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A NOTE ON JORDAN DERIVABLE LINEAR MAPS

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Abstract. Let H be a complex Hilbert space and let δ be a linear map which is Jordan derivable at a given idempotent $P \in \mathrm{B}(H)$ in the sense that $\delta(A^2) = \delta(A)A + A\delta(A)$ holds for all A with $A^2 = P$. If P has infinite rank and co-rank, then we prove that the restriction of δ to $\mathrm{B}(\operatorname{Im} P)$ is an inner derivation and the restriction to $\mathrm{B}(\operatorname{Ker} P)$ is a sum of inner derivation and multiplication by a scalar. We give an example that this is not necessarily true when rank and co-rank of P are finite.

1. Introduction and statement of the result

Let $\mathscr{A} = \mathrm{B}(H)$ be an algebra of bounded operators on a complex Hilbert space H. Recall that a linear map δ from $\mathscr{A}$ into itself is called a derivation if $\delta(AB) = \delta(A)B + A\delta(B)$ for all $A, B \in \mathscr{A}$. More generally, δ is called a Jordan derivation if $\delta(A^2) = \delta(A)A + A\delta(A)$ for all $A \in \mathscr{A}$. On $\mathrm{B}(H)$ this is equivalent to $\delta(AB + BA) = \delta(A)B + A\delta(B) + \delta(B)A + B\delta(A)$ for all $A, B \in \mathscr{A}$, which some authors use as a definition of a Jordan derivation (see, e.g., [2]). More on derivations and Jordan derivations can be found in, e.g., [1, 2].

Recently, conditions which automatically yield derivability of a map were investigated (see, e.g., [3, 8, 9] and references therein). It is the aim of this paper to consider a similar problem for linear maps which are Jordan derivable only at some point. Corresponding to the above two equivalent definitions of Jordan derivations, we can say that: (1) δ is Jordan derivable at some point $Z \in \mathscr{A}$ if $\delta(A^2) = \delta(A)A + A\delta(A)$ holds for any $A \in \mathscr{A}$ with $A^2 = Z$, or alternatively, we can say that: (2) δ is Jordan derivable at some point $Z \in \mathscr{A}$ if $\delta(AB + BA) = \delta(A)B + A\delta(B) + \delta(B)A + B\delta(A)$ holds for any $A, B \in \mathscr{A}$ with $AB + BA = Z$. Observe that if δ satisfies (2) then it also satisfies (1) by inserting $B = A/2$. It is easy to see that the two definitions are not equivalent, take for example $Z = \left(\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $\delta : \left(\begin{smallmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} 0 & A_{12} \\ 0 & 0 \end{smallmatrix}\right)$, and $A = \begin{pmatrix} -\frac{1}{2}I & I \\ \frac{1}{3}I & -\frac{1}{2}I \end{pmatrix}$, $B = \left(\begin{smallmatrix} I & 3I \\ I & 2I \end{smallmatrix}\right)$. So condition (1) is less restrictive than (2).

By now, the majority of the papers on the subject used definition (2), see, e.g., [4] and the references therein. However, less is known for the maps satisfying (1). In case

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H is an infinite-dimensional Hilbert space it was proved in [5, Theorem 2.6] that every linear map $\delta\colon \mathsf{B}(H) \to \mathsf{B}(H)$ which is Jordan derivable at identity operator I, in the sense of definition (1), is an inner derivation, that is, there exists some $T \in \mathsf{B}(H)$ such that $\delta(X) = [T,X]$ for all $X \in \mathsf{B}(H)$. Here $[T,X] = TX - XT$ is a Lie product.

It is our aim to generalize this theorem to idempotents, i.e., operators P with $P^2 = P$, having infinite rank and corank in place of identity. The following is our main result.

THEOREM 1.1. *Let $P \in \mathsf{B}(H)$ be an idempotent with infinite rank and co-rank. Then a linear map $\delta\colon \mathsf{B}(H) \to \mathsf{B}(H)$ is Jordan derivable at P (that is, $\delta(A^2) = \delta(A)A + A\delta(A)$ whenever $A^2 = P$) if and only if there exists an operator $T \in \mathsf{B}(H)$, a scalar $\lambda \in \mathbb{C}$, and a linear map $f\colon \mathsf{B}(H) \to \mathsf{B}(H)$ such that*

$$\delta(A) = [T,A] + \lambda(I-P)A(I-P) + f(PA(I-P)) + f((I-P)AP)$$

for all $A \in \mathsf{B}(H)$.

Before proving Theorem 1.1 let us first show that on finite-dimensional spaces this might not be true.

EXAMPLE 1.2. Let H be a complex Hilbert space $\mathbb{C}^3$ with a standard basis of column vectors $\{e_1, e_2, e_3\}$. Then $\mathsf{B}(H)$ can be identified with the space M_3 of 3×3 complex matrices. For an arbitrary vector $x \in \mathbb{C}^3$ we denote by x^* the transpose and conjugation of x. Then $e_i e_j^* \in M_3$ is a rank one matrix with 1 at (i,j)-th place, and with zeros everywhere else. Let $\delta\colon \mathsf{B}(H) \to \mathsf{B}(H)$ be a linear map defined by

$$\begin{cases} \delta(e_1e_1^*) = -2e_3e_1^*; \\ \delta(e_1e_2^*) = -2e_3e_2^*; \\ \delta(e_1e_3^*) = I_3 - 2e_3e_3^*; \\ \delta(e_ie_j^*) = 0 \quad \text{otherwise.} \end{cases}$$

The only square-zero matrices from M_3 are of rank at most one. So, if $R \in M_3$ is a square-zero matrix, then it can be written as $R = (\sum \alpha_i e_i)(\sum \beta_j e_j)^*$, where $\mathrm{Tr}(R) = \sum \alpha_i \overline{\beta_i} = 0$. Then

$$\delta(R)R + R\delta(R) = -2\alpha_1 \mathrm{Tr}(R) e_3 \left(\sum \beta_j e_j\right)^* = 0,$$

and so δ is Jordan derivable at an idempotent $P = 0$. However

$$\delta(e_1e_3^*) = I_3 - 2e_3e_3^*,$$

which is not a trace-zero operator. So δ is not of the form $X \mapsto [T,X] + \lambda X + f(PA(I-P)) + f((I-P)AP)$ no matter what the operator T and the scalar λ we choose.

2. Proofs

Throughout this section H is an infinite dimensional complex Hilbert space and $\delta\colon \mathsf{B}(H) \to \mathsf{B}(H)$ is a linear map Jordan derivable at some point in the sense of definition (1). Let us first state the result from [6, Theorem 4.2] which we will use in the sequel.

LEMMA 2.1. *Let $F(H) \subseteq \mathsf{B}(H)$ be the space of all finite-rank operators and let $\delta,\tau\colon F(H) \to \mathsf{B}(H)$ be linear mappings such that $\delta(P) = \delta(P)P + P\tau(P)$ and $\tau(P) = \tau(P)P + P\tau(P)$ for any idempotent $P \in F(H)$. Then there exist $S,T \in \mathsf{B}(H)$ such that $\delta(A) = TA - AS$ for every $A \in F(H)$.*

The next lemma is a slight generalization of [5, Corollary 2.3].

LEMMA 2.2. *If $\delta\colon \mathsf{B}(H) \to \mathsf{B}(H)$ is Jordan derivable at 0, then there exist an operator $T \in \mathsf{B}(H)$ and a scalar λ so that*

$$\delta(A) = [T,A] + \lambda A \quad \text{for all} \quad A \in \mathsf{B}(H).$$

Sketch of the proof. By the assumption, δ satisfies

$$\delta(N^2) = \delta(N)N + N\delta(N) \quad \text{for any } N \in \mathsf{B}(H) \text{ with } N^2 = 0. \tag{1}$$

Similarly as in Step 1 of the proof of [5, Theorem 2.2], we obtain, for any idempotent P with infinite range and co-range, that

$$2\delta(P) = 2\delta(P)P + 2P\delta(P) - P\delta(I) - \delta(I)P.$$

Multiplying by P from the left and then from the right and comparing the two results, we see that $P\delta(I) = \delta(I)P$, wherefrom

$$\delta(P) = \delta(P)P + P\delta(P) - P\delta(I)P. \tag{2}$$

In the same way as in Steps 2–4 of the proof of [5, Theorem 2.2], we see that (2) holds for any idempotent P.

Let $\tau\colon \mathsf{B}(H) \to \mathsf{B}(H)$ be a map, defined by $\tau(X) = \delta(X) - \delta(I)X$ for all $X \in \mathsf{B}(H)$. Then by (2), τ and δ satisfy the conditions in Lemma 2.1. Hence there exist $S,T \in \mathsf{B}(H)$ such that

$$\delta(A) = TA - AS, \qquad A \in F(H). \tag{3}$$

In particular, given any rank-one idempotent P, we have $\delta(P) = TP - PS$. And by (2), we obtain $TP - PS = (TP - PSP) + (PTP - PS) - P\delta(I)P$. Hence $P(T - S - \delta(I))P = 0$. Since this is true for every rank-one idempotent P, it follows that (3) holds also for $A = I$. Step 7 of the proof of [5, Theorem 2.2] then shows that (3) holds for every $A \in \mathsf{B}(H)$.

It remains to show that there exists a scalar λ such that $\delta(A) = AT - TA + \lambda A$ for all $A \in \mathsf{B}(H)$. By (1), $0 = (TN - NS)N + N(TN - NS)$, that is, $N(T - S)N = 0$

for all $N \in \mathsf{B}(H)$ with $N^2 = 0$. Taking only rank-one square-zero N, it follows that $T - S = \lambda I$ for some scalar λ, and hence $\delta(A) = TA - AS = TA - AT + \lambda A$, completing the proof. $\square$

LEMMA 2.3. *Let $\delta \colon \mathsf{B}(H) \to \mathsf{B}(H)$ be a linear map Jordan derivable at an idempotent P with infinite rank and co-rank. Then there exist an operator $T \in \mathsf{B}(H)$ and a scalar λ such that*

$$\delta(A) = [T,A] + \lambda A(I-P) \text{ for any } A \in \mathsf{B}(H) \text{ with } A = PAP + (I-P)A(I-P).$$

Proof. Denote by $\operatorname{Im}P$ and $\operatorname{Ker}P$ the range and the kernel of P, respectively. According to the decomposition $H = \operatorname{Im}P + \operatorname{Ker}P$, we may write $P = I \oplus 0$, $A = A_{11} \oplus A_{22}$, and

$$\delta(X) = \begin{pmatrix} \delta_{11}(X) & \delta_{12}(X) \\ \delta_{21}(X) & \delta_{22}(X) \end{pmatrix}.$$

Define two linear operators $\tau_{11} \colon \mathsf{B}(\operatorname{Im}P) \to \mathsf{B}(\operatorname{Im}P)$ and $\tau_{22} \colon \mathsf{B}(\operatorname{Ker}P) \to \mathsf{B}(\operatorname{Ker}P)$ by

$$\tau_{11}(A_{11}) = \delta_{11}(A_{11} \oplus 0) \quad \text{and} \quad \tau_{22}(A_{22}) = \delta_{22}(I \oplus A_{22})$$

for each $A_{11} \in \mathsf{B}(\operatorname{Im}P)$ and $A_{22} \in \mathsf{B}(\operatorname{Ker}P)$. Since δ is Jordan derivable at P, a straightforward computation reveals that τ_{11} and τ_{22} are Jordan derivable at $I \in \mathsf{B}(\operatorname{Im}P)$ and at $0 \in \mathsf{B}(\operatorname{Ker}P)$, respectively. Since $\operatorname{Im}P$ and $\operatorname{Ker}P$ are infinite-dimensional Hilbert spaces, by a result of Jing [5, Theorem 2.6], there exists $X_{11} \in \mathsf{B}(\operatorname{Im}P)$, such that $\tau_{11}(A_{11}) = [X_{11}, A_{11}]$, and by Lemma 2.2 there exist $X_{22} \in \mathsf{B}(\operatorname{Ker}P)$ and a scalar λ, such that $\tau_{22}(A_{22}) = [X_{22}, A_{22}] + \lambda A_{22}$. This gives that

$$\begin{aligned}
\delta(A) &= \delta((A_{11} - I) \oplus 0) + \delta(I \oplus A_{22}) \\
&= \begin{pmatrix} [X_{11}, A_{11} - I] & 0 \\ 0 & [X_{22}, A_{22}] + \lambda A_{22} \end{pmatrix} + \begin{pmatrix} \delta_{11}(I \oplus A_{22}) & 0 \\ 0 & \delta_{22}((A_{11} - I) \oplus 0) \end{pmatrix} + \begin{pmatrix} 0 & \delta_{12}(A) \\ \delta_{21}(A) & 0 \end{pmatrix} \\
&= \begin{pmatrix} [X_{11}, A_{11}] & 0 \\ 0 & [X_{22}, A_{22}] + \lambda A_{22} \end{pmatrix} + \begin{pmatrix} \delta_{11}(I \oplus A_{22}) & 0 \\ 0 & \delta_{22}((A_{11} - I) \oplus 0) \end{pmatrix} + \begin{pmatrix} 0 & \delta_{12}(A) \\ \delta_{21}(A) & 0 \end{pmatrix}. \qquad (4)
\end{aligned}$$

Note that $P\delta(P)P = 0$. So $\delta(P) = \left(\begin{smallmatrix} 0 & * \\ * & * \end{smallmatrix}\right)$ for appropriate operators, denoted by $*$. We proceed to show that the middle term in (4) vanishes. We will use the result by Pearcy and Topping [7] as follows. Let $A = I \oplus N$, where $N \in \mathsf{B}(\operatorname{Ker}P)$ is an arbitrary square-zero matrix. Then $A^2 = P$, and so $\delta(A)A + A\delta(A) = \delta(P) = \left(\begin{smallmatrix} 0 & * \\ * & * \end{smallmatrix}\right)$. Using $A = I \oplus N$ in (4) and comparing the $(1,1)$ entries yields $\delta_{11}(I \oplus N) = 0$ for every square-zero $N \in \mathsf{B}(\operatorname{Ker}P)$. Since $\ker P$ is infinite-dimensional, by Pearcy and Topping [7] every operator is a sum of five square-zero ones, hence $\delta_{11}(I \oplus X) = 0$ for every $X \in \mathsf{B}(\operatorname{Ker}P)$. Hence (4) simplifies to

$$\delta(A_{11} \oplus A_{22}) = \begin{pmatrix} [X_{11}, A_{11}] & 0 \\ 0 & [X_{22}, A_{22}] + \lambda A_{22} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \delta_{22}((A_{11} - I) \oplus 0) \end{pmatrix} + \begin{pmatrix} 0 & \delta_{12}(A) \\ \delta_{21}(A) & 0 \end{pmatrix},$$

so that

$$\delta(P) = \delta(I \oplus 0) = \begin{pmatrix} 0 & \delta_{12}(P) \\ \delta_{21}(P) & 0 \end{pmatrix}.$$

Choose any involution $V \in \mathrm{B}(\mathrm{Im}\, P)$ and any square-zero operator $N \in \mathrm{B}(\mathrm{Ker}\, P)$ to form $A = V \oplus N$ with $A^2 = P$. Then, comparing the $(2,2)$ position in

$$\delta(V \oplus N)(V \oplus N) + (V \oplus N)\delta(V \oplus N) = \delta(P) = \begin{pmatrix} 0 & \delta_{12}(P) \\ \delta_{21}(P) & 0 \end{pmatrix}, \tag{5}$$

we obtain

$$\delta_{22}((V-I) \oplus 0)N + N\delta_{22}((V-I) \oplus 0) = 0$$

for every square-zero N. As above, by Pearcy and Topping [7], this implies that $\delta_{22}((V-I) \oplus 0)I + I\delta_{22}((V-I) \oplus 0) = 0$, and so $\delta_{22}((V-I) \oplus 0) = 0$ for every involution V. Since δ_{22} is linear, we get $\delta_{22}(\frac{I-V}{2} \oplus 0) = 0$. Given an arbitrary idempotent Q, the operator $I - 2Q$ is an involution. So $\delta_{22}(Q \oplus 0) = \delta_{22}(\frac{I-(I-2Q)}{2} \oplus 0) = 0$. By [7], every operator is a sum of five idempotents, so $\delta_{22}(\mathrm{B}(\mathrm{Im}\, P) \oplus 0) = 0$. Therefore (4) simplifies to

$$\delta(A_{11} \oplus A_{22}) = \begin{pmatrix} [X_{11}, A_{11}] & 0 \\ 0 & [X_{22}, A_{22}] + \lambda A_{22} \end{pmatrix} + \begin{pmatrix} 0 & \delta_{12}(A) \\ \delta_{21}(A) & 0 \end{pmatrix}.$$

Comparing the $(1,2)$ position in (5), we obtain

$$\delta_{12}(I \oplus 0) = \delta_{12}(V \oplus N)N + V\delta_{12}(V \oplus N) \tag{6}$$

and this equation is valid for any square-zero N and any involution V. With $V = I$ we get $\delta_{12}(I \oplus 0) = \delta_{12}(I \oplus N)N + \delta_{12}(I \oplus N) = \delta_{12}(I \oplus N)N + \delta_{12}(I \oplus 0) + \delta_{12}(0 \oplus N)$. After postmultiplying with N and simplifying we see that $\delta_{12}(0 \oplus N)N = 0$ for every square-zero N, and consequently (6), with $V = I$, reads $\delta_{12}(I \oplus 0) = \delta_{12}(I \oplus 0)N + \delta_{12}(I \oplus N)$. This further simplifies to

$$\delta_{12}(0 \oplus N) = -\delta_{12}(I \oplus 0)N$$

for every square-zero N, hence by [7] again, this is true for every $A_{22} \in \mathrm{B}(\mathrm{Ker}\, P)$. Now, inserting $V \oplus 0$, $V^2 = I$, in (6) we additionally obtain $\delta_{12}(I \oplus 0) = V\delta_{12}(V \oplus 0)$, which after premultiplying with V simplifies into

$$\delta_{12}(V \oplus 0) = V\delta_{12}(I \oplus 0). \tag{7}$$

This holds for every involution V, hence also for an involution $(I - 2Q)$, where Q is an idempotent. Therefore, by linearity of δ_{12}, the equation (7) also holds for every idempotent Q, and so for every $X \in \mathrm{B}(\mathrm{Im}\, P)$ by [7]. Introducing $Z = -\delta_{12}(I \oplus 0)$, we obtain that

$$\delta_{12}(A_{11} \oplus A_{22}) = ZA_{22} - A_{11}Z.$$

Likewise for $W = \delta_{21}(I \oplus 0)$, one can obtain that

$$\delta_{21}(A_{11} \oplus A_{22}) = WA_{11} - A_{22}W.$$

Therefore, for a matrix $T = \begin{pmatrix} X_{11} & Z \\ W & X_{22} \end{pmatrix}$ we see that

$$\delta(A_{11} \oplus A_{22}) = [T, (A_{11} \oplus A_{22})] + \begin{pmatrix} 0 & 0 \\ 0 & \lambda A_{22} \end{pmatrix}. \quad \square$$

LEMMA 2.4. *Let* $H = H_1 \oplus H_2$ *and let* $A = \left(\begin{smallmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{smallmatrix}\right) \in \mathsf{B}(H)$. *If* $A^2 = I \oplus 0$, *then* $A_{12} = A_{21} = 0$.

Proof. This follows easily by noticing that A commutes with its square $A^2 = I \oplus 0$. $\square$

Proof of Theorem 1.1. Write P as an operator block matrix $P = \left(\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right)$. By Lemma 2.3, there exist operator T and scalar λ such that

$$\begin{aligned}
&\delta\big(PAP + (I-P)A(I-P)\big) \\
&\quad = [T, PAP + (I-P)A(I-P)] + \lambda(I-P)A(I-P) \\
&\quad = [T,A] - [T, PA(I-P) + (I-P)AP] + \lambda(I-P)A(I-P).
\end{aligned}$$

Define a linear map $f\colon \mathsf{B}(H) \to \mathsf{B}(H)$ by $f(X) = \delta(X) - [T,X]$. Then, $\delta(A) = [T,A] + \lambda(I-P)A(I-P) + f(PA(I-P)) + f((I-P)AP)$.

Inversely, given any operator T and scalar λ, the commutator $X \mapsto [T,X]$ and the map $X \mapsto \lambda(I-P)X(I-P)$ are clearly Jordan derivable at P. Also, given any linear map $f\colon \mathsf{B}(H) \to \mathsf{B}(H)$, it is clear that the map $X \mapsto f(PX(I-P)) + f((I-P)XP)$ is also Jordan derivable at P. Indeed, if $A^2 = P$, then $f(PP(I-P)) + f((I-P)PP) = 0$. By Lemma 2.4, $A = PAP + (I-P)A(I-P)$, therefore also $f(PA(I-P)) = 0$ and $f((I-P)AP) = 0$. Hence,

$$X \mapsto [T,X] + \lambda(I-P)X(I-P) + f(PX(I-P)) + f((I-P)XP)$$

is Jordan derivable at P. $\square$

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