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HILBERT–SCHAUDER FRAME OPERATORS

RUI LIU

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Abstract. We introduce a new concept of frame operators for Banach spaces we call a Hilbert-Schauder frame operator. This is a hybird between standard frame theory for Hilbert spaces and Schauder frame theory for Banach spaces. Most of our results involve basic structure properties of the Hilbert-Schauder frame operator. Examples of Hilbert-Schauder frames include standard Hilbert frames and classical bases of ℓ_p and L^p-spaces with $1 < p \leqslant 2$. Finally, we give a new isomorphic characterization of Hilbert spaces.

Introduction

In 1946, Gabor [14] introduced a fundamental approach to signal decomposition in terms of elementary signals. In 1952, while addressing some difficult problems from the theory of nonharmonic Fourier series, Duffin and Schaeffer [11] abstracted Gabors method to define frames for a Hilbert space. For some reason the work of Duffin and Schaeffer was not continued until 1986 when the fundamental work of Daubechies, Grossman and Meyer [10] brought this all back to life, right at the dawn of the wavelet era. Today, the theory of frames in Hilbert spaces presents a central tool in mathematics and engineering, and has developed rather rapidly in the past decade. The motivation has come from applications to signal analysis, as well as from applications to a wide variety of areas of mathematics, such as operator theory [16] and Banach space theory [8].

In 1991, Gröchenig [15] generalized Hilbert frames to Banach spaces and introduced atomic decompositions and Banach frames. Han and Larson [16] defined a Schauder frame for a Banach space to be a compression of a Schauder basis for a Banach space. In [8], Casazza, Han and Larson gave and studied various definitions of frames for Banach spaces including the Schauder frame. In 2009, Casazza, Dilworth, Odell, Schlumprecht and Zsák [7] studied the coefficient quantization for Schauder frames in Banach spaces. In [4], Carando and Lassalle considered the duality theory for atomic decompositions. Concentrating on Schauder frames independent of the associated bases, the author [21] gave out the concepts of minimal and maximal associated

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bases with respect to Schauder frames, closely connected them with the duality theory, and extended known James' results [19] on unconditional bases to unconditional frames. In [22], the author and Zheng gave an characterization of Schauder frames which are near-Schauder bases, which generalized Holub's results [18] from Hilbert frames to Schauder frames. In [5], Carando, Lassalle and Schmidberg considered the reconstruction formula for Banach frames, extended and improved some James' type duality results in [4, 21]. Recently, Larson, Han, Liu and the author [17] developed elements of a general dilation theory for operator-valued measures and bounded linear maps between operator algebras that are not necessarily completely-bounded, and proved main results by extending and generalizing some known results from the theory of frames and framings. In [3], Beanland, Freeman and the author proved that the upper and lower estimates theorems for finite dimensional decompositions of Banach spaces can be extended and modified to Schauder frames, and gave a complete characterization on duality for Schauder frames. Recently as well, a continuous version of classical dilation theorem in [16] was given for vector bundles and Riemannian manifolds [13].

From [16, 8], one thing worthwhile to note is that the notion of Hilbert frame transform makes perfect sense in more general Banach spaces, although the standard frame operator is necessarily a Hilbert space concept. Proposition 1.10 in [16] forces a similarity between a frame and its canonical dual, so there is an isomorphism between the underlying space and its dual space, and most Banach spaces are not isomorphic to their dual spaces. Actually, the essential reason here is that the standard frame operator is invertible, which is not necessary for general Banach spaces.

In this paper, we introduce a new concept in frame theory for Banach spaces we call a Hilbert-Schauder frame. This is a hybird between standard Hilbert space frame theory and Schauder frames for Banach spaces. Most of our results involve properties of the associated Hilbert-Schauder frame operator. Examples of unconditional Hilbert-Schauder bases include classical bases of ℓ_p and $L^p[0,1]$ with $1 < p \leqslant 2$, while ℓ_q with $2 < q < \infty$ has no Hilbert-Schauder frame. Finally, following the idea in [16], we give a new isomorphic characterization of Hilbert spaces.

Throughout this paper we only consider the real case for convenience. The complex case is more complicated, because we will need the concept of antilinear dual space, denoted by $\overline{X^*}$, to extend the notions of self-adjointness and positivity into $B(X, \overline{X^*})$. For more information, please see [23].

1. Preliminaries

DEFINITION 1.1. Let $\mathscr{H}$ be a Hilbert space. A sequence $\{f_j\}_{j\in\mathbb{J}}$ in $\mathscr{H}$ is called a *(standard) Hilbert frame* of $\mathscr{H}$ if there are $0 < a \leqslant b < \infty$ such that

$$a\|x\|^2 \leqslant \sum_{j\in\mathbb{J}} |\langle x, f_n\rangle|^2 \leqslant b\|x\|^2 \quad \text{for all } x \in \mathscr{H}.$$

For a Hilbert frame $\{f_j\}_{j\in\mathbb{J}}$ of $\mathscr{H}$, we consider the operator $A : \mathscr{H} \to \ell_2$ with $x \mapsto \{\langle x, f_j\rangle\}_{j\in\mathbb{J}}$, Its joint $A^* : \ell_2(\mathbb{J}) \to \mathscr{H}$ with $\{a_j\}_{j\in\mathbb{J}} \mapsto \sum_{j\in\mathbb{J}} a_j f_j$ and their product

$$S = A^*A : \mathscr{H} \to \mathscr{H}, \quad x \mapsto \sum_{j\in\mathbb{J}} \langle x, f_j\rangle f_j.$$

Since

$$a\|x\|^2 \leqslant \sum_{j\in\mathbb{J}} |\langle x, f_j\rangle|^2 = \Big\langle \sum_{j\in\mathbb{J}} \langle x, f_j\rangle f_j, x\Big\rangle = \langle Sx, x\rangle \leqslant b\|x\|^2.$$

S is a positive and invertible operator with $a\,\mathrm{Id}_{\mathscr{H}} \leqslant S \leqslant b\,\mathrm{Id}_{\mathscr{H}}$ and thus,

$$x = S^{-1}Sx = \sum_{j\in\mathbb{J}} \langle x, f_j\rangle S^{-1} f_j. \tag{1.1}$$

The operator S is the standard Hilbert frame operator.

For the introduction to the theory of Hilbert frames we refer the reader to [6] and [9]. We follow [8, 16, 7, 21, 22] for the theory of Schauder frames in Banach spaces.

DEFINITION 1.2. Let X be a Banach space. A sequence $\{x_j, f_j\}_{j\in\mathbb{J}}$ in $X\times X^*$ is called a *Schauder frame* of X if

$$x = \sum_{j\in\mathbb{J}} \langle x, f_j\rangle x_j \quad \text{for all } x\in X. \tag{1.2}$$

2. Hilbert-Schauder frame operators

DEFINITION 2.1. Let X be a separable Banach space. A bounded linear operator $S : X \to X^*$ is called a *Hilbert-Schauder frame operator*, or *HSf-operator* for brevity, if there is a Schauder frame $\{x_j, f_j\}_{j\in\mathbb{J}}$ of X such that $S(x_j) = f_j$ for all $j\in\mathbb{J}$.

A Schauder frame $\{x_j, f_j\}_{j\in\mathbb{J}}$ of X is called a *Hilbert-Schauder frame*, or *HS-frame* for brevity, if there is a bounded linear operator $S : X \to X^*$ such that $S(x_j) = f_j$ for all $j\in\mathbb{J}$.

DEFINITION 2.2. Let X be a Banach space and $T\in B(X, X^*)$. We say that

(i) T is *self-adjoint* if $T^*|_X = T$;

(ii) T is *positive* if $(Tx)(x) \geqslant 0$ for all $x\in X$.

PROPOSITION 2.3. *Every HSf-operator is self-adjoint, positive, and injective.*

Proof. Let S be a HSf-operator of a Banach space X with the Schauder frame $\{x_j, f_j\}_{j\in\mathbb{J}}$. To get that S is self-adjoint, it is sufficient to prove that $(Sx)(y) = (Sy)(x)$ for all $x, y\in X$ as follows

$$\begin{aligned}(Sx)(y) &= \Big(S\sum_j f_j(x)x_j\Big)(y) = \Big(\sum_j f_j(x)f_j\Big)(y) = \sum_j f_j(x)f_j(y)\\ &= \Big(\sum f_j(y)f_j\Big)(x) = (Sy)(x) = (S^*|_X x)(y).\end{aligned}$$

S is positive, because for all $x\in X$ we have

$$(Sx)(x) = \Big(S\sum_j f_j(x)x_j\Big)(x) = \Big(\sum_j f_j(x)f_j\Big)(x) = \sum_j |f_j(x)|^2 \geqslant 0. \tag{2.1}$$

If $S(x) = 0$, then $(Sx)(x) = 0$. By (2.1), we have $0 = (Sx)(x) = \sum_j |f_j(x)|^2$, that is, $f_j(x) = 0$ for all $j \in \mathbb{J}$. Thus, $x = \sum_j f_j(x)x_j = 0$. It follows that S is injective. $\square$

LEMMA 2.4. *Let X be a separable Banach space and $\{x_j, f_j\}_{j\in\mathbb{J}}$ be a HS-frame of X with the HSf-operator S. Then $\sum_{j\in\mathbb{J}} |f_j(x)|^2 \leqslant \|S\|\|x\|^2$ for all $x \in X$.*

Proof. By (2.1), we have $(Sx)(x) = \sum_j |f_j(x)|^2$. Thus, $\sum_j |f_j(x)|^2 = (Sx)(x) \leqslant \|S\|\|x\|^2$ for all $x \in X$. $\square$

Thus, the linear operator

$$A : X \to \ell_2(\mathbb{J}), \ \ x \mapsto \sum_j f_j(x)e_j$$

is well-defined and bounded with $\|A\| \leqslant \sqrt{\|S\|}$. A is called the *Hilbert-Schauder analysis operator*. Its adjoint operator A^* is given by

$$A^* : \ell_2(\mathbb{J}) \to X^*, \ \ \sum_j a_j e_j \mapsto \sum_j a_j f_j.$$

A^* is called the *Hilbert-Schauder pre-frame operator*.

PROPOSITION 2.5. *By composing A and A^*, we obtain that the HSf-operator*

$$S = A^*A.$$

Thus, the HSf-operator S factors through $\ell_2(\mathbb{J})$.

Proof. For all $x \in X$, we have

$$A^*A(x) = A^*\big(\sum_j f_j(x)e_j\big) = \sum_j f_j(x)f_j = S(x). \qquad \square$$

Recall that for Schauder frames, one sequence does not uniquely determine the other, because it is a redundant system not a stable basis [21, 22]. The following propositions will show that for HS-frames things are different: one sequence uniquely determines the other with respect to the HSf-operator.

PROPOSITION 2.6. *Let X be a Banach space. Suppose $\{x_j, f_{1,j}\}_{j\in\mathbb{J}}$ and $\{x_j, f_{2,j}\}_{j\in\mathbb{J}}$ are both HS-frames of X with the HSf-operators S_1 and S_2, respectively. Then $S_1 = S_2$ and $f_{1,j} = f_{2,j}$ for all $j \in \mathbb{J}$.*

Proof. It is sufficient to prove that $S_1 = S_2$. For all $x, y \in X$ we have

$$\begin{aligned}
(S_1x)(y) &= \Big(S_1 \sum_j f_{2,j}(x)x_j\Big)(y) = \Big(\sum_j f_{2,j}(x)f_{1,j}\Big)(y) \\
&= \sum_j f_{2,j}(x)f_{1,j}(y) = \Big(\sum_j f_{1,j}(y)f_{2,j}\Big)(x) = \Big(S_2 \sum_j f_{1,j}(y)x_j\Big)(x) \\
&= (S_2y)(x) = (S_2^*|_X x)(y).
\end{aligned}$$

Then, by Proposition 2.3, we obtain that $S_1 = S_2^*|_X = S_2$. $\square$

PROPOSITION 2.7. *Let X be a Banach space. Suppose $\{x_{1,j}, f_j\}_{j\in\mathbb{J}}$ and $\{x_{2,j}, f_j\}_{j\in\mathbb{J}}$ are both HS-frames of X with the HSf-operators S_1 and S_2, respectively. Then $S_1 = S_2$ and $x_{1,j} = x_{2,j}$ for all $j \in \mathbb{J}$.*

Proof. For all $x \in X$ we have

$$S_1(x) = S_1\Big(\sum_j f_j(x)x_{1,j}\Big) = \sum f_j(x)f_j = S_2\Big(\sum f_j(x)x_{2,j}\Big) = S_2(x).$$

Then, by Proposition 2.3, $S_1 = S_2$ and $x_{1,j} = x_{2,j}$ for all $j \in \mathbb{J}$. □

Actually, HS-frames have a better locally duality property to essentially establish its advantage over Schauder frames.

PROPOSITION 2.8. *Let $\{x_n, f_n\}$ be a HS-frame of X with a HSf-operator S. Then $\{f_n, x_n\}$ is a Schauder frame for the closure of* $\operatorname{span}\{f_n\}$.

Proof. For each f_n, by Proposition 2.3, we have

$$\begin{aligned} f_n &= S(x_n) = S(\sum_j f_j(x_n)x_j) = \sum_j (Sx_j)(x_n)\cdot S(x_j) \\ &= \sum_j (S^*x_n)(x_j)f_j = \sum_j (Sx_n)(x_j)f_j = \sum_j f_n(x_j)f_j. \end{aligned} \tag{2.2}$$

By proposition 2.8 in [21], the space

$$Y = \Big\{f \in X^* : f = \|\cdot\| - \lim_{n\to\infty} \sum_{j=1}^{n} f(x_j)f_j\Big\},$$

is a norm closed subspace of X^*. Then, by (2.2), we get $\overline{\operatorname{span}}(f_n : n \in \mathbb{N}) \subset Y$. On the other hand, it is clear from the definition of Y that $Y \subset \overline{\operatorname{span}}(f_n : n \in \mathbb{N})$. Therefore, $Y = \overline{\operatorname{span}}(f_n : n \in \mathbb{N})$. Thus, $\{f_n, x_n\}$ is a Schauder frame of $\overline{\operatorname{span}}(f_n : n \in \mathbb{N})$. □

However, it is false for Schauder frames. The following example is an unconditional and semi-normalized Schauder frame $\{x_n, f_n\}$ of ℓ_1 for which $\{f_n, x_n\}$ is not a Schauder frame of $\overline{\operatorname{span}}(f_n : n \in \mathbb{N})$.

EXAMPLE 2.9. [21] Let (e_n) denote the usual unit vector basis of ℓ_1 and let (e_n^*) be the corresponding coordinate functionals, and set $\mathbf{1} = (1,1,1,\ldots) \in \ell_\infty$. Then define a sequence $(x_n, f_n) \subset \ell_1 \times \ell_\infty$ by putting $x_{2n-1} = x_{2n} = e_n$ for all $n \in \mathbb{N}$ and

$$f_n = \begin{cases} \mathbf{1}, & \text{if } n = 1; \\ e_1^* - \mathbf{1}, & \text{if } n = 2; \\ e_k^* - e_1^*/2^k, & \text{if } n = 2k-1 \text{ for } k \in \mathbb{N}\setminus\{1\}; \\ e_1^*/2^k, & \text{if } n = 2k \text{ for } k \in \mathbb{N}\setminus\{1\}. \end{cases}$$

Actually, we have $\mathbf{1} \neq \|\cdot\| - \lim_{n\to\infty} \sum_{j=1}^{n} \mathbf{1}(x_j) f_j$. We leave the detail to the reader.

Now we give some important examples of HS-frames.

EXAMPLE 2.10. Every standard Hilbert frame operator is an example of a HSf-operator by formula (1.1)

$$x = \sum_{j\in\mathbb{J}} \langle x, f_j\rangle S^{-1} f_j = \sum_{j\in\mathbb{J}} \langle x, S(S^{-1} f_j)\rangle S^{-1} f_j.$$

DEFINITION 2.11. Let X be a separable Banach space. A bounded linear operator $S : X \to X^*$ is called a *Hilbert-Schauder basis operator*, or *HSb-operator* for brevity, if there is a Schauder basis $\{z_j, z_j^*\}_{j\in\mathbb{J}}$ of X such that $S(z_j) = z_j^*$ for all $j \in \mathbb{J}$.

A Schauder basis $\{z_j, z_j^*\}_{j\in\mathbb{J}}$ of X is called a *Hilbert-Schauder basis*, or *HS-basis* for brevity, if there is a bounded linear operator $S : X \to X^*$ such that $S(z_j) = z_j^*$ for all $j \in \mathbb{J}$.

PROPOSITION 2.12. *The unit vector basis of ℓ_p with $1 \leqslant p \leqslant 2$ is an unconditional HS-basis.*

Proof. Let $\{e_n\}$ be the unit vector basis of ℓ_p and $\{e_n^*\}$ be the biorthogonal functionals in the dual space, that is, the unit vector basis of ℓ_q with $1/p + 1/q = 1$. Then $\{e_n, e_n^*\}$ is an unconditional Schauder basis of ℓ_p. Moreover, since $1 < p \leqslant 2 \leqslant q < \infty$, we have $(\sum_n |a_n|^q)^{1/q} \leqslant (\sum_n |a_n|^p)^{1/p}$ for all scalars $\{a_n\}$. Thus, the operator

$$S : \ell_p \to \ell_q, \quad \sum_n a_n e_n \mapsto \sum_n a_n e_n^* \quad \text{for all } \{a_n\} \in \ell_p$$

is well-defined and bounded with norm $\|S\| = 1$. Clearly, $S(e_n) = e_n^*$ for all $n \in \mathbb{N}$. Thus, $\{e_n, e_n^*\}$ is a a HS-basis.

When $p = 1$, the argument is similar. $\square$

PROPOSITION 2.13. *The Haar basis of $L^p[0,1]$ with $1 < p \leqslant 2$ is an unconditional HS-basis.*

Proof. Let $\{h_n\}$ be the Haar system [1], which is an unconditional basis in $L^p[0,1]$ for $1 < p < \infty$. Notice that the Haar system is not normalized in $L^p[0,1]$ for $1 < p < \infty$. To normalize them in L^p one should take $h_n/\|h_n\|_p = |I_n|^{-1/p} h_n$, where I_n denotes the support of the Haar function h_n. Then for $1 < p < \infty$, we have that the dual functionals associated to the Haar system are given by $h_n^* = \frac{1}{|I_n|} h_n, n \in \mathbb{N}$. Thus, $\{h_n, h_n^*\} = \{h_n, |I_n|^{-1} h_n\}$ is an unconditional basis system in L^p. By re-scaling, we have that $\{|I_n|^{-1/p} h_n, |I_n|^{-1/q} h_n\}$ is a normalized unconditional basis system in L^p with $1/p + 1/q = 1$. By [2], for $1 < p < \infty$ there exist constants A_p, B_p such that, if $\{x_n\}$ is a normalized λ-unconditional basic sequence in L^p, then

$$\lambda^{-1}\Big(\sum_n |a_n|^p\Big)^{1/p} \leqslant \Big\|\sum_n a_n x_n\Big\|_p \leqslant \lambda B_p \Big(\sum_n |a_n|^2\Big)^{1/2}, \quad \text{if } 2 \leqslant p < \infty,$$

$$(\lambda A_p)^{-1}\Big(\sum_n |a_n|^2\Big)^{1/2} \leqslant \Big\|\sum_n a_n x_n\Big\|_p \leqslant \lambda\Big(\sum_n |a_n|^p\Big)^{1/p}, \quad \text{if } 1 < p \leqslant 2.$$

Thus, the operator S defined by

$$S : L^p \to L^q \text{ with } 1 < p \leqslant 2 \leqslant q < \infty \text{ and } 1/p + 1/q = 1,$$

$$S\Big(\sum_n a_n |I_n|^{-1/p} h_n\Big) = \sum_n a_n |I_n|^{-1/q} h_n, \qquad \text{for all } \sum_n a_n |I_n|^{-1/p} h_n \in L^p$$

is well-defined and bounded. Clearly, we have $S(|I_n|^{-1/p}h_n) = |I_n|^{-1/q}h_n$ for all $n \in \mathbb{N}$. Thus, $\{|I_n|^{-1/q}h_n\}$ in L^q is an unconditional Hilbert-Schauder basis for L^p. $\square$

By Proposition 2.12 and 2.13, ℓ_p $(1 \leqslant p \leqslant 2)$ and $L^p[0,1]$ $(1 < p \leqslant 2)$ do have perfect HS-frames, normalized unconditional HS-bases $\{e_n, e_n^*\}$ with $\|e_n\| = \|e_n^*\| = 1$ for all $n \in \mathbb{N}$, but the interesting thing is that ℓ_q with $2 < q < \infty$ has no HS-frame $\{x_n, f_n\}$ with $\liminf_n \|f_n\| > 0$.

LEMMA 2.14. *Let $\{x_n, f_n\}$ be a HS-frame of X with the HSf-operator S. Then there is some $K > 0$ such that*

$$K \cdot B_{X^*} \subset \overline{B_{X^*} \bigcap S(X)}^{w^*}.$$

Proof. It is direct by the proof of Proposition 2.4 in [21]. $\square$

PROPOSITION 2.15. *ℓ_q with $2 < q < \infty$ has no Hilbert-Schauder frame $\{x_n, f_n\}$ with* $\liminf_n \|f_n\| > 0$.

Proof. Since ℓ_q is reflexive, by Lemma 2.14,

$$X^* = \overline{S(X)}^{w^*} = \overline{S(X)}^{w} = \overline{S(X)}.$$

Then, by Proposition 2.8, we have $f = \sum_j f(x_j) f_j$ for all $f \in X^*$. By $\liminf_n \|f_n\| > 0$, there is a subsequence $\{f_{n_k}\}$ with $\inf_k \|f_{n_k}\| > 0$. So it is easy to know that $\{x_{n_k}\}$ weakly converges to 0. By Pitt's theorem [20, 1], the HSf-operator $S : \ell_q \to \ell_p$ with $1/q + 1/p = 1$ and $p < q$ must be compact. Together with Proposition 7.6 in [12], we have $f_{n_k} = S(x_{n_k}) \to 0$, which leads to a contradiction. $\square$

We give an isomorphic characterization of Hilbert spaces following the idea in [16].

THEOREM 2.16. *A Banach space X is isomorphic to a Hilbert space if and only if it has a HS-frame $\{x_n\}$ with a HSf-operator S such that $\{Sx_n\}$ is also a HS-frame of $\overline{[Sx_n]}$.*

Proof. The necessity is easy by standard Hilbert frame theory.

For sufficiency, assume that $(\{Sx_n\},T)$ is a HSf-system of $\overline{[Sx_n]}$, then for all $x \in X$ and $f \in \overline{[Sx_n]}$,

$$\begin{aligned}(TSx)(f) &= (TS\sum_n (Sx_n)(x)\cdot x_n)(f) = (\sum_n (Sx_n)(x)\cdot TSx_n)(f)\\ &= \sum_n (Sx_n)(x)\cdot (TSx_n)(f) = \sum_n (TSx_n)(f)\cdot (Sx_n)(x)\\ &= (\sum_n (TSx_n)(f)\cdot Sx_n)(x) = f(x).\end{aligned}$$

By Proposition 2.4 in [21], $\overline{[Sx_n]}$ is a norming subspace of X^*, it follows that

$$\|TSx\|_{\overline{[Sx_n]}^*} = \sup_{f\in\overline{[Sx_n]},\|f\|\leqslant 1} |(TSx)(f)| = \sup_{f\in\overline{[Sx_n]},\|f\|\leqslant 1} |f(x)| \approx \|x\|_X.$$

Thus, TS is an isomorphic embedding. By Proposition 2.5, we have $S = A^*A$. Then $\|Ax\| \geqslant \|TA^*\|^{-1}\|TSx\| \geqslant K\|x\|$ for some $K > 0$, that is, $A : X \to \ell_2$ is an isomorphic embedding, which implies that X is isomorphic to a Hilbert space. □

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Rui Liu
Department of Mathematics and LPMC
Nankai University
Tianjin 300071
P.R. China
e-mail: ruiliu@nankai.edu.cn