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CLASS A OPERATORS AND THEIR EXTENSIONS

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Abstract. In this paper, we study various properties of analytic extensions of class A operators. In particular, we show that every analytic extension of a class A operator has a scalar extension. As a corollary, we get that such an operator with rich spectrum has a nontrivial invariant subspace.

1. Introduction

Let $\mathscr{H}$ and $\mathscr{K}$ be separable complex Hilbert spaces and let $\mathscr{L}(\mathscr{H},\mathscr{K})$ denote the space of all bounded linear operators from $\mathscr{H}$ to $\mathscr{K}$. If $\mathscr{H}=\mathscr{K}$, we write $\mathscr{L}(\mathscr{H})$ in place of $\mathscr{L}(\mathscr{H},\mathscr{K})$. If $T\in\mathscr{L}(\mathscr{H})$, we write $\sigma(T)$, $\sigma_{ap}(T)$, and $\sigma_e(T)$ for the spectrum, the approximate point spectrum, and the essential spectrum of T, respectively.

An arbitrary operator $T\in\mathscr{L}(\mathscr{H})$ has a unique polar decomposition $T=U|T|$, where $|T|=(T^*T)^{\frac{1}{2}}$ and U is the appropriate partial isometry satisfying $\ker(U)=\ker(|T|)=\ker(T)$ and $\ker(U^*)=\ker(T^*)$. Associated with T is a related operator $|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$, called the *Aluthge transform* of T, and denoted throughout this paper by $\widehat{T}$. For an arbitrary operator $T\in\mathscr{L}(\mathscr{H})$, the sequence $\{\widehat{T}^{(n)}\}$ of Aluthge iterates of T is defined by $\widehat{T}^{(0)}=T$ and $\widehat{T}^{(n+1)}=\widehat{\widehat{T}^{(n)}}$ for every positive integer n.

An operator $T\in\mathscr{L}(\mathscr{H})$ is said to be p*-hyponormal* if $(T^*T)^p\geqslant(TT^*)^p$. If $p=1$, T is called *hyponormal* and if $p=\frac{1}{2}$, T is called *semi-hyponormal*. An operator T is said to be w*-hyponormal* if $|\widehat{T}|\geqslant|T|\geqslant|\widehat{T}^*|$. w-Hyponormal operators were introduced by Aluthge and Wang (see [2] and [3]). An operator $T\in\mathscr{L}(\mathscr{H})$ is said to be class A if $|T^2|-|T|^2\geqslant 0$, and T is said to be F*-quasiclass* A if $F(T)^*(|T^2|-|T|^2)F(T)\geqslant 0$ for some function F that is analytic and nonconstant on some neighborhood of $\sigma(T)$. We say that an operator $T\in\mathscr{L}(\mathscr{H})$ is p-quasiclass A if there exists a nonconstant polynomial p such that $p(T)^*(|T^2|-|T|^2)p(T)\geqslant 0$. In particular, if $p(z)=z^k$ for some positive integer k or $p(z)=z$, then T is said to be a k*-quasiclass* A operator or a *quasiclass* A operator, respectively. The class of these operators has been studied by many authors (see [10], [13], [14], [23], and [27], etc.). An operator

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$T \in \mathscr{L}(\mathscr{H})$ is called *normaloid* if $\|T\| = r(T)$ where $r(T) := \sup\{|\lambda| : \lambda \in \sigma(T)\}$ denotes the spectral radius of T. It is well known from [10] that

$$p\text{-hyponormal} \Rightarrow w\text{-hyponormal} \Rightarrow \text{class A} \Rightarrow \text{normaloid}.$$

We give the following example to indicate that there exists a k-quasiclass A operators which does not belong to class A.

EXAMPLE 1.1. Let $T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \in \mathscr{L}(\mathbb{C}^3)$. Then $|T^2| - |T|^2 \ngeq 0$, and so T is not a class A operator. However, $T^{k^*}(|T^2| - |T|^2)T^k = 0$ for every positive integer k, which implies that T is a k-quasiclass A operator for every positive integer k.

From the above example, it is natural to ask whether k-quasiclass A operators are normaloid or not. Next we give a k-quasiclass A operator which is not normaloid.

EXAMPLE 1.2. Let W_α be the unilateral weighted shift with weights $\alpha := \{\alpha_n\}_{n \geqslant 0}$ of positive real numbers. Then it is easy to compute that W_α belongs to k-quasiclass A if and only if

$$\alpha_k \leqslant \alpha_{k+1} \leqslant \alpha_{k+2} \leqslant \cdots.$$

Hence, if we take the weights α such that $\alpha_0 = 2$ and $\alpha_n = \frac{1}{2}$ for all $n \geqslant 1$, then W_α belongs to k-quasiclass A for all $k \in \mathbb{N}$, but it is not normaloid.

We also find an equivalent condition for some operator-valued bilateral weighted shifts to be k-quasiclass A operators.

EXAMPLE 1.3. Let $\mathscr{K} = \oplus_{n=-\infty}^{\infty} \mathscr{H}_n$ where $\mathscr{H}_n = \mathscr{H}$ for all integers n. Given two positive operators A and B in $\mathscr{L}(\mathscr{H})$, define an operator $T \in \mathscr{L}(\mathscr{K})$ by $Tx = y$ with the following relation; if $x = \oplus_{n=-\infty}^{\infty} x_n \in \mathscr{K}$, then $y = \oplus_{n=-\infty}^{\infty} y_n \in \mathscr{K}$ is given by

$$y_n = \begin{cases} Ax_{n-1} \text{ if } n \leqslant 1 \\ Bx_{n-1} \text{ if } n > 1. \end{cases}$$

By straightforward computations, we get that T is a k-quasiclass A operator if and only if

$$A^k[(AB^2A)^{\frac{1}{2}} - A^2]A^k \geqslant 0.$$

For instance, we shall provide an example by using the Maple program. Let $A = \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 0 \\ 0 & 2\sqrt{23} \end{pmatrix}$ be operators on $\mathscr{H} = \mathbb{R}^2$, and let $\mathscr{H}_n = \mathscr{H}$ for all positive integers n. Note that

$$(AB^2A)^{\frac{1}{2}} - A^2 = \begin{pmatrix} 0.17472\cdots & -3.1798\cdots \\ -3.1798\cdots & 11.770\cdots \end{pmatrix}$$

as computed in [10]. Then

$$A^3[(AB^2A)^{\frac{1}{2}} - A^2]A^3 = \begin{pmatrix} 70778.\cdots & -71500.\cdots \\ -71500.\cdots & 72227.\cdots \end{pmatrix}$$

and its eigenvalues are $143010.\cdots$ and $-1.1705\cdots$, and so

$$A^3[(AB^2A)^{\frac{1}{2}} - A^2]A^3 \not\geq 0.$$

Therefore if we define T on $\oplus_{n=-\infty}^{\infty}\mathscr{H}_n$ as in the above, then T is not a 3-quasiclass A operator.

An operator $T \in \mathscr{L}(\mathscr{H})$ is said to be *analytic* if there exists a nonconstant analytic function F on a neighborhood of $\sigma(T)$ such that $F(T) = 0$. We say that an operator $T \in \mathscr{L}(\mathscr{H})$ is *algebraic* if there is a nonconstant polynomial p such that $p(T) = 0$. In particular, if $T^k = 0$ for some positive integer k, then T is called *nilpotent*. An operator $T \in \mathscr{L}(\mathscr{H})$ is said to be *quasinilpotent* if $\sigma(T) = \{0\}$. If an operator $T \in \mathscr{L}(\mathscr{H})$ is analytic, then $F(T) = 0$ for some nonconstant analytic function F on a neighborhood D of $\sigma(T)$. Since F cannot have infinitely many zeros in D, we write $F(z) = G(z)p(z)$ where G is a function that is analytic and does not vanish on D and p is a nonconstant polynomial with zeros in D. By Riesz-Dunford calculus, $G(T)$ is invertible and then $p(T) = 0$, which means that T is algebraic (see [5]). *When p has degree k, we say that T is analytic with order k throughout this paper.*

An operator $T \in \mathscr{L}(\mathscr{H})$ is called *scalar* of order m if it possesses a spectral distribution of order m, i.e., if there is a continuous unital homomorphism of topological algebras

$$\Phi : C_0^m(\mathbb{C}) \to \mathscr{L}(\mathscr{H})$$

such that $\Phi(z) = T$, where as usual z stands for the identical function on $\mathbb{C}$, and $C_0^m(\mathbb{C})$ for the space of all continuously differentiable functions of order m which are compactly supported, $0 \leqslant m \leqslant \infty$. An operator is *subscalar* of order m if it is similar to the restriction of a scalar operator of order m to an invariant subspace.

In 1984, M. Putinar showed in [25] that every hyponormal operator is subscalar of order 2. In 1987, his theorem was used to show that hyponormal operators with thick spectra have a nontrivial invariant subspace, which was a result due to S. Brown (see [4]). In this paper, we study various properties of analytic extensions of class A operators. In particular, we show that every analytic extension of a class A operator has a scalar extension. As a corollary, we get that such an operator with rich spectrum has a nontrivial invariant subspace. In addition, we study some properties of analytic extensions of class A operators.

2. Preliminaries

An operator $T \in \mathscr{L}(\mathscr{H})$ is called *left semi-Fredholm* if T has closed range and $\dim(\ker(T)) < \infty$, and T is called *right semi-Fredholm* if T has closed range and $\dim(\mathscr{H}/\operatorname{ran}(T)) < \infty$. When T is either left semi-Fredholm or right semi-Fredholm, T

is called *semi-Fredholm*. In this case, the Fredholm index of T is defined by $\text{ind}(T) := \dim(\ker(T)) - \dim(\mathscr{H}/\text{ran}(T))$. Note that $\text{ind}(T)$ is an integer or $\pm\infty$. We say that T is *Fredholm* if it is both left and right semi-Fredholm. Especially, an operator $T \in \mathscr{L}(\mathscr{H})$ is said to be *Weyl* if it is Fredholm of index zero. The Weyl spectrum is given by $\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\}$ and we write $\pi_{00}(T) := \{\lambda \in \text{iso}\sigma(T) : 0 < \dim(\ker(T - \lambda)) < \infty\}$. We say that Weyl's theorem holds for T if $\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T)$. A hole in $\sigma_e(T)$ is a nonempty bounded component of $\mathbb{C} \setminus \sigma_e(T)$, and a pseudohole in $\sigma_e(T)$ is a nonempty component of $\sigma_e(T) \setminus \sigma_{le}(T)$ or of $\sigma_e(T) \setminus \sigma_{re}(T)$, where $\sigma_{le}(T)$ and $\sigma_{re}(T)$ denotes the left essential spectrum and the right essential spectrum of T, respectively. The *spectral picture* of T is the structure consisting of $\sigma_e(T)$, the collection of holes and pseudoholes in $\sigma_e(T)$, and it is denoted by $SP(T)$ (see [24] for more details).

An operator $T \in \mathscr{L}(\mathscr{H})$ is said to have the *single-valued extension property* (or SVEP) if for every open subset G of $\mathbb{C}$ and any analytic function $f : G \to \mathscr{H}$ such that $(T - z)f(z) \equiv 0$ on G, we have $f(z) \equiv 0$ on G. For $T \in \mathscr{L}(\mathscr{H})$ and $x \in \mathscr{H}$, the set $\rho_T(x)$ is defined to consist of elements z_0 in $\mathbb{C}$ such that there exists an analytic function $f(z)$ defined in a neighborhood of z_0, with values in $\mathscr{H}$, which verifies $(T - z)f(z) \equiv x$, and it is called *the local resolvent set* of T at x. We denote the complement of $\rho_T(x)$ by $\sigma_T(x)$, called *the local spectrum* of T at x, and define *the local spectral subspace* of T, $H_T(F) = \{x \in \mathscr{H} : \sigma_T(x) \subseteq F\}$ for each subset F of $\mathbb{C}$. An operator $T \in \mathscr{L}(\mathscr{H})$ is said to have *property* (β) if for every open subset G of $\mathbb{C}$ and every sequence $f_n : G \to \mathscr{H}$ of $\mathscr{H}$-valued analytic functions such that $(T - z)f_n(z)$ converges uniformly to 0 in norm on compact subsets of G, then $f_n(z)$ converges uniformly to 0 in norm on compact subsets of G. An operator $T \in \mathscr{L}(\mathscr{H})$ is said to have *Dunford's property* (C) if $H_T(F)$ is closed for each closed subset F of $\mathbb{C}$. It is well known from [18] that

$$\text{Property } (\beta) \Rightarrow \text{Dunford's property } (C) \Rightarrow \text{SVEP}.$$

Let z be the coordinate function in the complex plane $\mathbb{C}$ and $d\mu(z)$ the planar Lebesgue measure. Consider a bounded (connected) open subset U of $\mathbb{C}$. We shall denote by $L^2(U, \mathscr{H})$ the Hilbert space of measurable functions $f : U \to \mathscr{H}$, such that

$$\|f\|_{2,U} = \Big(\int_U \|f(z)\|^2 d\mu(z) \Big)^{\frac{1}{2}} < \infty.$$

The space of functions $f \in L^2(U, \mathscr{H})$ that are analytic in U is denoted by

$$A^2(U, \mathscr{H}) = L^2(U, \mathscr{H}) \cap \mathscr{O}(U, \mathscr{H})$$

where $\mathscr{O}(U, \mathscr{H})$ denotes the Fr*é*chet space of $\mathscr{H}$-valued analytic functions on U with respect to uniform topology. $A^2(U, \mathscr{H})$ is called the Bergman space for U. Note that $A^2(U, \mathscr{H})$ is a Hilbert space.

Now, let us define a special Sobolev type space. For a fixed non-negative integer m, the vector-valued Sobolev space $W^m(U, \mathscr{H})$ with respect to $\bar{\partial}$ and of order m will be the space of those functions $f \in L^2(U, \mathscr{H})$ whose derivatives $\bar{\partial} f, \cdots, \bar{\partial}^m f$ in the sense of distributions still belong to $L^2(U,\mathscr{H})$. Endowed with the norm

$$\|f\|^2_{W^m} = \sum_{i=0}^{m} \|\bar{\partial}^i f\|^2_{2,U},$$

$W^m(U,\mathscr{H})$ becomes a Hilbert space contained continuously in $L^2(U,\mathscr{H})$.

We can easily show that the linear operator M of multiplication by z on $W^m(U,\mathscr{H})$ is continuous and it has a spectral distribution Φ of order m defined by the following relation; for $\varphi \in C_0^m(\mathbb{C})$ and $f \in W^m(U,\mathscr{H})$, $\Phi(\varphi)f = \varphi f$. Hence M is a scalar operator of order m.

3. Main results

In this section, we will show that every analytic extension of a class A operator has a scalar extension. For this, we begin with the following lemmas.

LEMMA 3.1. ([25]) For a bounded open disk D in the complex plane $\mathbb{C}$ there is a constant C_D such that for any operator $T \in \mathscr{L}(\mathscr{H})$ and $f \in W^m(D,\mathscr{H})$ $(m \geqslant 2)$ we have

$$\|(I-P)\bar{\partial}^i f\|_{2,D} \leqslant C_D\big(\|(T-z)^*\bar{\partial}^{i+1} f\|_{2,D} + \|(T-z)^*\bar{\partial}^{i+2} f\|_{2,D}\big)$$

for $i = 0,1,\cdots,m-2$, where P denotes the orthogonal projection of $L^2(D,\mathscr{H})$ onto the Bergman space $A^2(D,\mathscr{H})$.

LEMMA 3.2. ([25]) Let $T \in \mathscr{L}(\mathscr{H})$ be a hyponormal operator and let D be a bounded disk in $\mathbb{C}$. If $\{f_n\}$ is a sequence in $W^m(D,\mathscr{H})$ $(m > 2)$ such that

$$\lim_{n\to\infty} \|(T-z)\bar{\partial}^i f_n\|_{2,D} = 0$$

for $i = 1,2,\cdots,m$, then $\lim_{n\to\infty}\|\bar{\partial}^i f_n\|_{2,D_0} = 0$ for $i = 1,2,\cdots,m-2$ where D_0 is a disk strictly contained in D.

LEMMA 3.3. Let D be a bounded disk in $\mathbb{C}$ and let m be a positive integer with $m > 12$. If $T \in \mathscr{L}(\mathscr{H})$ is a class A operator and f_n is a sequence in $W^m(D,\mathscr{H})$ such that

$$\lim_{n\to\infty} \|(T-z)\bar{\partial}^i f_n\|_{2,D} = 0$$

for $i = 1,2,\cdots,m$, then it holds that

$$\lim_{n\to\infty} \|(I-P)\bar{\partial}^i f_n\|_{2,D_1} = 0$$

for $i = 0,1,2,\cdots,m-12$, where P denotes the orthogonal projection of $L^2(D,\mathscr{H})$ onto $A^2(D,\mathscr{H})$ and D_1 is any disk relatively compact in D. Furthermore, we have

$$\lim_{n\to\infty} \|\bar{\partial}^i f_n\|_{2,D_2} = 0$$

for $i = 1,2,\cdots,m-12$, where D_2 is any disk relatively compact in D_1.

Proof. As in [14, Lemma 3.1], we can show that

$$\lim_{n\to\infty} \|(I-P)\overline{\partial}^i f_n\|_{2,D_1} = 0$$

for $i = 0,1,2,\cdots,m-12$, where P denotes the orthogonal projection of $L^2(D,\mathscr{H})$ onto $A^2(D,\mathscr{H})$ and D_1 is any disk relatively compact in D. Then it follows that

$$\lim_{n\to\infty} \|(T-z)P\overline{\partial}^i f_n\|_{2,D_1} = 0$$

for $i = 1,2,\cdots,m-12$. Since T has property (β) from [14], we get that

$$\lim_{n\to\infty} \|P\overline{\partial}^i f_n\|_{2,D_2} = 0$$

for $i = 1,2,\cdots,m-12$, where D_2 is any disk relatively compact in D_1. Hence we complete our proof. □

The next lemma is the key step to prove the subscalarity for analytic extensions of class A operators.

LEMMA 3.4. Let $T \in \mathscr{L}(\mathscr{H}\oplus\mathscr{K})$ be an analytic extension of a class A operator, i.e., $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ where T_1 is a class A operator and T_3 is analytic with order k and let D be a bounded disk in $\mathbb{C}$ containing $\sigma(T)$. Define the map $V : \mathscr{H}\oplus\mathscr{K} \to H(D)$ by

$$Vh = \widetilde{1\otimes h}\ (\ \equiv 1\otimes h + \overline{(T-z)W^{2k+12}(D,\mathscr{H})\oplus W^{2k+12}(D,\mathscr{K})}\ )$$

where

$$H(D) := W^{2k+12}(D,\mathscr{H})\oplus W^{2k+12}(D,\mathscr{K})/\overline{(T-z)W^{2k+12}(D,\mathscr{H})\oplus W^{2k+12}(D,\mathscr{K})}$$

and $1\otimes h$ denotes the constant function sending any $z \in D$ to h. Then V is one-to-one and has closed range.

Proof. Let $f_n = f_n^1 \oplus f_n^2 \in W^{2k+12}(D,\mathscr{H})\oplus W^{2k+12}(D,\mathscr{K})$ and $h_n = h_n^1 \oplus h_n^2 \in \mathscr{H}\oplus\mathscr{K}$ be sequences such that

$$\lim_{n\to\infty} \|(T-z)f_n + 1\otimes h_n\|_{W^{2k+12}(D,\mathscr{H})\oplus W^{2k+12}(D,\mathscr{K})} = 0. \tag{1}$$

Then from (1) we have the following equations:

$$\begin{cases} \lim_{n\to\infty} \|(T_1-z)f_n^1 + T_2 f_n^2 + 1\otimes h_n^1\|_{W^{2k+12}} = 0 \\ \lim_{n\to\infty} \|(T_3-z)f_n^2 + 1\otimes h_n^2\|_{W^{2k+12}} = 0. \end{cases} \tag{2}$$

By the definition of the norm for the Sobolev space, (2) implies that

$$\begin{cases} \lim_{n\to\infty} \|(T_1-z)\overline{\partial}^i f_n^1 + T_2\overline{\partial}^i f_n^2\|_{2,D} = 0 \\ \lim_{n\to\infty} \|(T_3-z)\overline{\partial}^i f_n^2\|_{2,D} = 0 \end{cases} \tag{3}$$

for $i = 1, 2, \cdots, 2k+12$. Since T_3 is analytic with order k, there exists a nonconstant analytic function F on a neighborhood of $\sigma(T_3)$ such that $F(T_3) = 0$. As remarked in section one, write $F(z) = G(z)p(z)$ where G is analytic and does not vanish on a neighborhood of $\sigma(T_3)$ and $p(z) = (z-z_1)(z-z_2)\cdots(z-z_k)$ is a polynomial of degree k. Set $q_j(z) = (z-z_{j+1})\cdots(z-z_k)$ for $j = 0, 1, 2, \cdots, k-1$ and $q_k(z) = 1$.

Claim. It holds for every $j = 0, 1, 2, \cdots, k$ that

$$\lim_{n\to\infty} \|q_j(T_3)\bar{\partial}^i f_n^2\|_{2,D_j} = 0$$

for $i = 1, 2, \cdots, 2k-2j+12$, where $\sigma(T) \subsetneqq D_k \subsetneqq \cdots \subsetneqq D_2 \subsetneqq D_1 \subsetneqq D$.

To prove the claim, we will use the induction on j. Since $0 = F(T_3) = G(T_3)p(T_3)$ and $G(T_3)$ is invertible, it follows that $q_0(T_3) = p(T_3) = 0$, and so the claim holds when $j = 0$. Suppose that the claim is true for some $j = r$ where $0 \leqslant r < k$. That is,

$$\lim_{n\to\infty} \|q_r(T_3)\bar{\partial}^i f_n^2\|_{2,D_r} = 0 \tag{4}$$

for $i = 1, 2, \cdots, 2k-2r+12$, where $\sigma(T) \subsetneqq D_r \subsetneqq \cdots \subsetneqq D_1 \subsetneqq D$. By the second equation of (3) and (4), we get that

$$\begin{aligned} 0 &= \lim_{n\to\infty} \|q_{r+1}(T_3)(T_3 - z)\bar{\partial}^i f_n^2\|_{2,D_r} \\ &= \lim_{n\to\infty} \|q_{r+1}(T_3)(T_3 - z_{r+1} + z_{r+1} - z)\bar{\partial}^i f_n^2\|_{2,D_r} \\ &= \lim_{n\to\infty} \|(z_{r+1} - z)q_{r+1}(T_3)\bar{\partial}^i f_n^2\|_{2,D_r} \end{aligned} \tag{5}$$

for $i = 1, 2, \cdots, 2k-2r+12$. Since $z_{r+1}I$ is hyponormal, by applying Lemma 3.2 we obtain that

$$\lim_{n\to\infty} \|q_{r+1}(T_3)\bar{\partial}^i f_n^2\|_{2,D_{r+1}} = 0 \tag{6}$$

for $i = 1, 2, \cdots, 2k-2r+10$, where $\sigma(T) \subsetneqq D_{r+1} \subsetneqq D_r$. Hence we complete the proof of our claim.

From the claim with $j = k$, we have

$$\lim_{n\to\infty} \|\bar{\partial}^i f_n^2\|_{2,D_k} = 0 \tag{7}$$

for $i = 1, 2, \cdots, 12$, which implies by Lemma 3.1 that

$$\lim_{n\to\infty} \|(I - P_2)f_n^2\|_{2,D_k} = 0 \tag{8}$$

where P_2 denotes the orthogonal projection of $L^2(D_k, \mathscr{K})$ onto $A^2(D_k, \mathscr{K})$. By combining (7) with the first equation of (3), we obtain that

$$\lim_{n\to\infty} \|(T_1 - z)\bar{\partial}^i f_n^1\|_{2,D_k} = 0 \tag{9}$$

for $i = 1, 2, \cdots, 12$. From Lemma 3.3, it follows that

$$\lim_{n\to\infty} \|(I - P_1) f_n^1\|_{2,D_{k,1}} = 0. \tag{10}$$

Set $Pf_n := \binom{P_1 f_n^1}{P_2 f_n^2}$. Combining (8) and (10) with (2), we have

$$\lim_{n\to\infty} \|(T - z)Pf_n + 1 \otimes h_n\|_{2,D_{k,1}} = 0.$$

Let Γ be a curve in $D_{k,1}$ surrounding $\sigma(T)$. Then

$$\lim_{n\to\infty} \|Pf_n(z) + (T - z)^{-1}(1 \otimes h_n)(z)\| = 0$$

uniformly for all $z \in \Gamma$. Applying Riesz-Dunford functional calculus, we obtain that

$$\lim_{n\to\infty} \|\frac{1}{2\pi i}\int_\Gamma Pf_n(z)\, dz + h_n\| = 0.$$

But by Cauchy's theorem, $\frac{1}{2\pi i}\int_\Gamma Pf_n(z)\, dz = 0$. Hence $\lim_{n\to\infty} \|h_n\| = 0$, and so V is one-to-one and has closed range. $\square$

Now we are ready to prove that every analytic extension of a class A operator has a scalar extension.

THEOREM 3.5. Every analytic extension of a class A operator is subscalar.

Proof. Let $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ be an operator matrix defined on $\mathscr{H} \oplus \mathscr{K}$, where T_1 is a class A operator and T_3 is analytic with order k. Let D be an arbitrary bounded open disk in $\mathbb{C}$ that contains $\sigma(T)$. As in Lemma 3.4, if we define an operator $V : \mathscr{H} \oplus \mathscr{K} \to H(D)$ by $Vh = \widetilde{1 \otimes h}$, then V is one-to-one and has closed range. The class of a vector f or an operator S on $H(D)$ will be denoted by $\widetilde{f}$, respectively $\widetilde{S}$. Let M be the operator of multiplication by z on $W^{2k+12}(D,\mathscr{H}) \oplus W^{2k+12}(D,\mathscr{K})$. Then M is a scalar operator of order $2k + 12$ and has a spectral distribution Φ. Since the range of $T - z$ is invariant under M, $\widetilde{M}$ can be well-defined. Moreover, consider the spectral distribution $\Phi : C_0^{2k+12}(\mathbb{C}) \to \mathscr{L}\big(W^{2k+12}(D,\mathscr{H}) \oplus W^{2k+12}(D,\mathscr{K})\big)$ defined by the following relation; for $\varphi \in C_0^{2k+12}(\mathbb{C})$ and $f \in W^{2k+12}(D,\mathscr{H}) \oplus W^{2k+12}(D,\mathscr{K})$, $\Phi(\varphi)f = \varphi f$. Then the spectral distribution Φ of M commutes with $T - z$, and so $\widetilde{M}$ is still a scalar operator of order $2k + 12$ with $\widetilde{\Phi}$ as a spectral distribution. Since

$$VTh = \widetilde{1 \otimes Th} = \widetilde{z \otimes h} = \widetilde{M}(\widetilde{1 \otimes h}) = \widetilde{M}Vh$$

for all $h \in \mathscr{H} \oplus \mathscr{K}$, $VT = \widetilde{M}V$. In particular, $\mathrm{ran}(V)$ is invariant under $\widetilde{M}$, where $\mathrm{ran}(V)$ is the range of V. Since $\mathrm{ran}(V)$ is closed, it is a closed invariant subspace of the scalar operator $\widetilde{M}$. Since T is similar to the restriction $\widetilde{M}|_{\mathrm{ran}(V)}$ and $\widetilde{M}$ is a scalar operator of order $2k + 12$, T is subscalar of order $2k + 12$. $\square$

As an application of our main theorem, we prove that every F-quasiclass A operator is subscalar with the following lemma.

LEMMA 3.6. Let $T \in \mathcal{L}(\mathcal{H})$ be F-quasiclass A and let $\mathcal{M}$ be an invariant subspace for T. Then the restriction $T|_{\mathcal{M}}$ is a p-quasiclass A operator.

Proof. Since T is an F-quasiclass A operator, $F(T)^*(|T^2|-|T|^2)F(T) \geqslant 0$ for some function F analytic and nonconstant on a neighborhood of $\sigma(T)$. Set $F(z) = G(z)p(z)$ where G is a nonvanishing analytic function on a neighborhood of $\sigma(T)$ and p is a nonconstant polynomial. Since $\mathcal{M}$ is a T-invariant subspace, we can write $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on the decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$, where $T_1 = T|_{\mathcal{M}}$, $T_3 = (I-P)T(I-P)|_{\mathcal{M}^{\perp}}$, and P denotes the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}$. Since $((T^2)^*T^2)^{\frac{1}{2}} \geqslant 0$, from [9] we can set

$$|T^2| = ((T^2)^*T^2)^{\frac{1}{2}} = \begin{pmatrix} B & C \\ C^* & D \end{pmatrix},$$

where $B \geqslant 0$, $D \geqslant 0$, and $C = B^{\frac{1}{2}}SD^{\frac{1}{2}}$ for some contraction $S : \mathcal{M}^{\perp} \to \mathcal{M}$. Then a simple calculation gives that

$$(T^2)^*T^2 = |T^2|^2 = \begin{pmatrix} B & C \\ C^* & D \end{pmatrix}^2 = \begin{pmatrix} B^2+CC^* & BC+CD \\ C^*B+DC^* & C^*C+D^2 \end{pmatrix}.$$

Since

$$(T^2)^*T^2 = \begin{pmatrix} (T_1^2)^*T_1^2 & * \\ * & * \end{pmatrix},$$

we get that $B^2 + CC^* = (T_1^2)^*T_1^2$. Hence

$$|T_1^2| = ((T_1^2)^*T_1^2)^{\frac{1}{2}} = (B^2+CC^*)^{\frac{1}{2}} \geqslant B.$$

Also, since

$$|T|^2 = T^*T = \begin{pmatrix} T_1^*T_1 & * \\ * & * \end{pmatrix} = \begin{pmatrix} |T_1|^2 & * \\ * & * \end{pmatrix},$$

we have

$$\begin{aligned} 0 &\leqslant F(T)^*(|T^2|-|T|^2)F(T) \\ &= F(T)^* \begin{pmatrix} B-|T_1|^2 & * \\ * & * \end{pmatrix} F(T) = G(T)^* \begin{pmatrix} p(T_1)^*(B-|T_1|^2)p(T_1) & * \\ * & * \end{pmatrix} G(T) \end{aligned}$$

by Riesz-Dunford's functional calculus. Since $G(T)$ is invertible, we obtain from [9] that $p(T_1)^*(B-|T_1|^2)p(T_1) \geqslant 0$, which completes our proof. $\square$

THEOREM 3.7. Every F-quasiclass A operator is subscalar. In particular, every k-quasiclass A operator is subscalar of order $2k+12$.

Proof. Suppose that $T \in \mathcal{L}(\mathcal{H})$ satisfies that $F(T)^*(|T^2|-|T|^2)F(T) \geqslant 0$ for some analytic function F on a neighborhood of $\sigma(T)$. If the range of $F(T)$ is norm dense in $\mathcal{H}$, then T is a class A operator. Hence T is subscalar of order 12 by

Theorem 3.5. So it suffices to assume that the range of $F(T)$ is not norm dense in $\mathscr{H}$. Since $F(T)$ commutes with T, $\overline{\operatorname{ran}(F(T))}$ is a T-invariant subspace, and so we can express T as $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $\mathscr{H} = \overline{\operatorname{ran}(F(T))} \oplus \ker(F(T)^*)$ where $T_1 = T|_{\overline{\operatorname{ran}(F(T))}}$, $T_3 = (I-P)T(I-P)|_{\ker(F(T)^*)}$, and P denotes the projection of $\mathscr{H}$ onto $\overline{\operatorname{ran}(F(T))}$. Note that $F(z) = G(z)p(z)$ where G is a nonvanishing analytic function on a neighborhood of $\sigma(T)$ and p is a nonconstant polynomial. Then $G(T)$ is invertible and thus we obtain that $\ker(F(T)^*) = \ker(p(T)^*)$. Since $p(T_3) = (I-P)p(T)(I-P)|_{\ker(F(T)^*)}$, it holds for any $x \in \ker(F(T)^*)$ that

$$\langle p(T_3)x, x\rangle = \langle p(T)x, x\rangle = \langle x, p(T)^*x\rangle = 0.$$

Hence $p(T_3) = 0$ and so T_3 is analytic. In addition, since $P(|T^2| - |T|^2)P \geqslant 0$, we have

$$|T_1^2| - |T_1|^2 \geqslant B - |T_1|^2 \geqslant 0$$

from the proof of Lemma 3.6 and [9]. This means that T_1 is a class A operator. Therefore if T_3 is analytic with order k, then T is subscalar of order $2k+12$ by Theorem 3.5. □

In the next corollary, we obtain a partial solution to the invariant subspace problem for analytic extensions of class A operators, which is a generalization of S. Brown's result mentioned in section one.

COROLLARY 3.8. Let $T \in \mathscr{L}(\mathscr{H} \oplus \mathscr{K})$ be an analytic extension of a class A operator. If $\sigma(T)$ has nonempty interior in $\mathbb{C}$, then T has a nontrivial invariant subspace.

Proof. The proof follows from Theorem 3.5 and [8]. □

For the following corollary, note that an operator $T \in \mathscr{L}(\mathscr{H})$ is said to be *power regular* if $\{\|T^n x\|^{\frac{1}{n}}\}_{n=0}^{\infty}$ converges for each $x \in \mathscr{H}$ and $r_T(x)$ denotes the *local spectral radius* of T at x given by $r_T(x) := \limsup_{n\to\infty} \|T^n x\|^{\frac{1}{n}}$. Moreover, we recall that for an operator $T \in \mathscr{L}(\mathscr{H})$, *a spectral maximal space* of T is defined to be a closed T-invariant subspace $\mathscr{M}$ of $\mathscr{H}$ with the property that $\mathscr{M}$ contains any closed T-invariant subspace $\mathscr{N}$ of $\mathscr{H}$ such that $\sigma(T|_{\mathscr{N}}) \subset \sigma(T|_{\mathscr{M}})$. Furthermore, recall that an operator $X \in \mathscr{L}(\mathscr{H}, \mathscr{K})$ is called a *quasiaffinity* if it has trivial kernel and dense range. An operator $S \in \mathscr{L}(\mathscr{H})$ is said to be a *quasiaffine transform* of an operator $T \in \mathscr{L}(\mathscr{K})$ if there is a quasiaffinity $X \in \mathscr{L}(\mathscr{H}, \mathscr{K})$ such that $XS = TX$. Also, operators $S \in \mathscr{L}(\mathscr{H})$ and $T \in \mathscr{L}(\mathscr{K})$ are *quasisimilar* if there are quasiaffinities $X \in \mathscr{L}(\mathscr{H}, \mathscr{K})$ and $Y \in \mathscr{L}(\mathscr{K}, \mathscr{H})$ such that $XS = TX$ and $SY = YT$.

COROLLARY 3.9. If $T \in \mathscr{L}(\mathscr{H} \oplus \mathscr{K})$ is an analytic extension of a class A operator, then the following statements hold.
(i) T has property (β), Dunford's property (C), and the single-valued extension property.

(ii) T is power regular.
(iii) $r_T(x) = \lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}}$ for all $x \in \mathscr{H}$.
(iv) $H_T(E)$ is a spectral maximal space of T and $\sigma(T|_{H_T(E)}) \subset \sigma(T) \cap E$ for any closed subset E in $\mathbb{C}$.
(v) If S is a quasiaffine transform of T such that $XS = TX$ where X is a quasiaffinity, then S has the single-valued extension property and $XH_S(E) \subseteq H_T(E)$ for any subset E in $\mathbb{C}$.

Proof. (i) From section two, it suffices to prove that T has property (β). Since property (β) is transmitted from an operator to its restrictions to closed invariant subspaces, we are reduced by Theorem 3.5 to the case of a scalar operator. Since every scalar operator has property (β) (see [25]), T has property (β).

(ii) From Theorem 3.5, T is similar to the restriction of a scalar operator to one of its invariant subspaces. Since a scalar operator is power regular and the restrictions of power regular operators to their invariant subspaces are still power regular, T is also power regular.

(iii) The proof follows from (i) and [18].

(iv) Since T has property (C) from (i), $H_T(E)$ is closed for any closed subset E in $\mathbb{C}$. Hence the proof follows from [6] or [18].

(v) Let $f : G \to \mathscr{H} \oplus \mathscr{K}$ be an analytic function on an open set G in $\mathbb{C}$ such that $(S-z)f(z) \equiv 0$. Then $(T-z)Xf(z) = X(S-z)f(z) \equiv 0$ on G. Since T has the single-valued extension property, $Xf(z) \equiv 0$ on G. Since X is a quasiaffinity, $f(z) \equiv 0$ on G. Hence S has the single-valued extension property. To prove the last conclusion, it suffices to show that $\sigma_T(Xx) \subseteq \sigma_S(x)$ for any $x \in \mathscr{H} \oplus \mathscr{K}$; in fact, if it holds, then $x \in H_S(E)$ implies $\sigma_T(Xx) \subset E$, which means that $Xx \in H_T(E)$. If $z_0 \in \rho_S(x)$, then we can choose an $\mathscr{H} \oplus \mathscr{K}$-valued analytic function f on some neighborhood of z_0 for which $(S-z)f(z) \equiv x$. Since $XS = TX$, we have $(T-z)Xf(z) = X(S-z)f(z) \equiv Xx$, and so $z_0 \in \rho_T(Xx)$. $\square$

COROLLARY 3.10. Let C and D be operator matrices in $\mathscr{L}(\mathscr{H} \oplus \mathscr{K})$ which are analytic extensions of class A operators. If C and D are quasisimilar, then $\sigma(C) = \sigma(D)$ and $\sigma_e(C) = \sigma_e(D)$.

Proof. Since C and D satisfy property (β) from Corollary 3.9, the proof follows from [26]. $\square$

COROLLARY 3.11. Let $T \in \mathscr{L}(\mathscr{H} \oplus \mathscr{K})$ be an analytic extension of a class A operator. If there exists a nonzero vector $x \in \mathscr{H} \oplus \mathscr{K}$ such that $\sigma_T(x) \subsetneqq \sigma(T)$, then T has a nontrivial hyperinvariant subspace.

Proof. Set $\mathscr{M} := H_T(\sigma_T(x))$, i.e., $\mathscr{M} = \{y \in \mathscr{H} \oplus \mathscr{K} : \sigma_T(y) \subseteq \sigma_T(x)\}$. Since T has Dunford's property (C) by Corollary 3.9, $\mathscr{M}$ is a T-hyperinvariant subspace from [6] or [18]. Since $x \in \mathscr{M}$, we get $\mathscr{M} \neq \{0\}$. Suppose $\mathscr{M} = \mathscr{H} \oplus \mathscr{K}$. Since T has the single-valued extension property by Corollary 3.9, it follows from [18] that

$$\sigma(T) = \bigcup\{\sigma_T(y) : y \in \mathscr{H} \oplus \mathscr{K}\} \subseteq \sigma_T(x) \subsetneqq \sigma(T),$$

which is a contradiction. Hence $\mathscr{M}$ is a nontrivial T-hyperinvariant subspace. $\square$

Next we show that every analytic extension $T \in \mathscr{L}(\mathscr{H} \oplus \mathscr{K})$ of a class A operator is isoloid (i.e., $\operatorname{iso}\sigma(T) \subseteq \sigma_p(T)$ where $\operatorname{iso}\sigma(T)$ denotes the set of all isolated points of $\sigma(T)$). If $T \in \mathscr{L}(\mathscr{H})$ is analytic, then there exists a nonconstant polynomial $p(z)$ such that $p(T) = 0$. If $q(z)$ is a minimal polynomial satisfying $q(T) = 0$, it is obvious that $q(z)$ is a factor of $p(z)$.

LEMMA 3.12. Suppose that $T \in \mathscr{L}(\mathscr{H} \oplus \mathscr{K})$ is an analytic extension of a class A operator, i.e., $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ is an operator matrix on $\mathscr{H} \oplus \mathscr{K}$ where T_1 is a class A operator and $F(T_3) = 0$ for a nonconstant analytic function F on a neighborhood D of $\sigma(T_3)$. Then the spectrum $\sigma(T) = \sigma(T_1) \cup \sigma(T_3)$ and $\sigma(T_3)$ is a subset of $\{z \in \mathbb{C} : p(z) = 0\}$ where $F(z) = G(z)p(z)$, G is analytic and does not vanish on D, and p is a polynomial.

Proof. Since $p(T_3) = 0$, choose a minimal polynomial q such that $q(T_3) = 0$ and $q(z)$ is a factor of $p(z)$ as remarked in the above. Then $\{z \in \mathbb{C} : q(z) = 0\}$ is nonempty and is contained in $\{z \in \mathbb{C} : p(z) = 0\}$. First we will show that $\sigma(T_3) = \sigma_p(T_3) = \{z \in \mathbb{C} : q(z) = 0\}$. Since $q(T_3) = 0$, we have $q(\sigma(T_3)) = \sigma(q(T_3)) = \{0\}$ by the spectral mapping theorem. This means that $\sigma(T_3) \subseteq \{z \in \mathbb{C} : q(z) = 0\}$. Moreover if we assume that $z_1, \cdots, z_k$ are all the roots of $q(z) = 0$, not necessarily distinct, then $(T_3 - z_1)(T_3 - z_2)\cdots(T_3 - z_k)x = 0$ for all $x \in \mathscr{K}$. By the minimality of the degree of q, we can select a vector $x_0 \in \mathscr{K}$ such that $(T_3 - z_2)\cdots(T_3 - z_k)x_0 \neq 0$, and so $z_1 \in \sigma_p(T_3)$. Similarly, $z_i \in \sigma_p(T_3)$ for all $i = 1, 2, \cdots, k$. Hence $\sigma(T_3) = \sigma_p(T_3) = \{z \in \mathbb{C} : q(z) = 0\}$. Since $\{z \in \mathbb{C} : q(z) = 0\}$ is a finite set, $\sigma(T_1) \cap \sigma(T_3)$ is also finite, which implies that $\sigma(T_1) \cap \sigma(T_3)$ has no interior point. By using [11], we get $\sigma(T) = \sigma(T_1) \cup \sigma(T_3)$, which completes the proof. $\square$

THEOREM 3.13. Every analytic extension of a class A operator is isoloid.

Proof. Suppose that $T \in \mathscr{L}(\mathscr{H} \oplus \mathscr{K})$ is an analytic extension of a class A operator. Then we get by Lemma 3.12 that $\sigma(T) = \sigma(T_1) \cup \sigma(T_3)$ and $\sigma(T_3)$ is a finite set. Let $\lambda \in \mathbb{C}$ be an isolated point of $\sigma(T)$. Then either λ is an isolated point of $\sigma(T_1)$ or $\lambda \in \sigma(T_3)$. If λ is an isolated point of $\sigma(T_1)$, then $\lambda \in \sigma_p(T_1) \subseteq \sigma_p(T)$ because every class A operator is isoloid by [13]. Thus we may assume that $\lambda \in \sigma_p(T_3)$ and $\lambda \notin \sigma(T_1)$. Since $\lambda \in \sigma_p(T_3)$, we get $\ker(T_3 - \lambda) \neq \{0\}$. In addition it holds for any $x \in \ker(T_3 - \lambda)$ that $(T - \lambda)(-(T_1 - \lambda)^{-1}T_2 x \oplus x) = 0$. Hence $\lambda \in \sigma_p(T)$. $\square$

COROLLARY 3.14. Let $T \in \mathscr{L}(\mathscr{H} \oplus \mathscr{K})$ be an analytic extension of a class A operator. If T is quasinilpotent, then it is nilpotent.

Proof. Since $\sigma(T) = \{0\}$, Lemma 3.12 implies that $\sigma(T_1) = \{0\}$ and T_3 is nilpotent. Since T_1 is a class A operator, it is normaloid by [10]. Hence we get $\|T_1\| = r(T_1) = 0$. Therefore, T is nilpotent. $\square$

PROPOSITION 3.15. Suppose that $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \in \mathscr{L}(\mathscr{H} \oplus \mathscr{K})$ is an analytic extension of a class A operator, i.e., T_1 is a class A operator and $F(T_3) = 0$ for some nonconstant analytic function F on a neighborhood D of $\sigma(T_3)$ with the representation $F(z) = G(z)p(z)$ where G is analytic and does not vanish on D and $p(z) = (z-z_1)(z-z_2)\cdots(z-z_k)$ is a polynomial. Then
(i) $H_T(E) \supset H_{T_1}(E) \oplus \{0\}$ for every subset E of $\mathbb{C}$, and
(ii) if E is a closed subset of $\mathbb{C}$ with $z_i \notin E$ for some $i = 1, 2, \cdots, k$ and $\{T_j\}_{j=1}^{3}$ are mutually commuting, then

$$H_T(E) \subseteq \{x_1 \oplus x_2 \in \mathscr{H} \oplus \mathscr{K} : p_i(T_3)x_1 \in H_{T_1}(E) \text{ and } x_2 \in \ker(p_i(T_3))\}$$

where $p_i(z) = (z-z_1)\cdots(z-z_{i-1})(z-z_{i+1})\cdots(z-z_k)$.

Proof. (i) Let E be any subset of $\mathbb{C}$ and let $x_1 \in H_{T_1}(E)$ be given. Since T has the single-valued extension property by Corollary 3.9, there exists an $\mathscr{H}$-valued analytic function f_1 on $\mathbb{C} \setminus E$ for which $(T_1 - z)f_1(z) \equiv x_1$ on $\mathbb{C} \setminus E$. Hence $(T - z)(f_1(z) \oplus 0) \equiv x_1 \oplus 0$ on $\mathbb{C} \setminus E$, and so $x_1 \oplus 0 \in H_T(E)$.

(ii) We may assume that E is any closed subset of $\mathbb{C}$ with $z_1 \notin E$, and let $x_1 \oplus x_2 \in H_T(E)$ be given. Since T has the single-valued extension property by Corollary 3.9, we can choose an $\mathscr{H} \oplus \mathscr{K}$-valued analytic function $f(z) = f_1(z) \oplus f_2(z)$ defined on $\mathbb{C} \setminus E$ such that $(T - z)f(z) = x_1 \oplus x_2$ for all $z \in \mathbb{C} \setminus E$. Then we have

$$\begin{cases} (T_1 - z)f_1(z) + T_2 f_2(z) = x_1 \\ (T_3 - z)f_2(z) = x_2 \end{cases} \tag{11}$$

for all $z \in \mathbb{C} \setminus E$. Since $p(T_3) = (T_3 - z_1)p_1(T_3) = 0$, it follows from (11) that

$$(z - z_1)p_1(T_3)f_2(z) + p_1(T_3)x_2 \equiv 0 \text{ on } \mathbb{C} \setminus E. \tag{12}$$

By taking $z = z_1$ in (12), we obtain that $p_1(T_3)x_2 = 0$, which means $x_2 \in \ker(p_1(T_3))$. Moreover, $(T_1 - z)p_1(T_3)f_1(z) \equiv p_1(T_3)x_1$ on $\mathbb{C} \setminus E$ from (11), which implies $p_1(T_3)x_1 \in H_{T_1}(E)$. $\square$

In the following proposition, we will consider the Putnam's type inequality corresponding to the analytic extension of a class A operator. Note that the Putnam's inequality holds for class A operators;

$$\| |T^2| - |T|^2 \| \leqslant \frac{1}{\pi}\mu(\sigma(T))$$

where μ denotes the planar Lebesgue measure (see [23]).

PROPOSITION 3.16. Suppose that $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \in \mathscr{L}(\mathscr{H} \oplus \mathscr{K})$ is an analytic extension of a class A operator, i.e., T_1 is a class A operator and $F(T_3) = 0$ for some nonconstant analytic function F on a neighborhood D of $\sigma(T)$ with the representation

$F(z) = G(z)p(z)$ where G is analytic and does not vanish on D and $p(z)$ is a polynomial.
(i) If T is compact, then both $p(T)$ and $F(T)$ are expressed as the sum of a normal operator and a nilpotent operator of order 2.
(ii) The following inequality holds;

$$||P(|T^2| - |T|^2)P|| \leqslant \frac{1}{\pi}\mu(\sigma(T))$$

where P is the orthogonal projection of $\mathscr{H} \oplus \mathscr{K}$ onto $\mathscr{H} \oplus \{0\}$. Moreover, if $\sigma(T)$ is a Lebesgue null set, then T_1 is normal.

Proof. (i) We have $F(T) = \begin{pmatrix} F(T_1) & S \\ 0 & 0 \end{pmatrix}$ for some operator $S : \mathscr{K} \to \mathscr{H}$. Since T is compact and T_1 is the restriction of T to the invariant subspace $\mathscr{H} \oplus \{0\}$, T_1 is also compact. Thus T_1 is normal by [14], and so is $F(T_1)$. Since $F(T) - F(T_1) \oplus 0$ is a nilpotent operator of order 2, we complete the proof for $F(T)$, and the proof for $p(T)$ is analogous.

(ii) Since $PTP = TP$, we get that $|T_1^2| = (P|T^2|^2P)^{\frac{1}{2}} \geqslant P|T^2|P$ by Hansen's inequality (see [10]). Since $|T_1|^2 = (TP)^*(TP) = P|T|^2P$, we have $|T_1^2| - |T_1|^2 \geqslant P(|T^2| - |T|^2)P$. Since $\sigma(T) = \sigma(T_1) \cup \sigma(T_3)$ and $\sigma(T_3)$ is a finite set by Lemma 3.12, it follows from [23] that

$$||P(|T^2| - |T|^2)P|| \leqslant |||T_1^2| - |T_1|^2|| \leqslant \frac{1}{\pi}\mu(\sigma(T_1)) = \frac{1}{\pi}\mu(\sigma(T)).$$

Moreover, if $\mu(\sigma(T)) = 0$, then $\mu(\sigma(T_1)) = 0$, and hence T_1 is normal from [28]. □

COROLLARY 3.17. Under the same hypotheses as in Proposition 3.16, let $\sigma(T)$ be a Lebesgue null set. If T_1 has dense range, then T is the direct sum of a normal operator and an analytic operator.

Proof. Since T_1 is normal by Proposition 3.16, it suffices to show that $T_2 = 0$. Since $\sigma(T)$ is a Lebesgue null set, we know that $P(|T^2| - |T|^2)P = 0$ and $|T_1^2| = |T_1|^2$ from Proposition 3.16. From easy computations, we get that

$$|T^2|^2 = \begin{pmatrix} |T_1^2|^2 & * \\ * & * \end{pmatrix} \quad \text{and} \quad |T|^4 = \begin{pmatrix} |T_1|^4 + T_1^*T_2T_2^*T_1 & * \\ * & * \end{pmatrix}.$$

Hence $|T_1^2|^2 = |T_1|^4 + T_1^*T_2T_2^*T_1$. Since $|T_1^2| = |T_1|^2$, $T_1^*T_2T_2^*T_1 = 0$. Since T_1 has dense range, $T_2 = 0$. Thus $T = T_1 \oplus T_3$. □

Next we show that the spectral mapping theorem for the Weyl spectrum and Weyl's theorem hold for an analytic extension T of a class A operator, more generally for $f(T)$ where f is any analytic function on some neighborhood of $\sigma(T)$.

THEOREM 3.18. If $T \in \mathscr{L}(\mathscr{H} \oplus \mathscr{K})$ is an analytic extension of a class A operator, then
(i) it satisfies Weyl's theorem, and
(ii) $f(\sigma_w(T)) = \sigma_w(f(T))$ for any analytic function f on some neighborhood of $\sigma(T)$.

Proof. Suppose that $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \in \mathscr{L}(\mathscr{H} \oplus \mathscr{K})$ is an analytic extension of a class A operator, i.e., T_1 is a class A operator and $F(T_3) = 0$ for some nonconstant analytic function F on a neighborhood D of $\sigma(T_3)$.

(i) Note that every class A operator is isoloid and satisfies Weyl's theorem by [5]. Furthermore, since every analytic operator is algebraic as noted in section one or [5], T_3 is isoloid and it satisfies Weyl's theorem by [22]. Since $\sigma_w(T_1) \cap \sigma_w(T_3)$ has no interior points by Lemma 3.12, Weyl's theorem holds for $T_1 \oplus T_3$ from [20]. If $\lambda_0 \notin \sigma_{le}(T_3) \cap \sigma_{re}(T_3)$ and $\lambda_0 \in \sigma_e(T_3)$, then $T_3 - \lambda_0$ is semi-Fredholm and $\lambda_0 \in \sigma(T_3)$. Since T_3 is algebraic, λ_0 is an isolated point of $\sigma(T_3)$. By [7], $T_3 - \lambda_0$ is Fredholm and $\operatorname{ind}(T_3 - \lambda_0) = 0$, which is a contradiction. Thus we have $\sigma_e(T_3) = \sigma_{le}(T_3) \cap \sigma_{re}(T_3)$, which induces $\sigma_e(T_3) = \sigma_{le}(T_3) = \sigma_{re}(T_3)$. Therefore $SP(T_3)$ has no pseudoholes, and so we finally get that Weyl's theorem holds for T by [19].

(ii) If f is analytic on some neighborhood of $\sigma(T)$, then $\sigma_w(f(T_1)) = f(\sigma_w(T_1))$ by [5]. Moreover since T_3 is algebraic, we know that $\sigma_w(f(T_3)) = f(\sigma_w(T_3))$ and $\sigma_w(T_1) \cap \sigma_w(T_3)$ is finite and so has no interior points. Since $\sigma_w(T_1) \cap \sigma_w(T_3)$ is finite, $\sigma_w(f(T_1)) \cap \sigma_w(f(T_3)) = f(\sigma_w(T_1)) \cap f(\sigma_w(T_3))$ also has no interior points. Hence, we obtain from [20] that

$$\begin{aligned}\sigma_w(f(T)) &= \sigma_w(f(T_1)) \cup \sigma_w(f(T_3)) = f(\sigma_w(T_1)) \cup f(\sigma_w(T_3)) \\ &= f(\sigma_w(T_1) \cup \sigma_w(T_3)) = f(\sigma_w(T)).\end{aligned}$$

Thus we complete our proof. □

COROLLARY 3.19. Let $T \in \mathscr{L}(\mathscr{H} \oplus \mathscr{K})$ be an analytic extension of a class A operator. Then Weyl's theorem holds for $f(T)$ where f is any analytic function on some neighborhood of $\sigma(T)$.

Proof. If T is an analytic extension of a class A operator, then T is isoloid by Theorem 3.13. Let f be an analytic function on some neighborhood of $\sigma(T)$. Then it follows from [21] that

$$\sigma(f(T)) \setminus \pi_{00}(f(T)) = f(\sigma(T) \setminus \pi_{00}(T)).$$

Since Weyl's theorem holds for T and $f(\sigma_w(T)) = \sigma_w(f(T))$ by Theorem 3.18,

$$\sigma(f(T)) \setminus \pi_{00}(f(T)) = f(\sigma(T) \setminus \pi_{00}(T)) = f(\sigma_w(T)) = \sigma_w(f(T)).$$

Accordingly, Weyl's theorem holds for $f(T)$. □

REFERENCES

[1] A. ALUTHGE, *On p-hyponormal operators for $0 < p < 1$*, Int. Eq. Op. Th. **13** (1990), 307–315.

[2] A. ALUTHGE AND D. WANG, *w-Hyponormal operators*, Int. Eq. Op. Th. **36** (2000), 1–10.

[3] A. ALUTHGE AND D. WANG, *w-Hyponormal operators II*, Int. Eq. Op. Th. **37** (2000), 324–331.

[4] S. BROWN, *Hyponormal operators with thick spectrum have invariant subspaces*, Ann. of Math. **125** (1987), 93–103.

[5] X. CAO, *Analytically class A operators and Weyl's theorem*, J. Math. Anal. Appl. **320** (2006), 795–803.

[6] I. COLOJOARĂ AND C. FOIAŞ, *Theory of generalized spectral operators*, Gordon and Breach, New York, 1968.

[7] J. B. CONWAY, *A course in functional analysis*, Springer-Verlag New York, 1990.

[8] J. ESCHMEIER, *Invariant subspaces for subscalar operators*, Arch. Math. **52** (1989), 562–570.

[9] C. FOIAS AND A. E. FRAZHO, *The commutant lifting approach to interpolation problem*, Operator Theory Adv. Appl., vol. 44, Birkh$\ddot{a}$user, Boston, 1990.

[10] T. FURUTA, *Invitation to linear operators*, Taylor and Francis, 2001.

[11] J. K. HAN, H. Y. LEE, AND W. Y. LEE, *Invertible completions of 2×2 upper triangular operator matrices*, Proc. Amer. Math. Soc. **128** (1999), 119–123.

[12] M. ITO AND T. YAMAZAKI, *Relations between two inequalities $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{p}{p+r}} \geqslant B^r$ and $A^p \geqslant (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}}$ and their applications*, Int. Eq. Op. Th. **44** (2002), 442–450.

[13] I. H. JEON AND B. P. DUGGAL, *On operators with an absolute value conditions*, J. Korean Math. Soc. **41** (2004), 617–627.

[14] S. JUNG, E. KO, AND M. LEE, *On class A operators*, Studia Math. **198** (2010), 249–260.

[15] I. B. JUNG, E. KO, AND C. PEARCY, *Aluthge transforms of operators*, Int. Eq. Op. Th. **37** (2000), 449–456.

[16] I. H. KIM, *On (p,k)-quasihyponormal operators*, Math. Ineq. Appl. **7** (2004), 629–638.

[17] E. KO, *Kth roots of p-hyponormal operators are subscalar operators of order $4k$*, Int. Eq. Op. Th. **59** (2007), 173–187.

[18] K. LAURSEN AND M. NEUMANN, *An introduction to local spectral theory*, Clarendon Press, Oxford, 2000.

[19] W. Y. LEE, *Weyl's theorem for operator matrices*, Int. Eq. Op. Th. **32** (1998), 319–331.

[20] W. Y. LEE, *Weyl spectra of operator matrices*, Proc. Amer. Math. Soc. **129** (2000), 131–138.

[21] W.Y. LEE AND S. H. LEE, *A spectral mapping theorem for the Weyl spectrum*, Glasgow Math. J. **38** (1996), 61–64.

[22] M. OUDGHIRI, *Weyl's theorem and perturbations*, Int. Eq. Op. Th. **53** (2005), 535–545.

[23] S. M. PATEL, M. CHO, K. TANAHASHI, AND A. UCHIYAMA, *Putnam's inequality for class A operators and an operator transform by Cho and Yamazaki*, Scientiae Mathematicae Japonicae **67** (2008), 393–402.

[24] C. M. PEARCY, *Some recent developments in operator theory*, Conference board of the mathematical sciences regional conference series in mathematics, **36** (1975).

[25] M. PUTINAR, *Hyponormal operators are subscalar operators*, J. Op. Th. **12** (1984), 385–395.

[26] M. PUTINAR, *Quasisimilarity of tuples with Bishop's property (β)*, Int. Eq. Op. Th. **15** (1992), 1047–1052.

[27] K. TANAHASHI, A. UCHIYAMA, AND M. CHO, *Isolated points of spectrum of (p,k)-quasihyponormal operators*, Linear Alge. Appl. **382** (2004), 221–229.

[28] D. WANG AND J. I. LEE, *Spectral properties of class A operators*, Trends in Math, Information Center for Math. Science, **6** (2003), 93–98.

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