\langle v,u\rangle = 0 = \langle 0,h\rangle$$

for all $(u,h) \in T'$ which implies that $(0,v) \in T'^* = T'$. Hence, $v \in \mathscr{H}_\infty \cap \mathscr{H} = \{0\}$ so that $\operatorname{dom}(T')$ is dense.

THEOREM 2.8. *$T = T' \cap \mathscr{H} \oplus \mathscr{H}$ is the graph of a self-adjoint operator.*

Proof. Firstly, T is the graph of a function rather than a relation since $(f,h),(f,h') \in T$ implies $(0,h-h') \in T \subset T'$ and hence $h-h' \in \mathscr{H}_\infty \cap \mathscr{H} = \{0\}$.

To show that T is self-adjoint note that $T'^* \cap \mathscr{H} \oplus \mathscr{H} = T' \cap \mathscr{H} \oplus \mathscr{H} = T$. Thus, if $(v,h) \in T$ then $\langle v,f\rangle = \langle h,u\rangle$ for all $(u,f) \in T'$. Since $T \subset T'$ this statement holds particularly for all $(u,f) \in T$, which means $T \subset T^*$.

Conversely, let $(v,h) \in T^*$ then $(v,h) \in \mathscr{H} \oplus \mathscr{H}$ and $\langle v,f_0\rangle = \langle h,u_0\rangle$ for all $(u_0,f_0) \in T$. We need to show that $(v,h) \in T' = T'^*$, *i.e.*, $\langle v,f\rangle = \langle h,u\rangle$ for all $(u,f) \in T'$. Hence pick an arbitrary $(u,f) \in T'$. Then $u = u_0 \in \operatorname{dom}(T) \subset \mathscr{H}$ and $f = f_0 + f_\infty$ with $f_0 \in \mathscr{H}$ and $f_\infty \in \mathscr{H}_\infty$. This gives

$$\langle v,f\rangle = \langle v,f_0\rangle + \langle v,f_\infty\rangle = \langle h,u_0\rangle = \langle h,u\rangle$$

and completes the proof. □

2.4. The resolvent operator

The resolvent $(T-\lambda)^{-1} : \mathscr{H} \to \mathscr{H}$ is denoted by R_λ. We extend the domain of R_λ to $\mathscr{H}_1$ by setting $R_\lambda h = 0$ when $h \in \mathscr{H}_\infty$. The range of R_λ is $\operatorname{dom}(T-\lambda) = \operatorname{dom}(T)$, which is a dense set in $\mathscr{H}$. Note that

$$u = R_\lambda f \text{ if and only if } (u, \lambda u + f) \in T'. \tag{2.10}$$

Using the kernel g_0 of the evaluation operator we have by (2.4)

$$(R_\lambda u)(k) = \langle R_\lambda u, g_0(k,\cdot)\rangle = \langle u, R_{\overline{\lambda}} g_0(k,\cdot)\rangle.$$

Thus we may view $G(\lambda,k,\cdot) = \overline{R_{\overline{\lambda}} g_0(k,.)} = R_\lambda g_0(k,.)$ as the Green's function for our operator T. Using this function, introduce the kernel

$$g(\lambda,k,j) = G(\lambda,k,j) + g_0(k,j)/\lambda.$$

To give a precise description of $g(\lambda,k,j)$, we introduce, for $\lambda \neq 0$, the solutions $\phi(\lambda,\cdot)$ and $\theta(\lambda,\cdot)$ of $Lu = \lambda wu$ which satisfy the initial conditions

$$\lambda\theta(\lambda,0) = \cos\alpha \text{ and } \theta'(\lambda,0) - q(0)\theta(\lambda,0) = -\sin\alpha$$

and

$$\lambda\phi(\lambda,0) = \sin\alpha \text{ and } \phi'(\lambda,0) - q(0)\phi(\lambda,0) = \cos\alpha.$$

These functions satisfy $[\theta(\lambda,\cdot),\phi(\lambda,\cdot)](n) = 1/\lambda$.

THEOREM 2.9. *Suppose T is the self-adjoint operator in $\mathscr{H}$ determined by the relation T_1 and the boundary condition* (2.9)*. Then there exists a unique complex-valued function m defined on $\mathbb{C}-\mathbb{R}$, such that*

$$\psi(\lambda,\cdot) = \theta(\lambda,\cdot) + m(\lambda)\phi(\lambda,\cdot)$$

is in $\mathscr{H}_1$. Furthermore,

$$g(\lambda,k,j) = \phi(\lambda,\min(k,j))\psi(\lambda,\max(k,j)), \tag{2.11}$$

for all $k,j\in\mathbb{N}_0$,

$$(Lg(\lambda,k,\cdot))(n)=\lambda w(n)g(\lambda,k,n)+\frac{\delta_k(n)}{\lambda}, \tag{2.12}$$

for $k\in\mathbb{N}_0$ *and* $n\in\mathbb{N}$, *and*

$$g'(\lambda,k,0)-q(0)g(\lambda,k,0)=\begin{cases}\psi(\lambda,k)\cos\alpha, & \text{if } k\in\mathbb{N}\\ (m(\lambda)\cos\alpha-\sin\alpha)\sin\alpha/\lambda & \text{if } k=0.\end{cases} \tag{2.13}$$

The function m is called the Titchmarsh-Weyl m-function and $\psi(\lambda,\cdot)$ is called the Titchmarsh-Weyl solution of $Lu=\lambda wu$.

Proof. Since $\dim D_\lambda=1$ for $\operatorname{Im}\lambda\neq 0$, there exists a solution $\psi(\lambda,\cdot)$ of $Lu=\lambda wu$ which is in $\mathscr{H}_1$. Of course, ψ can be written as a linear combination of θ and ϕ, *i.e.*, $\psi(\lambda,\cdot)=A\theta(\lambda,\cdot)+B\phi(\lambda,\cdot)$. Now, A cannot be zero, because this would mean that $\phi(\lambda,\cdot)$ is an eigenfunction associated with a non-real eigenvalue of a self-adjoint operator. So by renormalizing $\psi(\lambda,\cdot)$ we may assume $A=1$. Since, due to $\dim D_\lambda=1$, there can be only one such solution we have that $m(\lambda)$ is well defined for any non-real λ.

Now, for a fixed $k\in\mathbb{N}_0$ and a fixed non-real λ, let $F(j)=f(j)-g_0(k,j)/\lambda$ where $f(j)=\phi(\lambda,\min(k,j))\psi(\lambda,\max(k,j))$. Our goal is to show that $(F,\lambda F+g_0(k,\cdot))=(F,\lambda f)\in T'$. For $j\in\mathbb{N}$ one computes $(Lf)(j)=\lambda w(j)f(j)+\delta_k(j)/\lambda$ since $[\psi(\lambda,\cdot),\phi(\lambda,\cdot)](j)=1/\lambda$. For $j=0$ we find instead

$$(f'-qf)(0)=\begin{cases}\psi(\lambda,k)\cos\alpha, & \text{if } k\in\mathbb{N}\\ (m(\lambda)\cos\alpha-\sin\alpha)\sin\alpha/\lambda & \text{if } k=0.\end{cases}$$

Employing now the identity (2.7) we find that F satisfies the equation $(LF)(j)=\lambda w(j)f(j)$ for all $j\in\mathbb{N}$ even for $j=k$. Thus $(F,\lambda f)\in T_1$. One also checks that $(F,\lambda f)$ satisfies the boundary condition (2.9) using $\lambda f(0)=\psi(\lambda,k)\sin\alpha$ and identity (2.8).

We have now shown that $(F,\lambda f)=(F,\lambda F+g_0(k,\cdot))\in T'$. Using (2.10) we get $F=R_\lambda g_0(k,\cdot)=G(\lambda,k,\cdot)$. □

Recall that an analytic function f defined on the upper half plane is called a Nevanlinna or a Herglotz function if everywhere $\operatorname{Im}(f(z))\geqslant 0$. These functions have the following representation

$$f(z)=a+bz+\int_{\mathbb{R}}\left(\frac{1}{t-z}-\frac{t}{t^2+1}\right)d\rho(t)$$

where $a\in\mathbb{R}$, $b\geqslant 0$, and $\rho:\mathbb{R}\to\mathbb{R}$ is monotone non-decreasing on $\mathbb{R}$, and satisfies $\int_{\mathbb{R}}d\rho(t)/(t^2+1)<\infty$. The numbers a and b as well as the function ρ are uniquely determined by f if we require that ρ is right-continuous and $\rho(0)=0$. To any such right-continuous monotone non-decreasing function ρ there corresponds a Lebesgue-Stieltjes measure which we will also denote by ρ. As the latter takes sets as its arguments, confusion between the monotone function and the measure cannot arise.

We show next that the Titchmarsh-Weyl m-function associated with T is a Herglotz function. The measure corresponding to m is called the spectral measure associated with T.

THEOREM 2.10. *The function m is analytic outside $\mathbb{R}$ and maps the upper half plane into itself, and hence is Herglotz. Moreover, m satisfies $m(\overline{\lambda}) = \overline{m(\lambda)}$.*

Proof. Computing $g(\lambda,0,0)$, given by equation (2.11), using the initial values for θ and $\psi = \theta + m\phi$ shows that

$$m(\lambda) = -\cot\alpha + \lambda^2 g(\lambda,0,0)/(\sin\alpha)^2$$

if $\sin\alpha \neq 0$. If instead $\sin\alpha = 0$ one computes $g(\lambda,1,1)$ to find

$$m(\lambda) = g(\lambda,1,1) - (1+q(0))/\lambda.$$

Denoting the spectral decomposition of T by $\omega \mapsto E(\omega)$ we define the cumulative distribution function $\mu_{k,k}(t) = \langle E_{(-\infty,t]} g_0(k,\cdot), g_0(k,\cdot)\rangle = \|E_{(-\infty,t]} g_0(k,\cdot)\|^2$. Then the spectral theorem implies that

$$G(\lambda,k,k) = \langle R_\lambda g_0(k,\cdot), g_0(k,\cdot)\rangle = \int_{\mathbb{R}} \frac{1}{t-\lambda} d\mu_{k,k}(t)$$

From this it follows immediately that m is analytic away from the real axis.

The fact that $m(\overline{\lambda}) = \overline{m(\lambda)}$ follows since, as the difference equation shows, this property is shared by $\theta(\lambda,\cdot)$ and $\phi(\lambda,\cdot)$.

Using formula (2.2) we find

$$\lambda \sum_{n=1}^{N} (|\psi'(\lambda,n)|^2 + q(n)|\psi(\lambda,n)|^2) = C(\lambda,N) + |\lambda|^2 \sum_{n=1}^{N} w(n)|\psi(\lambda,n)|^2, \qquad (2.14)$$

where

$$C(\lambda,N) = \lambda\psi(\lambda,N+1)\overline{\psi'(\lambda,N)} - \lambda\psi(\lambda,0)\overline{(\psi'(\lambda,0) - q(0)\psi(\lambda,0))}.$$

Because of the initial conditions satisfied by θ and ϕ one sees that

$$\operatorname{Im}(\lambda\psi(\lambda,0)\overline{(\psi'(\lambda,0) - q(0)\psi(\lambda,0))}) = -\operatorname{Im}(m(\lambda)).$$

One may also imitate the proof of part (2) of Lemma 2.4 (as we did already in the proof of Lemma 2.5) to show that $\operatorname{Im}(\lambda\psi(\lambda,N+1)\overline{\psi'(\lambda,N)})$ tends to zero as N tends to infinity. Therefore, taking imaginary parts on both sides of (2.14) and then taking N to infinity gives

$$\operatorname{Im}(\lambda)\|\psi(\lambda,\cdot)\|^2 = \operatorname{Im}(m(\lambda))$$

proving that m is a Herglotz function. $\square$

3. The Fourier transform

Let ρ be the measure associated with the Titchmarsh-Weyl m-function, *i.e.*, the spectral measure of T. This measure determines a Hilbert space $L^2(\mathbb{R},\rho)$ with the inner product

$$\langle F,G\rangle_\rho = \int_{\mathbb{R}} F\overline{G}d\rho.$$

We shall define a generalized Fourier transform $\mathfrak{F}:\mathscr{H}_1 \to L^2(\mathbb{R},\rho)$. We define it first for finitely supported functions and extend it later to all $u \in \mathscr{H}_1$. If u is finitely supported and $t \neq 0$ we set

$$(\mathfrak{F}u)(t) = \sum_{k=0}^{\infty}(u'(k)\phi'(t,k)+q(k)u(k)\phi(t,k)).$$

Formula (2.2) shows that

$$(\mathfrak{F}u)(t) = -u(0)\cos\alpha + \sum_{n=1}^{\infty} u(n)w(n)t\phi(t,n). \tag{3.1}$$

Let $p_n(t) = t\phi(t,n)$. One shows by induction that the p_n are polynomials and that their degree is at most n. This allows us to define $(\mathfrak{F}u)(0)$ by requiring $\mathfrak{F}u$ to be continuous. Equation (3.1) gives, in particular,

$$(\mathfrak{F}\delta_n)(t) = \begin{cases} w(n)t\phi(t,n), & n\in\mathbb{N},\\ -\cos\alpha, & n=0. \end{cases} \tag{3.2}$$

We also note that $(\mathfrak{F}u)(0) = -u(0)$ if $\alpha = 0$ since then, again by induction, $p_n(0) = 0$ for all $n \in \mathbb{N}_0$.

Let, as before, $\omega \mapsto E_\omega$ be the spectral resolution of T. We extend the domain of definition of each projection E_ω from $\mathscr{H}$ to $\mathscr{H}_1$ by setting $E_\omega u = 0$ when $u \in \mathscr{H}_\infty$. In particular, $E_{\mathbb{R}}$ is then the orthogonal projection from $\mathscr{H}_1$ onto $\mathscr{H}$.

LEMMA 3.1. *Let $\mu:\mathbb{R}\to\mathbb{R}$ be a monotone nondecreasing and right-continuous function, differentiable at* 0. *Then*

$$\int_{-1}^{1}\int_{(-1,1]}(t^2+s^2)^{-1/2}d\mu(t)ds < \infty.$$

This is Lemma 14.3 in Everitt and Bennewitz [3]. For the proof we refer the reader to this source even though it is short and elementary.

LEMMA 3.2. *If u and v are finitely supported sequences in $\mathscr{H}_1$ then $\mathfrak{F}u,\mathfrak{F}v \in L^2(\mathbb{R},\rho)$ and*

$$\langle E_I u, v\rangle = \langle \chi_I \mathfrak{F}u, \mathfrak{F}v\rangle_\rho$$

for any interval $I \subset \mathbb{R}$. In particular,

$$\langle E_{\mathbb{R}} u, v\rangle = \langle \mathfrak{F}u, \mathfrak{F}v\rangle_\rho.$$

Proof. By the polarization identity we have that

$$\langle E_{(-\infty,t]}u,v\rangle = \sum_{k=0}^{3} i^k \|E_{(-\infty,t]}(u+i^k v)\|^2.$$

The functions $t \mapsto \mu_k(t) = \|E_{(-\infty,t]}(u+i^k v)\|^2$ are right-continuous, monotone non-decreasing and hence differentiable away from a set of measure zero. Similarly the spectral measure ρ of T is differentiable almost everywhere. Now fix $A,B \in \mathbb{R}$, $A < B$ so that each of these distribution functions is differentiable at both A and B and let Γ be the positively orientated boundary of the rectangle with vertices at $A \pm i$ and $B \pm i$. We will show that

$$\frac{1}{2\pi i}\int_\Gamma \langle R_\lambda u,v\rangle d\lambda = -\langle E_{[A,B]}u,v\rangle \tag{3.3}$$

as long as u,v are finitely supported functions. Under these conditions we also have

$$\frac{1}{2\pi i}\int_\Gamma m(\lambda)(\mathfrak{F}u)(\lambda)\overline{(\mathfrak{F}v)(\overline{\lambda})}d\lambda = -\langle \chi_{[A,B]}\mathfrak{F}u,\mathfrak{F}v\rangle_\rho. \tag{3.4}$$

The function

$$\lambda \mapsto \langle R_\lambda u,v\rangle - m(\lambda)(\mathfrak{F}u)(\lambda)\overline{(\mathfrak{F}v)(\overline{\lambda})} \tag{3.5}$$

is, as we will also show, a polynomial. It will then follow from Cauchy's theorem that

$$\langle E_{[A,B]}u,v\rangle = \langle \chi_{[A,B]}\mathfrak{F}u,\mathfrak{F}v\rangle_\rho$$

for all intervals whose endpoints are chosen from a certain set of full measure. By right-continuity the equation holds, in fact, for all finite and infinite intervals.

We begin with the proof of (3.3). By the spectral theorem

$$\langle R_\lambda u,v\rangle = \int_{\mathbb{R}} \frac{d\langle E_{(-\infty,t]}u,v\rangle}{t-\lambda},$$

so that

$$\frac{1}{2\pi i}\int_\Gamma \langle R_\lambda u,v\rangle d\lambda = \frac{1}{2\pi i}\int_\Gamma\int_{\mathbb{R}} \frac{d\langle E_{(-\infty,t]}u,v\rangle}{t-\lambda}d\lambda.$$

This integral is absolutely convergent, which one sees in the following way: after splitting Γ into its four pieces and using the polarization identity on $\langle E_{(-\infty,t]}u,v\rangle$ one obtains a number of integrals of the form treated in Lemma 3.1 with $\mu(t) = \|E_{(-\infty,t]}(u+i^k v)\|^2$, $k = 0,\ldots,3$. Each of these is absolutely convergent for our choice of A and B. Thus we may apply Fubini's theorem to obtain

$$\frac{1}{2\pi i}\int_\Gamma\int_{\mathbb{R}} \frac{d\langle E_{(-\infty,t]}u,v\rangle}{t-\lambda}d\lambda = \frac{1}{2\pi i}\int_{\mathbb{R}} d\langle E_{(-\infty,t]}u,v\rangle \int_\Gamma \frac{1}{t-\lambda}d\lambda.$$

Cauchy's integral formula, the fact that A and B do not carry mass, and the identity $\int_{\mathbb{R}} \chi_{[A,B]} d\langle E_{(-\infty,t]}u,v\rangle = \langle E_{[A,B]}u,v\rangle$ give equation (3.3).

The proof of (3.4) is similar after replacing $m(\lambda)$ by its Herglotz representation and employing Lemma 3.1 with $\mu = \rho$.

It remains to establish that the expression (3.5) is a polynomial in λ. The explicit form of the Green's function gives

$$R_\lambda \delta_j(k) = \langle g(\lambda,k,\cdot),\delta_j\rangle - \frac{1}{\lambda}\delta_j(k)$$

for $j,k \in \mathbb{N}_0$. Equations (2.5), (2.12) and (2.13) give, for any $k \in \mathbb{N}_0$,

$$R_\lambda \delta_j(k) = \begin{cases} \lambda w(j) g(\lambda,j,k) & \text{if } j \in \mathbb{N} \\ -\psi(\lambda,k)\cos\alpha & \text{if } j = 0. \end{cases}$$

This, (2.5), and (2.10) entail

$$\langle R_\lambda \delta_j, \delta_k\rangle = w(k)(\lambda(R_\lambda \delta_j)(k) + \delta_j(k)) = \lambda^2 w(k) w(j) g(\lambda,j,k) + w(k)\delta_j(k)$$

as long as $j,k \in \mathbb{N}$. Similarly, for $k \in \mathbb{N}$,

$$\langle R_\lambda \delta_0, \delta_k\rangle = -\lambda w(k)\psi(\lambda,k)\cos\alpha = \overline{\langle R_{\overline{\lambda}}\delta_0, \delta_k\rangle} = \langle R_\lambda \delta_k, \delta_0\rangle$$

and

$$\langle R_\lambda \delta_0, \delta_0\rangle = -\sin\alpha\cos\alpha + m(\lambda)(\cos\alpha)^2.$$

We now use these results and (3.2) in (3.5) when $u = \delta_j$ and $v = \delta_k$. In the resulting expression we replace ψ and g by their equivalents in terms of θ, ϕ and m to see that all contributions involving m will cancel and that the remaining term is a polynomial in λ. The general case for finitely supported u and v follows then immediately. □

The previous lemma allows us to extend the definition of the Fourier transform to all of $\mathscr{H}_1$. To this end let u be any element of $\mathscr{H}_1$ and $n \mapsto u_n$ a sequence of finitely supported functions converging to u. Consequently,

$$\|\mathfrak{F}(u_n - u_m)\|_\rho = \|E_{\mathbb{R}}(u_n - u_m)\| \leqslant \|u_n - u_m\|.$$

Thus $n \mapsto \mathfrak{F}u_n$ is a Cauchy sequence in $L^2(\mathbb{R},\rho)$ and hence convergent. The limit does not depend on the sequence chosen to approximate u and is, by definition, $\mathfrak{F}u$.

THEOREM 3.3. *The following form of Parseval's identity holds for the Fourier transform* $\mathfrak{F}: \mathscr{H}_1 \to L^2(\mathbb{R},\rho)$*: for all* $u,v \in \mathscr{H}_1$

$$\langle E_{\mathbb{R}} u, v\rangle = \langle \mathfrak{F}u, \mathfrak{F}v\rangle_\rho.$$

Moreover, $\mathfrak{F}$ *has kernel* $\mathscr{H}_\infty$.

Proof. Parseval's identity holds because of the continuity of inner products, the boundedness of $E_{\mathbb{R}}$, and the very definition of $\mathfrak{F}u$ as a limit of transforms of finitely supported sequences. Thus, $u \in \ker\mathfrak{F}$ if and only if $\|E_{\mathbb{R}}u\| = 0$, *i.e.*, if and only if $u \in \mathscr{H}_\infty$. □

It will prove useful later to consider the following two trivial examples.

EXAMPLE 3.4. Assume $\cos\alpha = 0$ and $w(k) = 0$ for all $k \in \mathbb{N}$. Then $\mathscr{H}_1 = \mathscr{H}_\infty$ which implies $\mathscr{H} = \{0\}$. In this case $\psi(\lambda,\cdot) = -\psi_0$ and hence $\psi(\lambda,0) = -\psi_0(0)$ which gives $m(\lambda) = -\lambda\psi_0(0)$. The Herglotz representation of m is

$$m(\lambda) = -\lambda\psi_0(0) = a + b\lambda + \int_{\mathbb{R}} \frac{1+t\lambda}{(t-\lambda)(1+t^2)} d\rho(t).$$

Therefore, $a = 0$, $b = -\psi_0(0)$, and $\rho = 0$ so that $L^2(\mathbb{R},\rho) = \{0\}$. In this case the Fourier transform is $\mathfrak{F} = 0$.

EXAMPLE 3.5. Assume $\cos\alpha \neq 0$ but $w(k) = 0$ for all $k \in \mathbb{N}$. Then $\mathscr{H}_\infty = \{h \in \mathscr{H}_1 : h(0) = 0\}$. In particular $\ell_0 \subset \mathscr{H}_\infty$ so that $\mathscr{H} \subset D_0$. Since, by part (5) of Lemma 2.6, $\langle h, \psi_0 \rangle = -h(0)$ we see that, in fact, $D_0 = \mathscr{H}$, which is one-dimensional. We have $\mathfrak{F}u = -u(0)\cos\alpha$ for any $u \in \mathscr{H}_1$. To determine $L^2(\mathbb{R},\rho)$ we consider again $\psi(\lambda,\cdot)$ which is in D_0 and hence a multiple of ψ_0. Specifically, employing the initial conditions for $\psi(\lambda,\cdot)$ and ψ_0,

$$\psi(\lambda,\cdot) = (-\sin\alpha + m(\lambda)\cos\alpha)\psi_0$$

which implies

$$m(\lambda) = \frac{\lambda\psi_0(0)\sin\alpha + \cos\alpha}{\lambda\psi_0(0)\cos\alpha - \sin\alpha}.$$

Thus m is a Möbius transform and this is only possible if ρ is a Dirac measure at the (sole) eigenvalue $\lambda_0 = \tan\alpha/\psi_0(0)$ of T. One also finds that

$$\rho(\{\lambda_0\}) = -\psi_0(0)^{-1}(\cos\alpha)^{-2}.$$

Thus $L^2(\mathbb{R},\rho)$ is the set of all equivalence classes of complex-valued functions on $\mathbb{R}$ which agree at the point λ_0. We have $\|f\|_\rho^2 = |f(\lambda_0)|^2\rho(\{\lambda_0\})$ if $f \in L^2(\mathbb{R},\rho)$.

LEMMA 3.6. *If λ is not real and $u \in \mathscr{H}_1$, then the Fourier transform of $R_\lambda u$ is $t \mapsto \mathfrak{F}u(t)/(t-\lambda)$.*

Proof. By the spectral theorem

$$\langle R_\lambda u, v \rangle = \int_{\mathbb{R}} \frac{d\langle E_{(-\infty,t]}u, v\rangle}{t-\lambda} = \int_{\mathbb{R}} \frac{\hat{u}(t)\overline{\hat{v}(t)}}{t-\lambda} d\rho(t)$$

where $\hat{u} = \mathfrak{F}u$ and $\hat{v} = \mathfrak{F}v$. Employing the identities $R_\lambda - R_{\overline{\lambda}} = (\lambda - \overline{\lambda})R_{\overline{\lambda}}R_\lambda$ and $\langle R_\lambda u, R_\lambda u \rangle = \langle R_{\overline{\lambda}}R_\lambda u, u \rangle$ one arrives at

$$\left\| \frac{\hat{u}(t)}{t-\lambda} \right\|_{\rho(t)}^2 = \langle R_\lambda u, R_\lambda u \rangle = \int_{\mathbb{R}} \frac{\hat{u}(t)\overline{\mathfrak{F}(R_\lambda u)(t)}}{t-\lambda} d\rho(t).$$

Thus

$$\|\mathfrak{F}(R_\lambda u) - \frac{\hat{u}(t)}{t-\lambda}\|_{\rho(t)}^2 = \|\mathfrak{F}(R_\lambda u)\|_\rho^2 - 2\|R_\lambda u\|^2 + \|R_\lambda u\|^2$$

which is zero by Parseval's identity. □

LEMMA 3.7. $1 \in \mathfrak{F}(\mathscr{H}_1) \subset L^2(\mathbb{R},\rho) \subset L^1(\mathbb{R},\rho)$.

Proof. If all $w(k)$ are equal to zero and $\cos\alpha = 0$ we know from Example 3.4 that $\rho = 0$ and hence all functions (including 1) are equivalent to $0 \in \mathfrak{F}(\mathscr{H}_1)$. Otherwise let $k_0 = 0$ if $\cos\alpha \neq 0$ or else let k_0 be the first positive integer for which $w(k_0) \neq 0$. It follows then by induction that the functions $t \mapsto t\phi(t,n)$ are constant for $n \leqslant k_0$. Equation (3.1) shows next that $\mathfrak{F}\delta_{k_0}$ is constant. This proves the first inclusion. The second inclusion was established already and the last inclusion follows from the Cauchy-Schwarz inequality and the fact that $1 \in L^2(\mathbb{R},\rho)$. □

COROLLARY 3.8. *Suppose $\lambda \in \mathbb{C} - \mathbb{R}$. Then $t \mapsto 1/(t-\lambda)$ is in $L^1(\mathbb{R},\rho)$.*

Proof. $|1/(t-\lambda)| \leqslant 1/|\mathrm{Im}(\lambda)| \in L^1(\mathbb{R},\rho)$. □

LEMMA 3.9. *The Fourier transform $\mathfrak{F} : \mathscr{H}_1 \to L^2(\mathbb{R},\rho)$ is surjective.*

Proof. Assume that $\hat{h} \in L^2(\mathbb{R},\rho)$ is orthogonal to all transforms. We will show below that then $\hat{h} = 0$ almost everywhere with respect to ρ so that the range of $\mathfrak{F}$ is dense in $L^2(\mathbb{R},\rho)$. Now suppose $\hat{f}$ is any element of $L^2(\mathbb{R},\rho)$. Then there is a sequence $n \mapsto u_n$ such that $\mathfrak{F}u_n$ converges to $\hat{f}$. This implies that $n \mapsto \mathfrak{F}u_n$ and hence $n \mapsto E_{\mathbb{R}}u_n$ are Cauchy sequences. But $u = \lim_{n\to\infty} E_{\mathbb{R}}u_n \in \mathscr{H}$ will map to $\hat{f}$ under $\mathfrak{F}$, since $\mathfrak{F}$ is continuous.

It remains to show that $\hat{h} = 0$ almost everywhere with respect to ρ if it is orthogonal to all transforms. Because 1 is a transform (by Lemma 3.7) so are both $1/(t-\lambda)$ and $1/(t-\overline{\lambda})$. Thus we get

$$\int_{\mathbb{R}} \frac{\hat{h}(t)\nu}{(t-\mu)^2+\nu^2} d\rho(t) = 0$$

where $\mu = \mathrm{Re}(\lambda)$ and $\nu = \mathrm{Im}(\lambda) > 0$. We may now integrate this expression over μ between A and B. Since $\hat{h} \in L^1(\mathbb{R},\rho)$ we may apply Fubini's theorem to obtain

$$0 = \int_{\mathbb{R}} \int_A^B \frac{\hat{h}(t)\nu}{(t-\mu)^2+\nu^2} d\mu\, d\rho(t) = \int_{\mathbb{R}} \hat{h}(t) \left(\arctan\left(\frac{B-t}{\nu}\right) - \arctan\left(\frac{A-t}{\nu}\right) \right) d\rho(t).$$

As $\nu > 0$ tends to zero $(\arctan((B-t)/\nu) - \arctan((A-t)/\nu))/\pi$ tends to the characteristic function of $[A,B]$, except at A and B. This proves that $\int_{[A,B]} \hat{h} d\rho = 0$ provided that A and B are points of continuity of ρ. It follows now from right-continuity that $t \mapsto \int_{(-\infty,t]} \hat{h} d\rho = 0$ for all $t \in \mathbb{R}$ and this proves that $\hat{h} = 0$ almost everywhere with respect to ρ. □

THEOREM 3.10. *If $u \in \mathrm{dom}(T)$ then $\mathfrak{F}(Tu)(t) = t(\mathfrak{F}u)(t)$. Conversely, if $\hat{u}$ and $t \mapsto t\hat{u}(t)$ are in $L^2(\mathbb{R},\rho)$, then u, the preimage of $\hat{u}$ under $\mathfrak{F}$ in $\mathscr{H}$, is in* $\mathrm{dom}(T)$.

Proof. u is in $\operatorname{dom}(T)$ precisely when there exists a $v \in \mathscr{H}_1$ such that $Tu = v$ or, equivalently, $u = R_\lambda(v - \lambda u)$. Taking the Fourier transform on both sides we get $\mathfrak{F}u(t) = (\mathfrak{F}v(t) - \lambda \mathfrak{F}u(t))/(t - \lambda)$ or, after simplification, $t\mathfrak{F}u(t) = \mathfrak{F}v(t)$. □

LEMMA 3.11. *Each of the following three statements implies the other two.*

1. $\alpha = 0$.
2. *The operator T has eigenvalue 0 with eigenfunction ψ_0.*
3. $\rho(\{0\}) \neq 0$.

Moreover, if $\alpha = 0$, then $\mathfrak{F}\psi_0 = (\mathfrak{F}\psi_0)(0)\chi_{\{0\}}$ where $\chi_{\{0\}}$ is the characteristic function at 0, and

$$(\mathfrak{F}\psi_0)(0) = -\psi_0(0) = \|\psi_0\|^2 = 1/\rho(\{0\}).$$

Proof. If $\alpha = 0$, then $0\cos\alpha - (\psi_0' - q\psi_0)(0)\sin\alpha = 0$ which means ψ_0 is an eigenfunction of T associated with the eigenvalue 0. If $T\psi_0 = 0$ then, by Theorem 3.10, we have $t(\mathfrak{F}\psi_0)(t) = 0$. Thus $\mathfrak{F}\psi_0$ is a multiple of $\chi_{\{0\}}$, the characteristic function at 0, and $\rho(\{0\})$ can not be zero since $\|\mathfrak{F}\psi_0\|_\rho = \|\psi_0\| \neq 0$. Finally, suppose $\rho(\{0\}) \neq 0$. The function $\chi_{\{0\}} \in L^2(\mathbb{R},\rho)$ has a preimage $u \neq 0$ in $\mathscr{H}$. In fact $u \in \operatorname{dom}(T)$ by Theorem 3.10 and $Tu = 0$. This implies that u is a multiple of ψ_0 and the boundary condition gives $0\cos\alpha - (\psi_0' - q\psi_0)(0)\sin\alpha = -\sin\alpha = 0$. Hence $\alpha = 0$.

To prove the last statement let $n \mapsto u_n$ be a sequence of finitely supported functions converging to ψ_0. We know that $(\mathfrak{F}u_n)(0) = -u_n(0) = \langle u_n, \psi_0\rangle$ and, by Lemma 2.1, $\lim_{n\to\infty} u_n(0) = \psi(0)$. By Parseval's identity

$$|(\mathfrak{F}\psi_0)(0) - (\mathfrak{F}u_n)(0)|^2\rho(\{0\}) \leqslant \|\mathfrak{F}(\psi_0 - u_n)\|_\rho^2 = \|E_\mathbb{R}(\psi_0 - u_n)\|^2 \leqslant \|\psi_0 - u_n\|^2$$

which tends to zero. Hence $(\mathfrak{F}\psi_0)(0) = -\psi_0(0) = \langle\psi_0, \psi_0\rangle$. Now Parseval's identity and evaluation of the resulting integral complete the proof. □

We now describe the inverse Fourier transform. Let

$$e(\cdot,k) = \mathfrak{F}g_0(k,\cdot)$$

and define $\mathfrak{G} : L^2(\mathbb{R},\rho) \to \mathscr{H}_1$ by

$$(\mathfrak{G}\hat{u})(k) = \langle\hat{u}, e(\cdot,k)\rangle_\rho.$$

That $\mathfrak{G}\hat{u}$ is indeed in $\mathscr{H}_1$, in fact in $\mathscr{H}$, is a part of the proof of the following theorem.

THEOREM 3.12. *$\mathfrak{G}$ is the adjoint of $\mathfrak{F}$ and the inverse of $\mathfrak{F}|_{\mathscr{H}}$. In particular, $\mathfrak{F}\circ\mathfrak{G} = I$ and $\mathfrak{G}\circ\mathfrak{F} = E_\mathbb{R}$.*

Moreover,

$$e(t,k) = \begin{cases} \phi(t,k), & \text{if } t \neq 0 \\ \psi_0(k)\cos\alpha, & \text{if } t = 0 \end{cases}$$

almost everywhere with respect to ρ.

Proof. Assume $u \in \mathscr{H}_1$ and let $\hat{u} = \mathfrak{F}u$. Then

$$(\mathfrak{G}\hat{u})(k) = \langle \hat{u}, e(\cdot,k)\rangle_\rho = \langle E_\mathbb{R} u, g_0(k,\cdot)\rangle = (E_\mathbb{R} u)(k)$$

by Parseval's identity. Hence $\mathfrak{G} \circ \mathfrak{F} = E_\mathbb{R}$. Similarly, assume $\hat{u} \in L^2(\mathbb{R},\rho)$. Since $\mathfrak{F}$ is surjective there is a $u \in \mathscr{H}_1$ such that $\hat{u} = \mathfrak{F}u$. Therefore,

$$(\mathfrak{F} \circ \mathfrak{G})(\hat{u}) = \mathfrak{F}(E_\mathbb{R} u) = \mathfrak{F}(u) = \hat{u},$$

which means $\mathfrak{F} \circ \mathfrak{G} = I$.

Since $\mathfrak{G}$ maps into $\mathscr{H}$ we have shown that $\mathfrak{F}|_\mathscr{H} \circ \mathfrak{G} = \mathfrak{G} \circ \mathfrak{F}|_\mathscr{H}$ is the identity on $\mathscr{H}$, *i.e.*, $\mathfrak{G} = \mathfrak{F}|_\mathscr{H}^{-1}$. To show that $\mathfrak{G} = \mathfrak{F}^*$ we point out that

$$\langle \mathfrak{F}u, \hat{v}\rangle_\rho = \langle \mathfrak{F}u, \mathfrak{F}(\mathfrak{G}\hat{v})\rangle_\rho = \langle u, \mathfrak{G}\hat{v}\rangle$$

using again that $\mathfrak{G}$ maps into $\mathscr{H}$ to justify the application of Parseval's identity.

Now we compute the kernel e. If $\cos\alpha = 0$ and $w(k) = 0$ for all $k \in \mathbb{N}$, then all functions are equivalent to zero so there is nothing to show. Otherwise, as in Lemma 3.7, let k_0 be 0 if $\cos\alpha \neq 0$ or else the smallest positive integer k for which $w(k) \neq 0$.

Now suppose $t \neq 0$. The linearity of $\mathfrak{F}$, (2.7), (2.8), and (3.2) give that $e(t,\cdot)$ satisfies the equations

$$Le(t,\cdot) = tw\phi(t,\cdot) = L\phi(t,\cdot) \tag{3.6}$$

and

$$e'(t,0) - q(0)e(t,0) = \cos\alpha = \phi'(t,0) - q(0)\phi(t,0). \tag{3.7}$$

Hence $u(t,\cdot) = e(t,\cdot) - \phi(t,\cdot)$ satisfies the homogenous equation $Ly = 0$ and is thus a linear combination of ϕ_0 and ψ_0. Taking also (3.7) and parts (1) and (2) of Lemma 2.6 into account shows that there is a number $a(t)$ for which $u(t,\cdot) = a(t)\phi_0$. We want to show that $a(t) = 0$. Suppose $\hat{v} \in L^2(\mathbb{R},\rho)$ is compactly supported. Then $v = \mathfrak{G}\hat{v}$ is in $\operatorname{dom}(T)$. Since $v(k) = \langle \hat{v}, e(\cdot,k)\rangle_\rho$ we get from (3.6) that

$$(Lv)(k) = \langle \hat{v}(t), tw(k)\phi(t,k)\rangle_{\rho(t)} \tag{3.8}$$

and from (3.7) that

$$(v' - qv)(0) = \langle \hat{v}, \cos\alpha\rangle_\rho. \tag{3.9}$$

But, using (2.4), we also have

$$(Lv)(k) = w(k)(Tv)(k) = w(k)\langle Tv, g_0(k,\cdot)\rangle = w(k)\langle t\hat{v}(t), e(t,k)\rangle_{\rho(t)} \tag{3.10}$$

and

$$(Tv)(0) = \langle Tv, g_0(0,\cdot)\rangle = \langle t\hat{v}(t), e(t,0)\rangle_{\rho(t)}. \tag{3.11}$$

If $k_0 = 0$ we combine (3.9) and (3.11) to get

$$0 = (Tv)(0)\cos\alpha - (v' - qv)(0)\sin\alpha = \cos\alpha\langle \hat{v}(t), t(e(t,0) - \phi(t,0))\rangle_{\rho(t)}.$$

If $k_0 > 0$ we combine (3.8) and (3.10) to get

$$0 = w(k_0)\langle \hat{v}(t), t(e(t,k_0) - \phi(t,k_0))\rangle_{\rho(t)}.$$

In either case, since $t \neq 0$, we have $a(t)\phi_0(k_0) = e(t,k_0) - \phi(t,k_0) = 0$ almost everywhere with respect to ρ. We also know from part (3) of Lemma 2.6 that $\phi_0(k_0) \neq 0$ so that $a(t) = 0$ and $e(t,k) = \phi(t,k)$ except possibly for $t = 0$.

Finally, let $t = 0$. Only when $\alpha = 0$ are the values of $e(0,k)$ of any significance. We have then

$$\psi_0(n) = \langle -\psi_0(0)\chi_{\{0\}}, e(\cdot,n)\rangle_\rho = -\psi_0(0)e(0,n)\rho(\{0\}) = e(0,n)$$

according to Lemma 3.11. □

LEMMA 3.13. *We have the following identities:*

$$(\mathfrak{F}\psi_0)(t) = \begin{cases} -\sin\alpha/t & \text{if } \alpha \neq 0 \\ \chi_{\{0\}}(t)/\rho(\{0\}) & \text{if } \alpha = 0 \end{cases}$$

and, if $\operatorname{Im}(\lambda) \neq 0$*,*

$$(\mathfrak{F}\psi(\lambda,\cdot))(t) = \frac{1}{t-\lambda}.$$

Proof. We have already computed $\mathfrak{F}\psi_0$ for $\alpha = 0$ in Lemma 3.11. If $\alpha \neq 0$ we have $\psi_0 = -g_0(0,\cdot)$ and hence $(\mathfrak{F}\psi_0)(t) = -e(t,0) = -\phi(t,0) = -\sin\alpha/t$. Recall that we may disregard the point $t = 0$ when $\alpha \neq 0$.

Since $(\lambda R_\lambda + I)g_0(0,\cdot) = \lambda g(\lambda,0,\cdot)$ we find on taking Fourier transforms

$$\left(\frac{\lambda}{t-\lambda} + 1\right) e(t,0) = \lambda\phi(\lambda,0)(\mathfrak{F}\psi(\lambda,\cdot))(t).$$

This proves the second claim for $\alpha \neq 0$ since $te(t,0) = \lambda\phi(\lambda,0) = \sin\alpha \neq 0$. If $\alpha = 0$ we have $\psi(\lambda,\cdot) = -R_\lambda\delta_0$. Taking Fourier transforms

$$(\mathfrak{F}\psi(\lambda,\cdot))(t) = -\frac{(\mathfrak{F}\delta_0)(t)}{t-\lambda} = \frac{1}{t-\lambda}$$

according to (3.2). □

We end this section with a version of the classical Paley-Wiener theorem which relates growth behavior of the transform to the size of the support of a function in the original space.

THEOREM 3.14. *Assume* $w(n) \neq 0$ *for all* $n \in \mathbb{N}$*. If* $u \in \mathscr{H}_1$ *has its support in* $[0,N]$*, then* $\mathfrak{F}u$ *is a polynomial of degree at most* N*. If* $\cos\alpha = 0$ *the degree is in fact at most* $N-1$*. Conversely, suppose* $\hat{u} \in L^2(\mathbb{R},\rho)$ *has a polynomial continuation of degree* N*. Then the support of* $u = \mathfrak{F}^{-1}\hat{u}$ *is contained in* $[0,N]$ *provided that* $\cos\alpha \neq 0$*. If* $\cos\alpha = 0$ *then* $\operatorname{supp} u$ *is contained in* $[0,N+1]$.

Proof. An induction proof, using the initial conditions satisfied by $\phi(t,\cdot)$, shows that the leading term of the polynomial $p_n(t) = t\phi(t,n)$ is $t^n(\cos\alpha)\prod_{j=1}^{n-1}(-w(j))$ except when $n = 0$ in which case we have $p_0(t) = \sin\alpha$. The first part of the theorem is now immediate in view of equation (3.1).

Next suppose $\hat{u} \in L^2(\mathbb{R},\rho)$ extends to polynomial of degree N and that $\cos\alpha \neq 0$. Then $\hat{u}(t) = \nu_0 + \sum_{j=1}^{N} \nu_j p_j(t)$ with uniquely determined coefficients ν_j. It follows that

$$u = \mathfrak{F}^{-1}\hat{u} = \frac{-\nu_0}{\cos\alpha}\delta_0 + \sum_{j=1}^{N} \frac{\nu_j}{w(j)}\delta_j$$

and this sequence has its support in $[0,N]$. The proof is similar if $\cos\alpha = 0$ when one takes into account that in this case p_n has degree $n-1$ with leading coefficient $(1+q(0))\prod_{j=1}^{n-1}(-w(j))$ so that $\hat{u}(t) = \sum_{j=1}^{N+1} \nu_j p_j(t)$. $\square$

4. The inverse spectral problem

In this section we consider the following inverse problem: suppose there is another operator $\breve{T}$ of the same type as T, with Hilbert space $\breve{\mathscr{H}}_1$, boundary condition parameter $\breve{\alpha}$, and coefficients $\breve{q}$ and $\breve{w}$. Suppose $\breve{T}$ and T have the same spectral measure, *i.e.*, $\breve{\rho} = \rho$. Then what can we say about the relationship between the operators T and $\breve{T}$?

From now on we will use the subscripts $\mathscr{H}_1$ and $\breve{\mathscr{H}}_1$ in our notation for the scalar products in these spaces to avoid confusion. We will also make the following assumption in this section.

HYPOTHESIS 4.1. $w(n) \neq 0$ and $\breve{w}(n) \neq 0$ for all $n \in \mathbb{N}$.

We then have two possibilities for $\mathscr{H}$. Either $\mathscr{H} = \mathscr{H}_1$ if $\cos\alpha \neq 0$ or $\mathscr{H} = \{\delta_0\}^{\perp} = \{u \in \mathscr{H}_1 : (u' - qu)(0) = 0\}$ if $\cos\alpha = 0$. In the later case neither δ_0 nor δ_1 are in $\mathscr{H}$. In fact, δ_0 spans $\mathscr{H}_\infty$ and δ_1 has a component in $\mathscr{H}_\infty$. Its projection onto $\mathscr{H}$ is

$$\varepsilon_0 = E_{\mathbb{R}}\delta_1 = \delta_1 + \frac{1}{1+q(0)}\delta_0.$$

Define

$$\mathscr{U} = \breve{\mathfrak{G}} \circ \mathfrak{F} : \mathscr{H}_1 \to \breve{\mathscr{H}}_1$$

and note that the range of $\mathscr{U}$ is $\breve{\mathscr{H}}$. Also $\mathscr{U}^* = \mathfrak{G} \circ \breve{\mathfrak{F}}$ maps $\breve{\mathscr{H}}_1 \to \mathscr{H}_1$ and has its range in $\mathscr{H}$. $\mathscr{U}$ is unitary as a map from $\mathscr{H}$ to $\breve{\mathscr{H}}$ since $\mathfrak{F}$ and $\breve{\mathfrak{G}}$ are. For later reference we note that

$$\mathfrak{G}(1) = \begin{cases} -\delta_0/\cos\alpha & \text{if } \cos\alpha \neq 0 \\ \varepsilon_0/(w(1)(1+q(0))) & \text{if } \cos\alpha = 0. \end{cases}$$

There is, of course, a corresponding expression for $\breve{\mathfrak{G}}(1)$. One shows easily that

$$(\mathscr{U}\delta_0)(k) = -(\cos\alpha)\breve{\mathfrak{G}}(1)(k) \text{ and } (\mathscr{U}^*\delta_0)(k) = -(\cos\breve{\alpha})\mathfrak{G}(1)(k) \tag{4.1}$$

for all $k \in \mathbb{N}_0$.

LEMMA 4.2. *Suppose $\rho = \breve{\rho}$ and $n,k \geqslant 1$. Then*

$$\breve{w}(k)(\mathscr{U}\delta_n)(k) = w(n)(\mathscr{U}^*\delta_k)(n)$$

and

$$(\mathscr{U}\delta_n)(0) = w(n)(\sin\breve{\alpha})\mathfrak{G}(1)(n) \text{ and } (\mathscr{U}^*\delta_k)(0) = \breve{w}(k)(\sin\alpha)\breve{\mathfrak{G}}(1)(k).$$

Proof. We have

$$(\mathscr{U}\delta_n)(k) = \langle \breve{\mathfrak{F}}\mathscr{U}\delta_n, \breve{\mathfrak{F}}\breve{g}_0(k,\cdot)\rangle_\rho = \langle \mathfrak{F}\delta_n, \breve{e}(\cdot,k)\rangle_\rho = w(n)\langle t\phi(t,n), \breve{e}(t,k)\rangle_{\rho(t)}$$

where we used that $\mathscr{U}\delta_n \in \breve{\mathscr{H}}$ and Parseval's identity for the first equation, $\breve{\mathfrak{F}}\mathscr{U} = \mathfrak{F}$ and the definition of $\breve{e}$, the kernel of the inverse transform $\breve{\mathfrak{G}}$, for the second, and equation (3.2) in the third. For almost all t with respect to ρ we have $t\phi(t,n) = te(t,n)$ and $t\breve{\phi}(t,k) = t\breve{e}(t,k)$ even if $t = 0$. Hence

$$(\mathscr{U}\delta_n)(k) = w(n)\langle e(t,n), t\breve{\phi}(t,k)\rangle_{\rho(t)} = \frac{w(n)}{\breve{w}(k)}\langle \breve{\mathfrak{F}}\delta_k, e(t,n)\rangle_{\rho(t)} = \frac{w(n)}{\breve{w}(k)}(\mathscr{U}^*\delta_k)(n).$$

A similar reasoning shows that $(\mathscr{U}\delta_n)(0) = w(n)(\sin\breve{\alpha})\mathfrak{G}(1)(n)$, if $n \geqslant 1$ and the corresponding expression for $\mathscr{U}^*\delta_k$. $\square$

For future reference we also state that

$$\langle \delta_n, \delta_m\rangle_{\mathscr{H}_1} = \begin{cases} 0, & |k-m| \geqslant 2, \\ -1, & |k-m| = 1, \\ 2+q(m), & k = m \in \mathbb{N}, \\ 1+q(1), & k = m = 0 \end{cases} \tag{4.2}$$

The analogous expression holds in $\breve{\mathscr{H}}_1$.

4.1. The case where $\cos\alpha \neq 0 \neq \cos\breve{\alpha}$

THEOREM 4.3. *Assume the validity of Hypothesis 4.1 and that $\cos\alpha \neq 0 \neq \cos\breve{\alpha}$. The spectral measures ρ and $\breve{\rho}$ are identical if and only there is a sequence $r \in \mathbb{C}^{\mathbb{N}_0}$ with the following properties*

1. $r(n)\overline{r(n+1)} = 1$ *for all* $n \in \mathbb{N}_0$,
2. $|r(n)|^2(2+\breve{q}(n)) = 2+q(n)$ *for all* $n \in \mathbb{N}$,
3. $|r(0)|^2(1+\breve{q}(0)) = 1+q(0)$,
4. $|r(n)|^2\breve{w}(n) = w(n)$ *for all* $n \in \mathbb{N}$, *and*
5. $r(0) = \sin\breve{\alpha}/\sin\alpha = \cos\alpha/\cos\breve{\alpha} \neq 0$,

where the last condition is to be interpreted as requiring that α and $\breve{\alpha}$ are either both different from zero or else both equal to zero in which case we ask that $r(0)=1$.

Proof. First assume that the conditions on r hold and define $\mathscr{L}:\mathbb{C}^{\mathbb{N}_0}\to\mathbb{C}^{\mathbb{N}_0}$: $u\mapsto ru$. An easy computation using properties (1), (2) and (4) shows that $\breve{L}y=\lambda\breve{w}y$ if $y=\mathscr{L}u$ and $Lu=\lambda wu$. Taking properties (3) and (5) into account we can relate the initial condition of $\mathscr{L}\phi$ and $\mathscr{L}\theta$ to those of ϕ and θ. This yields

$$\mathscr{L}\phi(\lambda,\cdot)=\breve{\phi}(\lambda,\cdot)$$

and

$$\mathscr{L}\theta(\lambda,\cdot)=\breve{\theta}(\lambda,\cdot)+c(\alpha,\breve{\alpha})\breve{\phi}(\lambda,\cdot),$$

where $c(\alpha,\breve{\alpha})=0$ if $\alpha=\breve{\alpha}=0$ and $c(\alpha,\breve{\alpha})=\cot\alpha-\cot\breve{\alpha}$ otherwise.

$\mathscr{L}\psi(\lambda,\cdot)$ is also a solution of the difference equation $\breve{L}y=\lambda\breve{w}y$. In fact, by the linearity of $\mathscr{L}$,

$$\mathscr{L}\psi(\lambda,\cdot)=\breve{\theta}(\lambda,\cdot)+(m(\lambda)+c(\alpha,\breve{\alpha}))\breve{\phi}(\lambda,\cdot).$$

We will prove that this is equal to $\breve{\psi}(\lambda,\cdot)$ as this implies that $\breve{m}(\lambda)=m(\lambda)+c(\alpha,\breve{\alpha})$ and hence, using the uniqueness of the Herglotz representation, that $\breve{\rho}=\rho$.

To show that $\mathscr{L}\psi(\lambda,\cdot)=\breve{\psi}(\lambda,\cdot)$ we simply need to argue that $\mathscr{L}|_{\mathscr{H}_1}$ maps into $\breve{\mathscr{H}}_1$. This is indeed so since $\mathscr{L}|_{\mathscr{H}_1}:\mathscr{H}_1\to\breve{\mathscr{H}}_1$ is unitary as the following computation shows. Pick arbitrary $u,v\in\mathscr{H}_1$. Then, because of the first condition on r,

$$\begin{aligned}\langle\mathscr{L}u,\mathscr{L}v\rangle_{\breve{\mathscr{H}}_1}=\sum_{n=0}^{\infty}[u'(n)\overline{v'(n)}+(|r(n+1)|^2-1)u(n+1)\overline{v(n+1)}\\ +(|r(n)|^2-1+|r(n)|^2\breve{q}(n))u(n)\overline{v(n)}].\end{aligned}$$

Shifting indices on the second term in this sum and using the second and third conditions on r yields

$$\langle\mathscr{L}u,\mathscr{L}v\rangle_{\breve{\mathscr{H}}_1}=\sum_{n=0}^{\infty}[u'(n)\overline{v'(n)}+q(n)u(n)\overline{v(n)}]=\langle u,v\rangle_{\mathscr{H}_1},$$

i.e., $\mathscr{L}|_{\mathscr{H}_1}:\mathscr{H}_1\to\breve{\mathscr{H}}_1$ is unitary. In summary, our conditions on r are sufficient for the equality of ρ and $\breve{\rho}$.

To show that they are also necessary, assume that $\breve{\rho}=\rho$ and hence that $\breve{m}(\lambda)=m(\lambda)+A\lambda+B$ for appropriate constants A and B. The Paley-Wiener theorem shows that $\operatorname{supp}(\mathscr{U}\delta_n)\subset[0,n]$ and $\operatorname{supp}(\mathscr{U}^*\delta_k)\subset[0,k]$. Lemma 4.2 and the fact that $\mathfrak{G}(1)(n)=0$ for $n\geqslant 1$ give now that $(\mathscr{U}\delta_n)(k)=0$ unless $n=k$. Define $r(n)=(\mathscr{U}\delta_n)(n)$ so that $\mathscr{U}u=ru$ for all $u\in\mathscr{H}_1$. Since $\mathscr{U}$ is unitary we have

$$\langle\delta_n,\delta_k\rangle_{\mathscr{H}_1}=\langle\mathscr{U}\delta_n,\mathscr{U}\delta_k\rangle_{\breve{\mathscr{H}}_1}=r(n)\overline{r(k)}\langle\delta_n,\delta_k\rangle_{\breve{\mathscr{H}}_1}.$$

This, (4.2), and its analogue for $\breve{\mathscr{H}}_1$ give that the first three properties of r hold.

Since $\mathfrak{F}\psi(\lambda,\cdot) = 1/(t-\lambda) = \breve{\mathfrak{F}}\breve{\psi}(\lambda,\cdot)$ we have for non-real λ

$$\breve{\psi}(\lambda,\cdot) = \mathscr{U}\psi(\lambda,\cdot) = r(n)\psi(\lambda,\cdot). \tag{4.3}$$

The equations satisfied by $\psi(\lambda,\cdot)$ and $\breve{\psi}(\lambda,\cdot)$ give then

$$0 = \lambda(w(n) - |r(n)|^2\breve{w}(n))\psi(\lambda,n)$$

for $n \geqslant 1$. Since $\psi(\lambda,n) \neq 0$ (the contrary would mean a non-real eigenvalue for a self-adjoint operator) we obtain the fourth condition on r.

It remains to establish the fifth property. Since

$$\mathscr{U}\delta_0 = -\cos\alpha\mathscr{G}(1) = (\cos\alpha/\cos\breve{\alpha})\delta_0$$

we have $r(0) = \cos\alpha/\cos\breve{\alpha}$. Upon evaluation at 0 equation (4.3) and the fact that $\breve{m} - m$ is a linear polynomial imply

$$m(\lambda)(r(0)\sin\alpha - \sin\breve{\alpha}) = (A\lambda + B)\sin\breve{\alpha} + \cos\breve{\alpha} - r(0)\cos\alpha.$$

Since m itself cannot be a linear polynomial (ρ not being zero) we have $r(0)\sin\alpha - \sin\breve{\alpha} = 0$. This completes the proof. □

4.2. The cases where $\cos\alpha$ or $\cos\breve{\alpha}$ may vanish

THEOREM 4.4. *Assume the validity of Hypothesis 4.1 and that* $\cos\alpha = 0 = \cos\breve{\alpha}$. *Then the spectral measures* $\breve{\rho}$ *and* ρ *are identical if and only if* $\breve{T} = T$.

Proof. It is clear that $T = \breve{T}$ implies $\breve{\rho} = \rho$. Assume now that $\breve{\rho} = \rho$ which implies that $\breve{m}(\lambda) = A\lambda + B + m(\lambda)$. We employ again the Paley-Wiener theorem and Lemma 4.2 to prove that $\operatorname{supp}(\mathscr{U}\delta_n) = \{n\}$ but only when $n \geqslant 2$. The support of $\mathscr{U}\delta_1 = \mathscr{U}\varepsilon_0$ is $\{0,1\}$ and $\mathscr{U}\delta_0 = 0$. Defining $r(n) = (\mathscr{U}\delta_n)(n)$ whenever $n \in \mathbb{N}$ and $r(0) = r(1)(1+q(0))/(1+\breve{q}(0))$ we get $\mathscr{U}u = ru$ for all $u \in \mathscr{H}$. In particular, $u = \psi(\lambda,\cdot) - \delta_0/(1+q(0))$ is in $\mathscr{H}$ since $\langle\psi(\lambda,\cdot),\delta_0\rangle = 1$ and $\langle\delta_0,\delta_0\rangle = 1+q(0)$. Hence, using Lemma 3.13,

$$\breve{\psi}(\lambda,\cdot) - \frac{\delta_0}{1+\breve{q}(0)} = \mathscr{U}\left(\psi(\lambda,\cdot) - \frac{\delta_0}{1+q(0)}\right) = r\left(\psi(\lambda,\cdot) - \frac{\delta_0}{1+q(0)}\right). \tag{4.4}$$

Evaluating at 0 and 1 and using that $\breve{m}(\lambda) = A\lambda + B + m(\lambda)$ gives

$$m(r(0)-1) = \left(A + \frac{r(0)}{1+q(0)} - \frac{1}{1+\breve{q}(0)}\right)\lambda + B$$

and

$$m(q(0) - \breve{q}(0)) = (A(1+\breve{q}(0)) + r(1) - 1)\lambda + B(1+\breve{q}(0)).$$

Since m cannot be a linear polynomial (this would mean that $\rho = 0$) we get $r(0) = 1$ and $q(0) = \breve{q}(0)$. From this we have next $r(1) = 1$. Since $\mathscr{U}$, thought of as a map from $\mathscr{H}$ to $\breve{\mathscr{H}}$ is unitary we get

$$-1 = \langle\varepsilon_0,\delta_2\rangle_{\mathscr{H}_1} = \langle\mathscr{U}\varepsilon_0,\mathscr{U}\delta_2\rangle_{\breve{\mathscr{H}}_1} = -r(1)\overline{r(2)}$$

so that $r(2)=1$ and

$$2+q(1)-\frac{1}{1+q(0)}=\langle\varepsilon_0,\varepsilon_0\rangle_{\mathscr{H}_1}=\langle\mathscr{U}\varepsilon_0,\mathscr{U}\varepsilon_0\rangle_{\check{\mathscr{H}}_1}=|r(1)|^2\left(2+\check{q}(1)-\frac{1}{1+\check{q}(0)}\right)$$

showing that $q(1)=\check{q}(1)$. Since $\langle\delta_n,\delta_k\rangle_{\mathscr{H}_1}=\langle\mathscr{U}\delta_n,\mathscr{U}\delta_k\rangle_{\check{\mathscr{H}}_1}$ for $n,k\geqslant 2$ we get from equation (4.2) that $r(n)\overline{r(n+1)}=1$ and $|r(n)|^2(\check{q}(n)+2)=q(n)+2$ for all $n\geqslant 2$. All this implies that $r=1$ and $q=\check{q}$.

We have now $L=\check{L}$ and, from equation (4.4), $\psi(\lambda,\cdot)=\check{\psi}(\lambda,\cdot)$. Hence, for $n\geqslant 1$,

$$\lambda\check{w}(n)\check{\psi}(\lambda,n)=(\check{L}\check{\psi}(\lambda,\cdot))(n)=(L\psi(\lambda,\cdot))(n)=\lambda w(n)\psi(\lambda,n)$$

which shows that $w=\check{w}$, too. □

It remains to consider the case where precisely one of $\cos\alpha$ and $\cos\check{\alpha}$ vanishes. Without loss of generality we may assume that $\cos\alpha=0$ and $\cos\check{\alpha}\neq 0$.

THEOREM 4.5. *Assume the validity of Hypothesis 4.1 and that* $\cos\alpha=0\neq\cos\check{\alpha}$. *The spectral measures* ρ *and* $\check{\rho}$ *are identical if and only if there is a sequence* $r\in\mathbb{C}^{\mathbb{N}_0}$ *with the following properties*

1. $r(n)\overline{r(n+1)}=1$ *for all* $n\in\mathbb{N}_0$,
2. $|r(n)|^2(2+\check{q}(n))=2+q(n+1)$ *for all* $n\in\mathbb{N}$,
3. $|r(0)|^2(1+\check{q}(0))=1+q(1)+q(0)/(1+q(0))$,
4. $|r(n)|^2\check{w}(n)=w(n+1)$ *for all* $n\in\mathbb{N}$, *and*
5. $r(0)=\sin\check{\alpha}/(1+q(0))=-w(1)(1+q(0))/\cos\check{\alpha}$.

Proof. First assume that the conditions on r hold and define $\mathscr{L}:\mathbb{C}^{\mathbb{N}_0}\to\mathbb{C}^{\mathbb{N}_0}:u\mapsto ru(\cdot+1)$. As in the proof of Theorem 4.3 we find that $y=\mathscr{L}u$ satisfies the equation $\check{L}y=\lambda\check{w}y$ if u satisfies $Lu=\lambda wu$. Investigating initial conditions gives

$$\mathscr{L}\phi(\lambda,\cdot)=\check{\phi}(\lambda,\cdot)$$

and

$$\mathscr{L}\theta(\lambda,\cdot)=\check{\theta}(\lambda,\cdot)-\left(\frac{\lambda}{1+q(0)}+\cot\check{\alpha}\right)\check{\phi}(\lambda,\cdot).$$

(Note that $\check{\alpha}$ cannot be zero here since α is not.)

One also shows in much the same way as in the proof of Theorem 4.3 that $\mathscr{L}|_{\mathscr{H}}:\mathscr{H}\to\check{\mathscr{H}}_1$ is unitary. Since $\mathscr{L}\delta_0=0$ it follows that not only $\mathscr{H}$ but all of $\mathscr{H}_1$ is being mapped into $\check{\mathscr{H}}_1$ which implies that $\mathscr{L}\psi(\lambda,\cdot)=\check{\psi}(\lambda,\cdot)$ and $\check{m}(\lambda)=m(\lambda)-\lambda/(1+q(0))-\cot\check{\alpha}$. This is only possible when $\check{\rho}=\rho$.

We now turn to necessity, assuming that $\check{\rho}=\rho$ and hence that $\check{m}(\lambda)=A\lambda+B+m(\lambda)$. This time the Paley-Wiener theorem and Lemma 4.2 prove that $\operatorname{supp}(\mathscr{U}\delta_n)=$

$\{n-1\}$ when $n\in\mathbb{N}$. We also have $\mathscr{U}\delta_1=\mathscr{U}\varepsilon_0$ and $\mathscr{U}\delta_0=0$. These facts give us $\mathscr{U}u=ru(\cdot+1)$ for all $u\in\mathscr{H}$ and even for all $u\in\mathscr{H}_1$ if we define $r(n)=(\mathscr{U}\delta_{n+1})(n)$ for all $n\in\mathbb{N}_0$. Since the restriction of $\mathscr{U}$ to $\mathscr{H}$ is unitary we have $\langle u,v\rangle_{\mathscr{H}_1}=\langle\mathscr{U}u,\mathscr{U}v\rangle_{\check{\mathscr{H}}_1}$ whenever $u,v\in\mathscr{H}$. Choosing u and v from among $\varepsilon_0,\delta_2,\delta_3,\dots$ proves that r satisfies properties (1) through (3).

As in the proof of the previous theorem we have $\psi(\lambda,\cdot)-\delta_0/(1+q(0))\in\mathscr{H}$. Hence, using Lemma 3.13,

$$\check\psi(\lambda,\cdot)=\mathscr{U}\left(\psi(\lambda,\cdot)-\frac{\delta_0}{1+q(0)}\right)=r\psi(\lambda,\cdot+1).$$

Utilizing the difference equations satisfied by $\check\psi$ and ψ gives

$$\lambda r(n+1)(w(n+1)-|r(n)|^2\check w(n))\check\psi(\lambda,n)=0$$

for all $n\geqslant 1$ and hence the fourth property of r.

We obtain from Lemma 4.2 that $r(0)=(\mathscr{U}\delta_1)(0)=\sin\check\alpha/(1+q(0))$ and from equation (4.1) that $1/r(0)=\varepsilon_0(1)/r(0)=(\mathscr{U}^*\delta_0)(1)=-(\cos\check\alpha)/(w(1)(1+q(0)))$. This completes the proof. □

5. The inverse scattering problem

In this chapter we show that the scattering data, *i.e.*, eigenvalues, norming constants, and the scattering amplitude, for our left-definite problem determine the spectral measure and even the operator T uniquely.

5.1. Jost solutions

The main tool in scattering theory are the Jost solutions to the difference equation. These are solutions which behave asymptotically like z^n as n tends to infinity where z is an appropriate function of λ. Their existence can be established under the following assumption.

Hypothesis 5.1. There is a non-negative constant q_0 such that $q(n)-q_0$ and $w(n)-1$ are summable on $\mathbb{N}$. Moreover, $w(n)\neq 0$ for all $n\in\mathbb{N}$.

We begin by reminding the reader about the following standard result on a Volterra-type equation.

Lemma 5.2. *Suppose $K:(\mathbb{N}_0\times\mathbb{N})\to\mathbb{C}$ satisfies $|K(n,k)|\leqslant B(k)$ for all $n\in\mathbb{N}_0$ and a summable sequence B and that $h:\mathbb{N}_0\to\mathbb{C}$ is a bounded sequence. Then the equation*

$$g(n)=h(n)+\sum_{k=n+1}^{\infty}K(n,k)g(k)$$

has a unique solution $g:\mathbb{N}_0\to\mathbb{C}$ such that $\lim_{n\to\infty}(g(n)-h(n))=0$.

Proof. Define $P(n)=\sum_{k=n+1}^{\infty}B(k)$. Then $P'(n)=-B(n+1)$ and P is monotone non-increasing. Using this and the summation by parts formula (2.1) one shows that for $j\in\mathbb{N}_0$

$$\sum_{k=n+1}^{N+1}P'(k-1)P(k)^j=P(N+1)^{j+1}-P(n)^{j+1}-\sum_{k=n}^{N}P'(k)\sum_{\ell=0}^{j-1}P(k)^{\ell+1}P(k+1)^{j-1-\ell}$$

which implies

$$\sum_{k=n+1}^{N+1}P'(k-1)P(k)^j\geqslant -P(n)^{j+1}-j\sum_{k=n}^{N}P'(k)P(k+1)^j$$

and hence

$$\sum_{k=n+1}^{\infty}P'(k-1)P(k)^j\geqslant -\frac{1}{j+1}P(n)^{j+1}.$$

Thus we may define

$$g_0(n)=h(n)$$

and, recursively,

$$g_{j+1}(n)=\sum_{k=n+1}^{\infty}K(n,k)g_j(k)$$

where the above argument and induction over j guarantees absolute convergence of the series defining the g_j and produces the estimate

$$|g_j(n)|\leqslant c_0P(n)^j/j!$$

where c_0 is chosen such that $|h(n)|\leqslant c_0$.

Next we define $g(n)=\sum_{j=0}^{\infty}g_j(n)$, the series being again absolutely convergent. Due to absolute convergence it is easy to see that g satisfies the Volterra equation and that $\lim_{n\to\infty}(g(n)-h(n))=0$. In order to prove uniqueness assume that $\tilde g$ is another solution of the Volterra equation. Then $g-\tilde g$ is a bounded sequence. Let c_1 denote a bound. Induction shows that

$$|g(n)-\tilde g(n)|\leqslant c_1P(n)^j/j!$$

for any $j\in\mathbb{N}_0$. This is only possible if $g=\tilde g$. □

THEOREM 5.3. *Suppose Hypothesis 5.1 to hold. Fix z such that $0<|z|\leqslant 1$ and $z^2\neq 1$ and let $\lambda=2+q_0-z-1/z$. Then the equation $Ly=\lambda wy$ has a unique solution $f(z,\cdot)$ such that $\lim_{n\to\infty}z^{-n}f(z,n)=1$. Moreover, for every $n\in\mathbb{N}_0$, the function $f(\cdot,n)$ is analytic in the open unit disk and continuous in the closed unit disk except for the points $z=\pm1$.*

Proof. We begin by solving the Volterra equation

$$g(z,n)=1+\sum_{k=n+1}^{\infty}\frac{z(1-z^{2k-2n})}{1-z^2}Q(z,k)g(z,k), \tag{5.1}$$

where

$$Q(z,k) = q(k) - q_0 + \lambda(1 - w(k)).$$

We may apply Lemma 5.2 with $h(n) = 1$ and

$$K(n,k) = \frac{z(1 - z^{2k-2n})}{1 - z^2} Q(z,k)$$

after we note that the estimate

$$|K(n,k)| \leqslant C(|q(k) - q_0| + |w(k) - 1|) = B(k)$$

holds uniformly for all z in the closed unit disk having some positive minimum distance from ± 1. This implies the pointwise existence of g. Since $z\lambda$ and hence $zQ(z,k)$ are analytic this uniformity also guarantees that the g_j and g are analytic in the open unit disk and continuous in the closed unit disk except for the points $z = \pm 1$.

Now, one checks by computation that $g'(z,n-1) - z^2 g'(z,n) + zQ(z,n)g(z,n) = 0$ and that this implies that $f(z,n) = z^n g(z,n)$ satisfies $Ly = \lambda wy$. $\square$

5.2. The inverse scattering problem

Our goal now is to relate the Jost solutions to the Weyl-Titchmarsh solutions and hence to the m-function which in turn determines the spectral measure. First note that the map $\mathscr{C} : z \mapsto 2 + q_0 - z - 1/z$ maps the open unit disk bijectively onto $\mathbb{C} - [q_0, 4 + q_0]$. The open upper (lower) half of the unit disk is mapped to the upper (lower) half plane and the intervals $(-1,0)$ and $(0,1)$ are mapped to the intervals $(4 + q_0, \infty)$ and $(-\infty, q_0)$, respectively. $\mathscr{C}$ also maps the unit circle to the interval $[q_0, 4 + q_0]$ in a two-to-one manner (except at the endpoints) since z and $\overline{z} = 1/z$ have the same image.

If $|z| < 1$ then $f(z,\cdot) \in \mathscr{H}_1$ and hence it is a multiple of the Weyl-Titchmarsh solution $\psi(\lambda,\cdot)$ associated with the operator T. Thus, the Weyl-Titchmarsh m-function is uniquely defined for all $\lambda \in \mathbb{C} - [q_0, 4 + q_0]$ and there is a function F, called the Jost function, such that

$$f(z,n) = F(z)\psi(\lambda,n).$$

Employing the initial conditions satisfied by $\psi(\lambda,\cdot)$ we obtain

$$F(z) = \lambda f(z,0)\cos\alpha - (f'(z,0) - q(0)f(z,0))\sin\alpha \tag{5.2}$$

and

$$G(z) = F(z)m(\lambda) = \lambda f(z,0)\sin\alpha + (f'(z,0) - q(0)f(z,0))\cos\alpha. \tag{5.3}$$

F and G are analytic in the open unit disk except possibly for a simple pole at zero due to the presence of λ. They are continuous up to the unit circle except for the points $z = \pm 1$.

We first investigate the spectral measure ρ in $(-\infty, q_0) \cup (4 + q_0, \infty)$. This set is associated with the set $(-1,1) - \{0\}$ in the z-plane. Since $f(z,n)$ is real when z is real, we obtain that m is real and analytic in $\mathbb{R} - [q_0, 4 + q_0]$ except for the points corresponding to zeros of F where m has poles. These, in turn are the eigenvalues of T

as a comparison of the expression (5.2) for F with the boundary condition (2.9) shows. Stieltjes' inversion formula

$$\rho(\mu)-\rho(\nu)=\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\int_{\nu}^{\mu}\operatorname{Im}(m(t+i\varepsilon))dt, \tag{5.4}$$

which holds for points of continuity $\nu<\mu$, shows that the support of the measure ρ in $(-\infty,q_0)\cup(4+q_0,\infty)$ consists of the discrete set of eigenvalues of T in these intervals. The jump of the spectral function ρ at the eigenvalue λ is given as the measure of the set $\{\lambda\}$, *i.e.*, by $\rho(\{\lambda\})$. To determine this quantity notice that, for $\lambda\neq 0$, $\mathfrak{F}\phi(\lambda,\cdot)$ is a multiple of the characteristic function of the set $\{\lambda\}$ since the inversion formula for the Fourier transform (*cf.* Lemma 3.12) gives

$$(\mathfrak{G}\chi_{\{\lambda\}})(k)=\int_{\mathbb{R}}\chi_{\{\lambda\}}(t)e(t,k)d\rho(t)=\phi(\lambda,k)\rho(\{\lambda\}).$$

This and Parseval's identity imply

$$\rho(\{\lambda\})=\|\phi(\lambda,\cdot)\|_{\mathscr{H}_1}^{-2}.$$

Similarly, as we have already argued in Lemma 3.11,

$$\rho(\{0\})=\|\psi_0\|_{\mathscr{H}_1}^{-2},$$

if $\lambda=0$ is an eigenvalue.

It is possible that q_0 or $4+q_0$ are also eigenvalues of T. The above argument relating the jumps of the function ρ with the norming constants applies here, too.

We now turn to the interval $(q_0,4+q_0)$ which we parametrize as $\mu=2+q_0-2\cos\theta$ with $\theta\in(0,\pi)$. The continuity properties of $f(\cdot,n)$ show that the Weyl-Titchmarsh solutions $\psi(\mu\pm i\varepsilon,\cdot)$ have limits $\psi_\pm(\mu,\cdot)$ as $\varepsilon>0$ approaches zero, specifically

$$\psi_\pm(\mu,\cdot)=\frac{f(\mathrm{e}^{\pm i\theta},\cdot)}{F(\mathrm{e}^{\pm i\theta})}. \tag{5.5}$$

We may now define $m_\pm(\mu)$ by requiring that

$$\psi_\pm(\mu,\cdot)=\theta(\mu,\cdot)+m_\pm(\mu)\phi(\mu,\cdot). \tag{5.6}$$

Note that $m_\pm$ are the limits of $m(\lambda)$ as λ approaches $\mu\in(q_0,4+q_0)$ from the upper or lower half-plane. This implies that $m_-(\mu)=\overline{m_+(\mu)}$. The representations (5.5) and (5.6) give now that

$$-2i\operatorname{Im}(m_+(\mu))[\theta(\mu,\cdot),\phi(\mu,\cdot)]=[\psi_+(\mu,\cdot),\psi_-(\mu,\cdot)]=\frac{-2i\operatorname{Im}(z)}{F(z)F(\overline{z})},$$

where $|z|=1$ and $z^2\neq 1$. Since, in this case, $\overline{f(z,\cdot)}$ and $f(\overline{z},\cdot)$ satisfy the same equation and have the same asymptotic behavior they are in fact equal. It follows, in view of Stieltjes' inversion formula (5.4), that

$$\pi\rho'(\mu)=\operatorname{Im}(m_+(\mu))=\frac{\mu\operatorname{Im}(z)}{|F(z)|^2}.$$

Using the right-continuity of ρ and the known value of $\rho(q_0)$ we are now able to determine ρ everywhere on $\mathbb{R}$. The quantity $|F(z)|$ for $|z|=1$, $z^2 \neq 1$, is called the scattering amplitude. We have thus the following theorem.

THEOREM 5.4. *Assume Hypothesis 5.1 to hold. Then the eigenvalues, the corresponding norming constants, and the scattering amplitude,* i.e., *the absolute value of the Jost function $F(z)$ for $|z|=1$, $z^2 \neq 1$, determine uniquely the spectral measure ρ.*

THEOREM 5.5. *Assume Hypothesis 5.1 to hold. Then the eigenvalues, the corresponding norming constants, and the scattering amplitude,* i.e., *the absolute value of the Jost function $F(z)$ for $|z|=1$, $z^2 \neq 1$, determine uniquely the operator T,* i.e., *the sequences q and w and the boundary condition parameter α.*

Proof. Suppose there are two operators T and $\breve{T}$ with the given scattering data. By Theorem 5.4 the operators T and $\breve{T}$ have the same spectral measure. According to Theorems 4.3, 4.4, and 4.5 there is a sequence r such that $r(n+2)=r(n)$ and $|r(n)|^2 w(n)/\breve{w}(n+\sigma)$ where σ may equal to 0 or ± 1. Since $w(n)$ and $\breve{w}(n)$ tend to one as n tends to infinity this forces r to be identically equal to one. This proves then also that $q=\breve{q}$, $w=\breve{w}$ and $\alpha=\breve{\alpha}$. $\square$

6. The inverse resonance problem

In the previous section we introduced the Jost solutions $f(z,\cdot)$ for z in the unit disk. Under further restrictions on the class of operators considered one can prove that the functions $f(\cdot,n)$ extend from the unit disk to the entire complex plane. The zeros of the corresponding Jost function F are related to eigenvalues if they are located in the unit disk. The zeros of F located outside the unit disk are also of significance. If z is such a zero then $\lambda = 2+q_0-z-1/z$ is called a resonance. The goal of this section is to investigate to what extent the location of all eigenvalues and resonances determines the spectral measure of T. Throughout this section we make use of the identity $\lambda = 2+q_0-z-1/z$. We start by stating the hypothesis which will be in force throughout this section.

HYPOTHESIS 6.1. There are two constants $A>0$ and $\beta>1$ such that

$$\max\{|q(n)-q_0|, |w(n)-1|\} \leqslant A\exp(-n^{\beta}).$$

Moreover, $w(n) \neq 0$ for all $n \in \mathbb{N}$.

In the following we will consider derivatives with respect to the complex variables λ and z. These will be denoted by a dot. Specifically, $\dot{\phi}$ and $\dot{f}$ denote the derivatives of ϕ and f with respect to their first variable.

LEMMA 6.2. *Assume Hypothesis 6.1 to hold and let $f(z,\cdot)$ be the Jost solution of $Ly=\lambda wy$. Then the functions $f(\cdot,n)$ extend to entire functions of growth order zero for each $n \in \mathbb{N}_0$. Moreover, there is a constant c such that $|\dot{f}(z,n)| \leqslant cn$ for all $|z| \leqslant 1$ and all $n \in \mathbb{N}$.*

Proof. The Volterra equation (5.1) can be rewritten as

$$g(z,n) = 1 + \sum_{k=n+1}^{\infty} K(z,n,k) g(z,k)$$

where $K(z,n,k) = Q(z,k)\sum_{m=0}^{k-n-1} z^{2m+1}$. If $R \geqslant 2$ and $|z| \leqslant R$ we get

$$\sum_{m=0}^{k-n-1} |z|^{2m+1} \leqslant \frac{|z|}{R} \cdot R^{2k-2n}.$$

One may now apply Lemma 5.2 with $h(n) = 1$ and $B(k) = A(5+q_0)R^{2k+1}\exp(-k^{\beta})$ which is still summable since $\beta > 1$. The result is that $g(\cdot,n)$ exists for all $n \in \mathbb{N}_0$ and is analytic in the disk of radius R for any $R \geqslant 2$.

We now determine the growth order of $f(\cdot,n)$. Suppose $|z| \geqslant 2$ and define $N(z) = \lfloor (3\log|z|)^{1/(\beta-1)} \rfloor$. If $k \geqslant N(z)+1$, then $|B(k)| \leqslant C|z|^{-k}$ where $C = 2A(5+q_0)$. Hence, if $n \geqslant N(z)$,

$$|P(z,n)| \leqslant \sum_{k=n+1}^{\infty} C|z|^{-k} \leqslant \frac{C|z|^{-1}}{1-|z|^{-1}} \leqslant C$$

so that $|f(z,n)| \leqslant \mathrm{e}^{C}|z|^{n}$. For an appropriate constant $c \geqslant 1$ and any $n \in \mathbb{N}_0$ we have

$$|f(z,n)| \leqslant c|z||f(z,n+1)| + |f(z,n+2)|.$$

From this, it follows by induction that

$$|f(z,n)| \leqslant \sum_{k=0}^{N} \binom{N}{k} (c|z|)^{k} |f(z,n+2N-k)|$$

for any $N \geqslant 0$. Thus, if $N = N(z)$,

$$\begin{aligned} |f(z,n)| &\leqslant \sum_{k=0}^{N(z)} \binom{N(z)}{k} (c|z|)^{k} \mathrm{e}^{C} |z|^{n+2N(z)-k} \\ &\leqslant \mathrm{e}^{C}|z|^{2N(z)+n}(2c|z|)^{N(z)} \leqslant \mathrm{e}^{C}|z|^{n}|z|^{4N(z)} \end{aligned}$$

once $|z| \geqslant 2c$. Hence $f(\cdot,n)$ has growth order zero since

$$\log(|z|^{4N(z)}) = 4N(z)\log(|z|) \leqslant (4\log|z|)^{\beta/(\beta-1)}$$

grows slower than any power of $|z|$.

To prove the last statement define

$$h(z,n) = \sum_{k=n+1}^{\infty} \dot{K}(z,n,k) g(z,k).$$

Then $\dot{g}(z,\cdot)$ satisfies the Volterra equation

$$y(n) = h(z,n) + \sum_{k=n+1}^{\infty} K(z,n,k)y(k).$$

Applying again Lemma 5.2 gives that $\dot{g}(z,n) - h(z,n)$ tends to zero as n tends to infinity. Since $h(z,n)$ itself tends to zero as n tends to infinity and since all estimates needed are uniform for $|z| \leqslant 1$ we obtain the last statement of the lemma. □

LEMMA 6.3. *If Hypothesis 6.1 holds, then T has no eigenvalues in $[q_0, 4+q_0]$ except when $q_0 = 0$ and $\alpha = 0$. Consequently, the preimages of all nonzero eigenvalues of T under the map $z \mapsto \lambda = 2 + q_0 - z - 1/z$ are in the open unit disk.*

Proof. For $|z| = 1$ and $z^2 \neq 1$ we have that $f(z,\cdot)$ and $f(1/z,\cdot)$ are two linearly independent solutions of $Ly = \lambda wy$. Their asymptotic behavior prevents them or any of their linear combinations to be in $\mathscr{H}_1$. Thus there are no eigenvalues in $\lambda \in (q_0, 4+q_0)$. If $\lambda = 4 + q_0$ one solution of the difference equation is $f(-1,\cdot)$. By making use of the Volterra approach once more one can show that there is another solution with asymptotic behavior $(-1)^n n$. Thus the general solution for $\lambda = 4 + q_0$ has asymptotic behavior $(a + bn)(-1)^n$ where a and b are arbitrary. None of these can be in $\mathscr{H}_1$. Finally, for $\lambda = q_0$ we have that the general solution is asymptotically equal to $a + bn$. These are not in $\mathscr{H}_1$ if $q_0 > 0$. If $q_0 = 0$ then it is an eigenvalue if and only if $\alpha = 0$ as we know from Lemma 3.11. □

Define J by $J(z) = zF(z)$. Then, from (5.2),

$$J(z) = z\lambda f(z,0)\cos\alpha - z(f(z,1) - (1+q(0))f(z,0))\sin\alpha.$$

Since $f(\cdot,0)$, $f(\cdot,1)$, and $z\lambda$ are entire functions of growth order zero it follows that J is entire and of growth order zero. In order to understand the behavior of J at zero we note that the kernel K of the Volterra equation for g satisfies $K(0,n,k) = w(k) - 1$ so that

$$g(0,n) = 1 + \sum_{k=n+1}^{\infty} (w(k) - 1)g(0,k).$$

One checks easily that this equation is solved by

$$f(0,n) = g(0,n) = \prod_{k=n+1}^{\infty} w(k).$$

Note here that, since $w - 1$ is summable and w is never zero, the product $\prod_{n=1}^{\infty} |w_n|$ is absolutely convergent. Hence we obtain

$$J(0) = -(\cos\alpha)\prod_{k=1}^{\infty} w(k).$$

This is zero if and only if $\cos\alpha = 0$ in which case also the value $\dot{J}(0)$ becomes interesting. We find

$$\dot{J}(0) = F(0) = (1+q_0)\prod_{k=1}^{\infty} w(k) \neq 0$$

provided $\cos\alpha = 0$. Note that $J(0)$ and $\dot{J}(0)$ are real. Thus, if we denote the non-zero zeros of F (and J) by z_n, repeated according to their multiplicities, then Hadamard's factorization theorem gives that

$$F(z) = \prod_{k=1}^{\infty} w(k) \prod_{n=1}^{\infty}(1 - z/z_n)\begin{cases} -\cos\alpha/z & \text{if } \cos\alpha \neq 0 \\ 1+q(0) & \text{if } \cos\alpha = 0. \end{cases}$$

THEOREM 6.4. *Assume the validity of Hypothesis 6.1 and that $\lambda = 0$ is not an eigenvalue of T. The operator T is then uniquely determined from the following information:*

1. *The value q_0,*
2. *the eigenvalues and resonances of T including their multiplicities,*
3. *whether or not 0 is a pole of F, and*
4. *the value*

$$\Omega = \prod_{k=1}^{\infty} |w(k)| \begin{cases} |\cos\alpha| & \text{if } 0 \text{ is a pole of } F \\ 1+q(0) & \text{if } 0 \text{ is not a pole of } F. \end{cases}$$

Proof. As we have just argued, the given data allow us to recover the function F up to a sign as

$$F(z) = cz^m \Omega \prod_{n=1}^{\infty}(1 - z/z_n)$$

where $m = -1$ or 0 depending on whether zero is a pole of F or not and where $c = \pm 1$. We want to employ Theorem 5.4 or rather Theorem 5.5. The value of c will be irrelevant to determine the spectral measure ρ on $(q_0, 4+q_0)$ which requires only the modulus of F. We now have to show that F/c will also determine the norming constants $\|\phi(\lambda_0,\cdot)\|^2_{\mathscr{H}_1}$ whenever λ_0 is an eigenvalue of T.

Hence, let $\lambda_0 \neq 0$ be an eigenvalue and let z_0 be the associated point in the unit disk of the z-plane. Using integration by parts we show that

$$\sum_{n=0}^{N}(\phi'(\lambda_0,n)^2 + q(n)\phi(\lambda_0,n)^2) = \phi'(\lambda_0,N)\phi(\lambda_0,N+1)$$
$$- (\phi'(\lambda_0,0) - q(0)\phi(\lambda_0,0))\phi(\lambda_0,0) + \lambda_0 \sum_{n=1}^{N} w(n)\phi(\lambda_0,n)^2.$$

Note that $\dot\phi(\lambda,\cdot)$ satisfies the difference equation $L\dot\phi(\lambda,\cdot)=\lambda w\dot\phi(\lambda,\cdot)+w\phi(\lambda,\cdot)$. A simple calculation using this fact shows that

$$[\dot\phi(\lambda,\cdot),\phi(\lambda,\cdot)]'(n-1)=w(n)\phi(\lambda,n)^2.$$

Hence

$$\sum_{n=0}^{N}(\phi'(\lambda_0,n)^2+q(n)\phi(\lambda_0,n)^2)=\phi'(\lambda_0,N)\phi(\lambda_0,N+1)+\lambda_0[\dot\phi(\lambda_0,\cdot),\phi(\lambda_0,\cdot)](N)$$
$$-(\phi'(\lambda_0,0)-q(0)\phi(\lambda_0,0))\phi(\lambda_0,0)-\lambda_0[\dot\phi(\lambda_0,\cdot),\phi(\lambda_0,\cdot)](0).$$

Here the last two terms on the right-hand side cancel each other. The first term tends to zero as N tends to infinity which follows from the fact that $\phi(\lambda_0,\cdot)$ is a multiple of $f(z_0,\cdot)$ and the asymptotic behavior of f. Consequently,

$$\|\phi(\lambda_0,\cdot)\|^2=\lim_{N\to\infty}\lambda_0[\dot\phi(\lambda_0,\cdot),\phi(\lambda_0,\cdot)](N).$$

Recall from (5.2) and (5.3) that $f(z,n)=F(z)\psi(\lambda,n)=F(z)\theta(\lambda,n)+G(z)\phi(\lambda,n)$ where $G(z)=m(\lambda)F(z)$ exists even at poles of m. This implies

$$[\dot f(z_0,\cdot),f(z_0,\cdot)](N)$$
$$=G(z_0)\dot F(z_0)[\theta(\lambda_0,\cdot),\phi(\lambda_0,\cdot)](N)+G(z_0)^2\dot\lambda_0[\dot\phi(\lambda_0,\cdot),\phi(\lambda_0,\cdot)](N)$$

using that $F(z_0)=0$ and the abbreviation $\dot\lambda_0=-1+1/z_0^2$. By Lemma 6.2 the left-hand side tends to zero as N tends to infinity so that

$$\|\phi(\lambda_0,\cdot)\|^2=\lim_{N\to\infty}-\frac{\lambda_0}{\dot\lambda_0 G(z_0)^2}G(z_0)\dot F(z_0)[\theta(\lambda_0,\cdot),\phi(\lambda_0,\cdot)](N)=-\frac{\dot F(z_0)}{\dot\lambda_0 G(z_0)}.$$

Since both $f(z,\cdot)$ and $f(1/z,\cdot)$ solve the same difference equation their Wronskian is a constant. The asymptotic behavior of f shows then that, in fact, $[f(z,\cdot),f(1/z,\cdot)](n)=1/z-z$. This gives

$$\frac{1}{z}-z=(F(z)G(1/z)-F(1/z)G(z))[\theta(\lambda,\cdot),\phi(\lambda,\cdot)].$$

Evaluating this at z_0 (and λ_0) gives

$$G(z_0)=-\lambda_0(\frac{1}{z}-z)/F(1/z_0)$$

so that

$$\|\phi(\lambda_0,\cdot)\|^2=\frac{z_0^3\dot F(z_0)F(1/z_0)}{\lambda_0(1-z_0^2)^2}.$$

Since F appears here quadratically the sign of c becomes irrelevant and we have finally expressed the norming constant $\|\phi(\lambda_0,\cdot)\|^2$ in terms of F/c. $\square$

If $\lambda = 0$ is an eigenvalue of T (which may only happen when $\alpha = 0$), the norming constant of the eigenfunction ψ_0 and hence the jump of the spectral function ρ at zero is not determined by F. To see this consider the example $q(n) = q_0$, $w(n) = 1$ for $n \in \mathbb{N}$ leaving the value $q(0)$ free. Then we have $f(z,n) = z^n$, $F(z) = \lambda f(z,0) = \lambda$ and $G(z) = f(z,1) - (1+q(0))f(z,0) = z - 1 - q(0)$. Let z_0 be the point which is mapped to $\lambda = 0$. Then $f(z_0,n) = G(z_0)\psi_0(n)$ and hence

$$\rho(\{0\}) = -\psi_0(0)^{-1} = \frac{G(z_0)}{f(z_0,0)} = 1 + q(0) - z_0.$$

Changing $q(0)$ will affect this jump but not the function F which depends only on q_0. Hence F does not determine this jump.

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R. AlAhmad
Department of Mathematics
University of Alabama at Birmingham
Birmingham, AL 35226-1170
USA
and
Mathematics Department
Yarmouk University
Irbid, Jordan 21163
e-mail: rami78@uab.edu

R. Weikard
Department of Mathematics
University of Alabama at Birmingham
Birmingham, AL 35226-1170
USA
e-mail: rudi@math.uab.edu