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WEYL THEORY AND EXPLICIT SOLUTIONS OF DIRECT AND INVERSE PROBLEMS FOR DIRAC SYSTEM WITH A RECTANGULAR MATRIX POTENTIAL

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Abstract. A non-classical Weyl theory is developed for Dirac systems with rectangular matrix potentials. The notion of the Weyl function is introduced and the corresponding direct problem is solved. Furthermore, explicit solutions of the direct and inverse problems are obtained for the case of rational Weyl matrix functions.

1. Introduction

Consider self-adjoint Dirac-type (also called Dirac, ZS or AKNS) system, which is a classical matrix differential equation:

$$\frac{d}{dx}y(x,z) = \mathrm{i}(zj + jV(x))y(x,z) \quad (x \geqslant 0), \tag{1.1}$$

$$j = \begin{bmatrix} I_{m_1} & 0 \\ 0 & -I_{m_2} \end{bmatrix}, \quad V = \begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix}, \tag{1.2}$$

where I_{m_k} is the $m_k \times m_k$ identity matrix and $v(x)$ is an $m_1 \times m_2$ matrix function, which is called the potential of system. Dirac-type systems are very well-known in mathematics and applications (see, for instance, books [6, 7, 30, 32, 47], recent publications [3, 4, 5, 9, 10, 17, 18, 48], and numerous references therein). The name ZS-AKNS is caused by the fact that system (1.1) is an auxiliary linear system for many important nonlinear integrable wave equations and as such it was studied, for instance, in [1, 2, 13, 21, 42, 52]. For the case that $m_1 \neq m_2$ systems of the form (1.1), (1.2) are, in particular, auxiliary linear systems for the coupled, multicomponent, and $m_1 \times m_2$ matrix nonlinear Schrödinger equations.

The Weyl and spectral theory of self-adjoint Dirac systems, where $m_1 = m_2$, was dealt with, for instance, in [5, 9, 24, 28, 32, 41, 47] (see also various references therein). The "non-classical" Weyl theory for the equally important case $m_1 \neq m_2$ and related questions are the subject of this paper.

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In Section 2 we treat the direct problem for the general-type Dirac system, that is, system (1.1), where the potential v is locally summable. A definition of the non-expansive generalized Weyl function is given, its existence and uniqueness are proved, and some basic properties are studied.

In Section 3 we consider Dirac systems with the so called *generalized pseudo-exponential* potentials (see Definition 3.1). Direct and inverse problems for such systems are solved there explicitly. For that purpose we follow the scheme from [16, 24, 39] and apply some classical results from system theory [26] and Riccati equations [31].

As usual, $\mathbb{N}$ stands for the set of natural numbers, $\mathbb{R}$ stands for the real axis, $\mathbb{C}$ stands for the complex plain, and $\mathbb{C}_+$ for the open upper semi-plane. If $a \in \mathbb{C}$, then $\overline{a}$ is its complex conjugate. The notation Im is used for image. An $m_2 \times m_1$ matrix α is said to be non-expansive, if $\alpha^*\alpha \leqslant I_{m_1}$ (or, equivalently, if $\alpha\alpha^* \leqslant I_{m_2}$).

We put $m_1 + m_2 =: m$. The fundamental solution of system (1.1) is denoted by $u(x,z)$, and this solution is normalized by the condition

$$u(0,z) = I_m. \tag{1.3}$$

2. Direct problem

We consider Dirac system (1.1) on the semi-axis $x \in [0,\infty)$ and assume that v is measurable and locally summable, that is, summable on all the finite intervals. In a way, which is similar, for instance, to the non-classical problem treated in [40] we shall use Möbius transformations and matrix balls to solve the direct problem for Dirac system.

Introduce a class of nonsingular $m \times m_1$ matrix functions $\mathscr{P}(z)$ with property-j, which are an immediate analog of the classical pairs of parameter matrix functions. Namely, the matrix functions $\mathscr{P}(z)$ are meromorphic in $\mathbb{C}_+$ and satisfy (excluding, possibly, a discrete set of points) the following relations

$$\mathscr{P}(z)^*\mathscr{P}(z) > 0, \quad \mathscr{P}(z)^* j \mathscr{P}(z) \geqslant 0 \quad (z \in \mathbb{C}_+). \tag{2.1}$$

DEFINITION 2.1. The set $\mathscr{N}(x,z)$ of Möbius transformations is the set of values at x, z of matrix functions

$$\varphi(x,z,\mathscr{P}) = \begin{bmatrix} 0 & I_{m_2} \end{bmatrix} u(x,z)^{-1}\mathscr{P}(z)\Big(\begin{bmatrix} I_{m_1} & 0 \end{bmatrix} u(x,z)^{-1}\mathscr{P}(z)\Big)^{-1}, \tag{2.2}$$

where $\mathscr{P}(z)$ are nonsingular matrix functions with property-j.

PROPOSITION 2.2. *Let Dirac system (1.1) on $[0,\infty)$ be given and assume that v is locally summable. Then the sets $\mathscr{N}(x,z)$ are well-defined. There is a unique matrix function $\varphi(z)$ in $\mathbb{C}_+$ such that*

$$\varphi(z) = \bigcap_{x<\infty} \mathscr{N}(x,z). \tag{2.3}$$

This function is analytic and non-expansive.

Proof. It is immediate from (1.1) that

$$\frac{d}{dx}\big(u(x,z)^*ju(x,z)\big) = \mathrm{i}(z-\overline{z})u(x,z)^*u(x,z) < 0, \quad z \in \mathbb{C}_+. \tag{2.4}$$

According to (1.3) and (2.4) we have

$$\mathfrak{A}(x,z) = \{\mathfrak{A}_{ij}(x,z)\}_{i,j=1}^2 := u(x,z)^*ju(x,z) \leqslant j, \quad z \in \mathbb{C}_+, \tag{2.5}$$

where $\mathfrak{A}$ is partitioned into four blocks so that $\mathfrak{A}_{ii}$ is an $m_i \times m_i$ matrix function ($i = 1,2$). Inequality (2.5) yields

$$\big(u(x,z)^*\big)^{-1} ju(x,z)^{-1} \geqslant j. \tag{2.6}$$

Thus, we get

$$\det\Big(\big[I_{m_1} \;\; 0\big]\, u(x,z)^{-1}\mathscr{P}(z)\Big) \neq 0, \tag{2.7}$$

and so $\mathscr{N}$ is well-defined via (2.2). Indeed, if (2.7) does not hold, there is a vector $f \in \mathbb{C}^{m_1}$ such that

$$\big[I_{m_1} \;\; 0\big]\, ju(x,z)^{-1}\mathscr{P}(z)f = \big[I_{m_1} \;\; 0\big]\, u(x,z)^{-1}\mathscr{P}(z)f = 0, \quad f \neq 0. \tag{2.8}$$

By (2.1) and (2.6) the subspace $\operatorname{Im}\big(u(x,z)^{-1}\mathscr{P}(z)\big)$ is a maximal j-nonnegative subspace. Clearly $\operatorname{Im}\big(\big[I_{m_1} \;\; 0\big]^*\big)$ is a maximal j-nonnegative subspace too. Therefore (2.8) implies $u(x,z)^{-1}\mathscr{P}(z)f \in \operatorname{Im}\big(\big[I_{m_1} \;\; 0\big]^*\big)$. But then it follows from the second equality in (2.8) that $f = 0$, which contradicts the inequality in (2.8).

Next, rewrite (2.2) in the equivalent form

$$\begin{bmatrix} I_{m_1} \\ \varphi(x,z,\mathscr{P}) \end{bmatrix} = u(x,z)^{-1}\mathscr{P}(z)\Big(\big[I_{m_1} \;\; 0\big]\, u(x,z)^{-1}\mathscr{P}(z)\Big)^{-1}. \tag{2.9}$$

In view of (2.1), (2.9), and of the definition of $\mathfrak{A}$ in (2.5), formula

$$\widehat{\varphi}(z) \in \mathscr{N}(x,z) \tag{2.10}$$

is equivalent to

$$\big[I_{m_1} \;\; \widehat{\varphi}(z)^*\big]\, \mathfrak{A}(x,z) \begin{bmatrix} I_{m_1} \\ \widehat{\varphi}(z) \end{bmatrix} \geqslant 0. \tag{2.11}$$

In a standard way, using formula (2.4) and the equivalence of (2.10) and (2.11), we get

$$\mathscr{N}(x_1,z) \subset \mathscr{N}(x_2,z) \quad \text{for} \quad x_1 > x_2. \tag{2.12}$$

Moreover, (2.11) at $x=0$ means that

$$\mathscr{N}(0,z)=\{\widehat{\varphi}(z):\, \widehat{\varphi}(z)^*\widehat{\varphi}(z)\leqslant I_{m_1}\}. \tag{2.13}$$

By Montel's theorem, formulas (2.12) and (2.13) imply the existence of an analytic and non-expansive matrix function $\varphi(z)$ such that

$$\varphi(z)\in\bigcap_{x<\infty}\mathscr{N}(x,z). \tag{2.14}$$

Indeed, because of (2.12) and (2.13) we see that the set of functions $\varphi(x,z,\mathscr{P})$ of the form (2.2) is uniformly bounded in $\mathbb{C}_+$. So, Montel's theorem is applicable and there is an analytic matrix function, which we denote by $\varphi_\infty(z)$ and which is a uniform limit of some sequence

$$\varphi_\infty(z)=\lim_{i\to\infty}\varphi(x_i,z,\mathscr{P}_i) \quad (i\in\mathbb{N}, \quad x_i\uparrow, \quad \lim_{i\to\infty}x_i=\infty) \tag{2.15}$$

on all the bounded and closed subsets of $\mathbb{C}_+$. Since $x_i\uparrow$ and equalities (2.9) and (2.12) hold, it follows that the matrix functions

$$\mathscr{P}_{ij}(z):=u(x_i,z)\begin{bmatrix} I_{m_1}\\ \varphi(x_j,z,\mathscr{P}_j)\end{bmatrix} \quad (j\geqslant i)$$

satisfy relations (2.1). Therefore, using (2.15) we derive that (2.1) holds for

$$\mathscr{P}_{i,\infty}(z):=u(x_i,z)\begin{bmatrix} I_{m_1}\\ \varphi_\infty(z)\end{bmatrix},$$

which implies that we can substitute $\mathscr{P}=\mathscr{P}_{i,\infty}$ and $x=x_i$ into (2.9) to get

$$\varphi_\infty(z)\in\mathscr{N}(x_i,z). \tag{2.16}$$

Since (2.16) holds for all $i\in\mathbb{N}$, we see that (2.14) is true for $\varphi(z)=\varphi_\infty(z)$.

Now, let us show that $\mathscr{N}$ is a matrix ball. It follows from (2.4) and (2.5) that

$$\frac{d}{dx}\mathfrak{A}\leqslant \mathrm{i}(\overline{z}-z)\mathfrak{A}\leqslant \mathrm{i}(\overline{z}-z)j, \quad \mathfrak{A}(0,z)=j.$$

Taking into account the relations above, we derive

$$-\mathfrak{A}_{22}(x,z)\geqslant\big(1+\mathrm{i}(\overline{z}-z)x\big)I_{m_2}. \tag{2.17}$$

Note also that (2.5) implies $\mathfrak{A}(x,z)^{-1}\geqslant j$ for $z\in\mathbb{C}_+$ (see [36]). Thus, we get

$$\big(\mathfrak{A}^{-1}\big)_{11}=\big(\mathfrak{A}_{11}-\mathfrak{A}_{12}\mathfrak{A}_{22}^{-1}\mathfrak{A}_{21}\big)^{-1}\geqslant I_{m_1}. \tag{2.18}$$

Since $-\mathfrak{A}_{22} > 0$, the square root $\Upsilon = \big(-\mathfrak{A}_{22}\big)^{1/2}$ is well-defined and we rewrite (2.11) in the form

$$\mathfrak{A}_{11} - \mathfrak{A}_{12}\mathfrak{A}_{22}^{-1}\mathfrak{A}_{21} - \big(\widehat{\varphi}^*\Upsilon - \mathfrak{A}_{12}\Upsilon^{-1}\big)\big(\Upsilon\widehat{\varphi} - \Upsilon^{-1}\mathfrak{A}_{21}\big) \geqslant 0,$$

where $\mathfrak{A}_{12} = \mathfrak{A}_{21}^*$. Equivalently, we have

$$\widehat{\varphi} = \rho_l \omega \rho_r - \mathfrak{A}_{22}^{-1}\mathfrak{A}_{21}, \quad \omega^*\omega \leqslant I_{m_2}, \tag{2.19}$$

$$\rho_l := \Upsilon^{-1} = \big(-\mathfrak{A}_{22}\big)^{-1/2}, \quad \rho_r := \big(\mathfrak{A}_{11} - \mathfrak{A}_{12}\mathfrak{A}_{22}^{-1}\mathfrak{A}_{21}\big)^{1/2}. \tag{2.20}$$

Here ω is an $m_2 \times m_1$ matrix function. Since (2.10) is equivalent to (2.19), the sets $\mathcal{N}(x,z)$ (where the values of x and z are fixed) are matrix balls, indeed. According to (2.17), (2.18), and (2.20) the next formula holds:

$$\rho_l(x,z) \to 0 \quad (x \to \infty), \quad \rho_r(x,z) \leqslant I_{m_1}. \tag{2.21}$$

Finally, relations (2.14), (2.19), and (2.21) imply (2.3). □

In view of Proposition 2.2 we define the Weyl function of Dirac system similar to the canonical system case [47].

DEFINITION 2.3. The Weyl-Titchmarsh (or simply Weyl) function of Dirac system (1.1) on $[0,\infty)$, where potential v is locally summable, is the function φ given by (2.3).

From Proposition 2.2 we see that the Weyl-Titchmarsh function always exists. Clearly, it is unique.

COROLLARY 2.4. *Let the conditions of Proposition 2.2 hold. Then the Weyl function is the unique function, which satisfies the inequality*

$$\int_0^\infty \begin{bmatrix} I_{m_1} & \varphi(z)^* \end{bmatrix} u(x,z)^* u(x,z) \begin{bmatrix} I_{m_1} \\ \varphi(z) \end{bmatrix} dx < \infty. \tag{2.22}$$

Proof. According to the equalities in (2.4) and (2.5) and to the inequality (2.11) we derive

$$\begin{aligned} &\int_0^r \begin{bmatrix} I_{m_1} & \varphi(z)^* \end{bmatrix} u(x,z)^* u(x,z) \begin{bmatrix} I_{m_1} \\ \varphi(z) \end{bmatrix} dx \\ &= \frac{\mathrm{i}}{z-\overline{z}} \begin{bmatrix} I_{m_1} & \varphi(z)^* \end{bmatrix} \big(\mathfrak{A}(0,z) - \mathfrak{A}(r,z)\big) \begin{bmatrix} I_{m_1} \\ \varphi(z) \end{bmatrix} \\ &\leqslant \frac{\mathrm{i}}{z-\overline{z}} \begin{bmatrix} I_{m_1} & \varphi(z)^* \end{bmatrix} \mathfrak{A}(0,z) \begin{bmatrix} I_{m_1} \\ \varphi(z) \end{bmatrix}. \end{aligned} \tag{2.23}$$

Inequality (2.22) is immediate from (2.23). Moreover, as $u^*u \geqslant -\mathfrak{A}$, the inequality (2.17) yields

$$\int_0^r \begin{bmatrix} 0 & I_{m_2} \end{bmatrix} u(x,z)^* u(x,z) \begin{bmatrix} 0 \\ I_{m_2} \end{bmatrix} dx \geqslant r I_{m_2}. \tag{2.24}$$

In view of (2.24), the function satisfying (2.22) is unique. □

REMARK 2.5. From Corollary 2.4, we see that inequality (2.22) can be used as an equivalent definition of the Weyl function. Definition of the form (2.22) is a more classical one and deals with solutions of (1.1) which belong to $L^2(0,\infty)$. Compare (2.22) with definitions of the Weyl-Titchmarsh or M-functions for discrete and continuous systems in [10, 32, 33, 37, 38, 47, 49, 50] (see also references therein).

Our last proposition in this section is dedicated to a property of the Weyl function, the analog of which may be used as a definition of generalized Weyl functions in more complicated non-self-adjoint cases (see, e.g., [17, 38, 40]).

PROPOSITION 2.6. *Let Dirac system* (1.1) *on* $[0,\infty)$ *be given, and assume that* v *is locally summable. Then, the following inequality*

$$\sup_{x \leqslant l,\, z \in \mathbb{C}_+} \left\| \mathrm{e}^{-\mathrm{i}xz} u(x,z) \begin{bmatrix} I_{m_1} \\ \varphi(z) \end{bmatrix} \right\| < \infty \tag{2.25}$$

holds on any finite interval $[0,l]$ *for the Weyl function* φ *of this system.*

Proof. We fix some l. Now, choose x such that $0 < x \leqslant l < \infty$. Because of (2.3), the Weyl function φ admits representations (2.2) (i.e., $\varphi(z) = \varphi(x,z,\mathscr{P})$). Hence, we can use (2.1) and (2.9) to get

$$\Psi(x,z)^* j \Psi(x,z) \geqslant 0, \quad \Psi(x,z) := \mathrm{e}^{-\mathrm{i}xz} u(x,z) \begin{bmatrix} I_{m_1} \\ \varphi(z) \end{bmatrix}. \tag{2.26}$$

On the other hand, equation (1.1) and definition of Ψ in (2.26) imply that

$$\begin{aligned} &\frac{d}{dx}\Big(\mathrm{e}^{-2xM}\Psi(x,z)^*(I_m + j)\Psi(x,z)\Big) \\ &= \mathrm{e}^{-2xM}\Psi(x,z)^*\Big(\mathrm{i}\big((I_m+j)jV - Vj(I_m+j)\big) - 2M(I_m+j)\Big)\Psi(x,z) \\ &= 2\mathrm{e}^{-2xM}\Psi(x,z)^* \begin{bmatrix} -2MI_{m_1} & \mathrm{i}v(x) \\ -\mathrm{i}v(x)^* & 0 \end{bmatrix} \Psi(x,z), \quad M := \sup_{x<l} \|V(x)\|. \end{aligned} \tag{2.27}$$

Using (2.26) and (2.27) we derive

$$\begin{aligned} &\frac{d}{dx}\Big(\mathrm{e}^{-2xM}\Psi(x,z)^*(I_m + j)\Psi(x,z)\Big) \\ &\leqslant 2\mathrm{e}^{-2xM}\Psi(x,z)^*\left(\begin{bmatrix} 0 & \mathrm{i}v(x) \\ -\mathrm{i}v(x)^* & 0 \end{bmatrix} - MI_m\right)\Psi(x,z) \leqslant 0. \end{aligned} \tag{2.28}$$

Finally, inequalities (2.26) and (2.28) lead us to

$$\Psi(x,z)^*\Psi(x,z) \leqslant \Psi(x,z)^*(I_m + j)\Psi(x,z) \leqslant 2\mathrm{e}^{2xM} I_{m_1}, \tag{2.29}$$

and (2.25) follows. □

3. Direct and inverse problems: explicit solutions

Various versions of Bäcklund-Darboux transformations are actively used to construct explicit solutions of linear and integrable nonlinear equations (see, e.g., [8, 20, 25, 34, 35, 43, 51] and numerous references therein). For the spectral and scattering results that follow from Bäcklund-Darboux transformations and related commutation and factorization methods see, for instance, publications [11, 12, 16, 19, 22, 24, 27, 29, 42]. Here we will give explicit solutions of our direct and inverse problems using the GBDT version of the Bäcklund-Darboux transformation (see [14, 16, 24, 39, 42, 43] and references therein).

To obtain explicit solutions, we consider $m_1 \times m_2$ potentials v of the form

$$v(x) = -2\mathrm{i}\vartheta_1^* \mathrm{e}^{\mathrm{i}x\alpha^*}\Sigma(x)^{-1}\mathrm{e}^{\mathrm{i}x\alpha}\vartheta_2, \tag{3.1}$$

where some $n \in \mathbb{N}$ is fixed and the $n \times n$ matrix function Σ is given by the formula

$$\Sigma(x) = \Sigma_0 + \int_0^x \Lambda(t)\Lambda(t)^* dt \quad (\Sigma_0 > 0), \quad \Lambda(x) = \begin{bmatrix} \mathrm{e}^{-\mathrm{i}x\alpha}\vartheta_1 & \mathrm{e}^{\mathrm{i}x\alpha}\vartheta_2 \end{bmatrix}. \tag{3.2}$$

Here α, ϑ_1, and ϑ_2 are $n \times n$, $n \times m_1$, and $n \times m_2$ parameter matrices, and the following matrix identity holds:

$$\alpha\Sigma_0 - \Sigma_0\alpha^* = \mathrm{i}(\vartheta_1\vartheta_1^* - \vartheta_2\vartheta_2^*). \tag{3.3}$$

Clearly, $\Sigma(x)$ is invertible for $x \geqslant 0$ and the potential v in (3.1) is well-defined.

DEFINITION 3.1. The $m_1 \times m_2$ potentials v of the form (3.1), where relations (3.2) and (3.3) hold, are called the generalized pseudo-exponential potentials. It is said that v is generated by the parameter matrices α, Σ_0, ϑ_1, and ϑ_2.

According to [39, Theorem 3] (see also [16]), the fundamental solution u of system (1.1), where V is given by (1.2), v is a generalized pseudo-exponential potential, and u is normalized by (1.3), admits representation

$$u(x,z) = w_\alpha(x,z)\mathrm{e}^{\mathrm{i}xzj}w_\alpha(0,z)^{-1}. \tag{3.4}$$

Here we have

$$w_\alpha(x,z) := I_m + \mathrm{i}j\Lambda(x)^*\Sigma(x)^{-1}(zI_n - \alpha)^{-1}\Lambda(x). \tag{3.5}$$

Note that the case $m_1 = m_2$ (i.e., the case of the pseudo-exponential potentials) was treated in greater detail in [24] (see [24] and references therein for the term *pseudo-exponential*, itself, too). Formulas (3.2) and (3.3) yield

$$\alpha\Sigma(x) - \Sigma(x)\alpha^* = \mathrm{i}\Lambda(x)j\Lambda(x)^*. \tag{3.6}$$

Identity (3.6), in turn, implies that $w_\alpha(z)$ is a transfer matrix function in Lev Sakhnovich form [44–47]. However, $w_\alpha(x,z)$ possesses an additional variable x and the way, in which this matrix function depends on x, is essential.

From [16, formula (2.9)], where W_{11} and W_{21} are $m_1 \times m_1$ and $m_2 \times m_1$ blocks of

$$w_\alpha(0,z) =: \{W_{ij}(z)\}_{i,j=1}^2, \tag{3.7}$$

we see that

$$W_{21}(z)W_{11}(z)^{-1} = -\mathrm{i}\vartheta_2^*\Sigma_0^{-1}(zI_n - \theta)^{-1}\vartheta_1, \quad \theta := \alpha - \mathrm{i}\vartheta_1\vartheta_1^*\Sigma_0^{-1}. \tag{3.8}$$

We note that [16, formulas (2.6) and (2.7)] imply that $W_{11}(z)$ is always well-defined and invertible for $z \notin \sigma(\alpha) \cup \sigma(\theta)$, where σ denotes the spectrum.

Relations (3.4), (3.7), and (3.8) are basic to solve the direct problem for Dirac systems with the generalized pseudo-exponential potentials (3.1).

THEOREM 3.2. *Let Dirac system* (1.1) *on* $[0,\infty)$ *be given and assume that* v *is a generalized pseudo-exponential potential, which is generated by the matrices* α, Σ_0, ϑ_1, *and* ϑ_2. *Then the Weyl function* φ *of system* (1.1) *has the form:*

$$\varphi(z) = -\mathrm{i}\vartheta_2^*\Sigma_0^{-1}(zI_n - \theta)^{-1}\vartheta_1, \quad \theta = \alpha - \mathrm{i}\vartheta_1\vartheta_1^*\Sigma_0^{-1}. \tag{3.9}$$

Proof. We compare (3.8) and (3.9) to see that

$$\varphi(z) = W_{21}(z)W_{11}(z)^{-1}. \tag{3.10}$$

Because of (3.4), (3.7), and (3.10) we have

$$u(x,z)\begin{bmatrix} I_{m_1} \\ \varphi(z) \end{bmatrix} = \mathrm{e}^{\mathrm{i}xz} w_\alpha(x,z) \begin{bmatrix} I_{m_1} \\ 0 \end{bmatrix} W_{11}(z)^{-1}. \tag{3.11}$$

To consider the matrix function $\Lambda^*\Sigma^{-1}$, which appears in the definition (3.5) of w_α, we derive from (3.2) that

$$\Sigma(x)^{-1}\Lambda(x)\Lambda(x)^*\Sigma(x)^{-1} = -\frac{d}{dx}\Sigma(x)^{-1}. \tag{3.12}$$

It is immediate also from (3.2) that $\Sigma(x) > 0$. Therefore, using (3.12) we get

$$\int_0^\infty \Sigma(t)^{-1}\Lambda(t)\Lambda(t)^*\Sigma(t)^{-1}dt \leqslant \Sigma_0^{-1}. \tag{3.13}$$

Furthermore, the last equality in (3.2) implies that

$$\sup_{\Im z > \|\alpha\| + \varepsilon} \|\mathrm{e}^{\mathrm{i}xz}\Lambda(x)\| < M_\varepsilon \quad (\varepsilon > 0). \tag{3.14}$$

It follows from (3.5), (3.13), and (3.14) that the entries of the right-hand side of (3.11) are well-defined and uniformly bounded in the $L^2(0,\infty)$ norm with respect to x for all z such that $\Im z \geqslant \max\big(\|\alpha\|, \|\theta\|\big)+\varepsilon$ and $\varepsilon > 0$. Hence, taking into account (3.11) we see that (2.22) holds for z from the mentioned above domain. So, according to the uniqueness statement in Corollary 2.4, $\varphi(z)$ of the form (3.9) coincides with the Weyl function in that domain. Since the Weyl function is analytic in $\mathbb{C}_+$, the matrix function φ coincides with it in $\mathbb{C}_+$ (i.e., φ is the Weyl function, indeed). $\square$

For the case that v is a generalized pseudo-exponential potential, where $\Sigma_0 > 0$, our Weyl function coincides with the reflection coefficient from [16] (see [16, Theorem 3.3]). Hence, the solution of our inverse problem can be considered as a particular case of the solution of the inverse problem from [16, Theorem 4.1], where the singular case $\Sigma_0 \not> 0$ was studied too.

Before we formulate the procedure to solve the inverse problem, some results on rational matrix functions and notions from system and control theories are required (see, e.g., [26, 31]). Let $\varphi(z)$ be a strictly proper rational matrix function, that is, such a rational matrix function that

$$\lim_{z\to\infty} \varphi(z) = 0. \tag{3.15}$$

Then φ admits representations (also called realizations):

$$\varphi(z) = C_N(zI_N - \mathscr{A}_N)^{-1}B_N, \tag{3.16}$$

where C_N, $\mathscr{A}_N$, and B_N are $m_2\times N$, $N\times N$, and $N\times m_1$, respectively, matrices. Here $N\in\mathbb{N}$, and m_1 (m_2) denotes the number of columns (rows) of φ.

DEFINITION 3.3. The minimal possible value of N in realizations (3.16) is called the McMillan degree of φ, and we denote this value by n. Realizations (3.16), where $N=n$, are called the minimal realizations.

From [31, Theorems 21.1.3, 21.2.1] we easily see that for a minimal realization

$$\varphi(z) = C(zI_n - \mathscr{A})^{-1}B \tag{3.17}$$

of a matrix φ, which is non-expansive on $\mathbb{R}$ and has no poles in $\mathbb{C}_+$, there is a positive solution $X > 0$ of the Riccati equation

$$XC^*CX + \mathrm{i}(X\mathscr{A}^* - \mathscr{A}X) + BB^* = 0. \tag{3.18}$$

Furthermore, all the hermitian solutions of (3.18) are positive.

THEOREM 3.4. *Let $\varphi(z)$ be a strictly proper rational matrix function, which is non-expansive on $\mathbb{R}$ and has no poles in $\mathbb{C}_+$. Assume that (3.17) is its minimal realization and that $X > 0$ is a solution of (3.18).*

Then $\varphi(z)$ is the Weyl function of the Dirac system, the potential of which is given by (3.1) and (3.2), where

$$\alpha = \mathscr{A} + \mathrm{i}BB^*X^{-1}, \quad \Sigma_0 = X, \quad \vartheta_1 = B, \quad \vartheta_2 = -\mathrm{i}XC^*. \tag{3.19}$$

This solution of the inverse problem is unique in the class of Dirac systems with the locally bounded potentials.

Proof. From (3.19) we see that

$$\alpha\Sigma_0 - \Sigma_0\alpha^* = \mathscr{A}X - X\mathscr{A}^* + 2\mathrm{i}BB^*, \quad \mathrm{i}(\vartheta_1\vartheta_1^* - \vartheta_2\vartheta_2^*) = \mathrm{i}BB^* - \mathrm{i}XC^*CX,$$

and so (3.3) is equivalent to (3.18). Since (3.3) holds, we apply Theorem 3.2. Theorem 3.2 states that the Weyl function of the Dirac system, where v is given by (3.1), has the form (3.9). Next, we substitute (3.19) into (3.9), to derive that the right-hand sides of (3.17) and the first equality in (3.9) coincide. In other words, the Weyl function of our system admits representation (3.17).

Finally, the uniqueness of the solution of the inverse problem follows from [15, Theorem 4.1]. □

We note that the corresponding uniqueness result in [16] was proved only for the class of systems with the generalized pseudo-exponential potentials.

Because of the second equality in (3.8) and identity (3.3), the matrix θ satisfies another identity: $\theta\Sigma_0 - \Sigma_0\theta^* = -\mathrm{i}(\vartheta_1\vartheta_1^* + \vartheta_2\vartheta_2^*)$, that is,

$$\Sigma_0^{-1}\theta - \theta^*\Sigma_0^{-1} = -\mathrm{i}\Sigma_0^{-1}(\vartheta_1\vartheta_1^* + \vartheta_2\vartheta_2^*)\Sigma_0^{-1}. \tag{3.20}$$

If $f \neq 0$ is an eigenvector of θ (i.e., $\theta f = \lambda f$), identity (3.20) implies that

$$(\lambda - \overline{\lambda})f^*\Sigma_0^{-1}f = -\mathrm{i}f^*\Sigma_0^{-1}(\vartheta_1\vartheta_1^* + \vartheta_2\vartheta_2^*)\Sigma_0^{-1}f. \tag{3.21}$$

Since $\Sigma_0 > 0$, we derive from (3.21) that

$$\sigma(\theta) \subset \mathbb{C}_- \cup \mathbb{R}. \tag{3.22}$$

Real eigenvalues of θ play a special role in the spectral theory of an operator, which corresponds to the Dirac system with a generalized pseudo-exponential potential (see, e.g., [23] for the case of square potentials). In our case the operator $\mathscr{H}$ corresponding to the Dirac system is defined in a way, which is similar to the definition from [23], but the initial condition is quite different. Namely, we determine $\mathscr{H}$ by the differential expression

$$\mathscr{H}_{de}y = -\mathrm{i}j\frac{d}{dx}y - Vy, \tag{3.23}$$

and by its domain $\mathscr{D}(\mathscr{H})$, which consists of all locally absolutely continuous $\mathbb{C}^m$-valued functions y in $L_m^2(0,\infty)$, such that

$$\mathscr{H}_{de}y \in L_m^2(0,\infty), \qquad y(0) = 0. \tag{3.24}$$

PROPOSITION 3.5. *Let the conditions of Theorem 3.2 hold, let θ be given by the second relation in (3.8), and let λ be a real eigenvalue of θ:*

$$\theta f = \lambda f, \quad f \neq 0, \quad \lambda \in \mathbb{R}. \tag{3.25}$$

Then, the matrix function

$$g(x) := j\Lambda(x)^*\Sigma(x)^{-1}f \tag{3.26}$$

is a bounded state of $\mathscr{H}$ *and* $\mathscr{H}g = \lambda g$.

Proof. First, we show that formulas (3.21) and (3.25) yield

$$\vartheta_1^*\Sigma_0^{-1}f = 0, \quad \vartheta_2^*\Sigma_0^{-1}f = 0, \quad \alpha f = \lambda f. \tag{3.27}$$

Indeed, the first two equalities in (3.27) easily follow from (3.21) for the case that $\lambda = \overline{\lambda}$. The equality $\alpha f = \lambda f$ is immediate from $\theta f = \lambda f$, definition of θ in (3.8), and equality $\vartheta_1^*\Sigma_0^{-1}f = 0$.

Next, we show that

$$\left(j\Lambda^*\Sigma^{-1}\right)' = \mathrm{i}j^2\Lambda^*\Sigma^{-1}\alpha + \left(\Lambda^*\Sigma^{-1}\Lambda - j\Lambda^*\Sigma^{-1}\Lambda j\right)j\Lambda^*\Sigma^{-1}. \tag{3.28}$$

Formula (3.28) follows from a general GBDT formula [43, (3.14)] and also from its Dirac system subcase [43, (2.13)], but it will be convenient to prove (3.28) directly. We note that formula (3.2) implies

$$\Lambda' = -\mathrm{i}\alpha\Lambda j, \qquad \Sigma' = \Lambda\Lambda^*, \tag{3.29}$$

and formula (3.6) can be rewritten as

$$\alpha^*\Sigma^{-1} = \Sigma^{-1}\alpha - \mathrm{i}\Sigma^{-1}\Lambda j\Lambda^*\Sigma^{-1}. \tag{3.30}$$

Since $j^2 = I_m$, using (3.29) and (3.30) we obtain (3.28).

Now, partitioning Λ into two blocks and using (3.1) and (3.2), we see that

$$v(x) = -2\mathrm{i}\Lambda_1(x)^*\Sigma(x)^{-1}\Lambda_2(x), \quad \Lambda =: \begin{bmatrix}\Lambda_1 & \Lambda_2\end{bmatrix}. \tag{3.31}$$

In view of (1.2) and (3.31) we have

$$\Lambda^*\Sigma^{-1}\Lambda - j\Lambda^*\Sigma^{-1}\Lambda j = \mathrm{i}jV. \tag{3.32}$$

Applying both sides of (3.28) to f and taking into account the last equality in (3.27) and relation (3.32), we derive

$$\left(j\Lambda(x)^*\Sigma(x)^{-1}f\right)' = \mathrm{i}\lambda j^2\Lambda(x)^*\Sigma(x)^{-1}f + \mathrm{i}jV(x)j\Lambda(x)^*\Sigma(x)^{-1}f. \tag{3.33}$$

Because of (3.23) and (3.26), we can rewrite (3.33) as

$$\mathscr{H}_{de}g = \lambda g, \tag{3.34}$$

and it remains to show that $g \in \mathscr{D}(\mathscr{H})$, that is, that $g \in L_m^2(0,\infty)$ and (3.24) holds for $y = g$. From (3.13) and (3.34) we see that $g, \mathscr{H}_{de}g \in L_m^2(0,\infty)$. Finally, the initial condition

$$g(0) = \begin{bmatrix}\vartheta_1 & -\vartheta_2\end{bmatrix}^*\Sigma_0^{-1}f = 0 \tag{3.35}$$

is immediate from (3.2), (3.26), and (3.27). $\square$

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