Operators and Matrices
Volume 7, Number 1 (2013), 197–204 doi:10.7153/oam-07-11

THE NUMERICAL RADII OF WEIGHTED SHIFT MATRICES AND OPERATORS

MAO-TING CHIEN AND HUE-AN SHEU

(*Communicated by C.-K. Li*)

Abstract. Let A be an operator on a separable Hilbert space. The numerical range of A is defined as $W(A) = \{\langle Ax, x\rangle : \|x\| = 1\}$. It is known that the numerical range of a weighted shift operator is a circular disk. In this paper, we compute and compare the numerical radii of certain weighted shift matrices and operators.

1. Introduction

Let A be an operator on a separable Hilbert space. The numerical range of A is defined to be the set

$$W(A) = \{\langle Ax, x\rangle : \|x\| = 1\}.$$

The numerical range is always nonempty, bounded and convex. Further, the range is compact for a finite-dimensional matrix. The numerical radius $w(A)$ is the supremum of the modulus of $W(A)$. (For reference on the numerical ranges of matrices and operators, see, for instance, [6].)

We consider a weighted shift operator on the Hilbert space $\ell^2(\mathbf{N})$ with bounded weights $(a_1, a_2, a_3, \ldots)$ represented by an infinite matrix of the form

$$A = A(a_1, a_2, \ldots) = \begin{pmatrix} 0 & 0 & 0 & 0 & \ldots \\ a_1 & 0 & 0 & 0 & \ldots \\ 0 & a_2 & 0 & 0 & \ldots \\ 0 & 0 & a_3 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{1.1}$$

In finite-dimensional case, an n-by-n weighted shift matrix with weights $(a_1, a_2, \ldots, a_{n-1})$ is the matrix

$$A = A(a_1, a_2, \ldots, a_{n-1}) = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 \\ a_1 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & a_{n-1} & 0 \end{pmatrix}. \tag{1.2}$$

Mathematics subject classification (2010): 15A60, 47A12.
Keywords and phrases: Numerical range; numerical radius; weighted shift operator.
The first author supported in part by Taiwan National Science Council under NSC 99-2115-M-031-004-MY2.

Paper OaM-07-11

It is known that the numerical range of a weighted shift operator is a circular disk about the origin (cf. [4], [5], [8], [9]), and the numerical range of a weighted shift matrix is a closed circular disc centered at the origin (cf. [2], [3]). In particular, $W(A(1,1,\ldots))$ is an open unit circular disk (cf. [9]), and $W(A(1,1,\ldots,1))$ of $A(1,1,\ldots,1) \in M_n$ is a circular disk about the origin with radius $\cos(\pi/(n+1))$, (cf. [7]). Further, Berger and Stampfli [1] showed that if $(1+h) > \sqrt{2}$,

$$w(A(1+h,1,1,\ldots)) = \frac{1}{2}\Big(((1+h)^2-1)^{1/2} + ((1+h)^2-1)^{-1/2}\Big).$$

It is easy to see that a weighted shift operator(and matrix) A is unitarily similar to $|A|$ (cf. [4]). Hence we may assume the weights are nonnegative for the study of the numerical range. In section 2, we determine the numerical radii of weighted shift matrices

$$A_k = A_k(1,\ldots,1,r,1,\ldots,1) \in M_n \tag{1.3}$$

with weights $(1,\ldots,1,r,1,\ldots,1)$, where $a_j = 1$ for all j expect one weight $a_k = r > 0$, $1 \leqslant k \leqslant n-1$. Moreover, we compare the numerical radii of weighted shift matrices $A_k, k = 1,2,\ldots,n$. In section 3, we compute the numerical radius of weighted shift operator $A(1,1+h,1,\ldots)$ with weights $(a_1,a_2,\ldots)$, where $a_j = 1$ for all j expect the weight $a_2 = 1+h$, and compare the numerical radius with the weighted shift operator $A(1+h,1,1,\ldots)$.

2. Weighted shift matrices

Firstly, we determine the numerical radii of weighted shift matrices $A_k = A_k(1,\ldots,1,r,1,\ldots,1)$ with weights $(1,\ldots,1,r,1,\ldots,1)$.

THEOREM 2.1. *Let $A_k = A_k(1,\ldots,1,r,1,\ldots,1), 1 \leqslant k \leqslant n-1$, be an n-by-n weighted shift matrix in (1.3).*

(*i*) *If $0 < r \leqslant 1$, then $w(A_k) = \cos\theta_k$, where $\theta_k \in (0,2\pi)$ is the minimum root of*

$$\sin(n+1)\theta + (1-r^2)\sum_{j=1}^{k}\sin(n+1-2j)\theta = 0. \tag{2.1}$$

(*ii*) *If $r \geqslant 2$, then $w(A_k) = \cosh\theta_k$, where θ_k is the maximum root of*

$$\sinh(n+1)\theta + (1-r^2)\sum_{j=1}^{k}\sinh(n+1-2j)\theta = 0. \tag{2.2}$$

Proof. Let $p_m(t)$ be the characteristic polynomial of the real part of the shift matrix $A(1,1,\ldots,1) \in M_m$. Setting $\psi_m(t) = 2^m p_m(t)$, then $\psi_m(t)$ is a Chebyshev polynomial of second kind, and thus

$$\psi_m(\cos\theta) = \sin(m+1)\theta/\sin\theta. \tag{2.3}$$

Let $q_{k,n}(t) = \det(tI - \Re(A_k))$.

Assume $r \leqslant 1$. Then $\rho(\Re(A_k)) \leqslant \|\Re(A_k)\|_{mc} \leqslant 1$, where $\|.\|_{mc}$ denotes the matrix norm of maximum column sum. Thus, every eigenvalue of $\Re(A_k)$ can be expressed as $\cos\theta$ for some θ. We claim that, for $1 \leqslant k \leqslant n-1$,

$$q_{k,n}(\cos\theta) = \frac{\sin(n+1)\theta + (1-r^2)\sum_{j=1}^{k}\sin(n+1-2j)\theta}{2^n \sin\theta}, \tag{2.4}$$

by proving that (2.4) holds for $k = 1,2$, and induction for $k \geqslant 3$.

Suppose $k = 1$. Then

$$q_{1,n}(t) = tp_{n-1}(t) - \frac{r^2}{4}p_{n-2}(t),$$

we have,

$$2^n q_{1,n}(t) = 2t\psi_{n-1}(t) - r^2\psi_{n-2}(t). \tag{2.5}$$

Substituting (2.3) into (2.5), we have that

$$\begin{aligned} 2^n q_{1,n}(\cos\theta) &= 2\cos\theta\frac{\sin n\theta}{\sin\theta} - r^2\frac{\sin(n-1)\theta}{\sin\theta} \\ &= \frac{\sin(n+1)\theta + (1-r^2)\sin(n-1)\theta}{\sin\theta}. \end{aligned} \tag{2.6}$$

Suppose $k = 2$. Then $q_{2,n}(t) = tq_{1,n-1}(t) - \frac{1}{4}p_{n-2}(t)$. Using (2.3) and (2.6), we have that

$$\begin{aligned} q_{2,n}(\cos\theta) &= \cos\theta\frac{1}{2^{n-1}}\left(\frac{\sin n\theta}{\sin\theta} + (1-r^2)\frac{\sin(n-2)\theta}{\sin\theta}\right) - \frac{1}{2^{n-2}}\frac{1}{4}\frac{\sin(n-1)\theta}{\sin\theta} \\ &= \frac{\sin(n+1)\theta + (1-r^2)\sin(n-1)\theta + (1-r^2)\sin(n-3)\theta}{2^n\sin\theta}. \end{aligned} \tag{2.7}$$

Suppose $k \geqslant 3$. Then

$$q_{k,n}(t) = tq_{k-1,n-1}(t) - \frac{1}{4}q_{k-2,n-2}(t). \tag{2.8}$$

For $k = 3$, substituting (2.6) and (2.7) into (2.8), we have that

$$\begin{aligned} &q_{3,n}(\cos\theta) \\ &= \frac{\cos\theta\Big(\sin n\theta + (1-r^2)\sin(n-2)\theta + (1-r^2)\sin(n-4)\theta\Big)}{2^{n-1}\sin\theta} \\ &\quad -\frac{1}{4}\frac{\sin(n-1)\theta + (1-r^2)\sin(n-3)\theta}{2^{n-2}\sin\theta} \\ &= \frac{\sin(n+1)\theta + (1-r^2)\sin(n-1)\theta + (1-r^2)\sin(n-3)\theta + (1-r^2)\sin(n-5)\theta}{2^n\sin\theta} \end{aligned}$$

Thus (2.4) holds. Suppose (2.4) holds for $k \leqslant m-1$. When $k = m$, according to (2.8), we compute that

$$\begin{aligned}
&q_{m,n}(\cos\theta)\\
&= \cos\theta q_{m-1,n-1}(\cos\theta) - \frac{1}{4} q_{m-2,n-2}(\cos\theta)\\
&= \frac{\cos\theta[\sin n\theta + (1-r^2)\sin(n-2)\theta + \cdots + (1-r^2)\sin((n-2(m-1))\theta)}{2^{n-1}\sin\theta}\\
&\quad -\frac{1}{4}\frac{\sin(n-1)\theta + (1-r^2)\sin(n-3)\theta + \cdots + (1-r^2)\sin(((n-1)-2(m-2))\theta)}{2^{n-2}\sin\theta}\\
&= \frac{\sin(n+1)\theta + (1-r^2)\sin(n-1)\theta + \cdots + (1-r^2)\sin((n+1)-2m)\theta}{2^n\sin\theta}.
\end{aligned}$$

This proves the induction. Hence $q_{k,n}(\cos\theta) = 0$ if and only if (2.1) holds. Therefore, the numerical radius $w(A_k) = \rho(\Re(A_k)) = \cos\theta_k$, where $\theta_k \in (0, 2\pi)$ is the minimum root of (2.1). Indeed, we will show later that $\theta_k \in (0, \pi/2)$.

Next, assume $r \geqslant 2$. Then $\rho(\Re(A_k)) = w(A_k) \geqslant r/2 \geqslant 1$. Thus some roots of $q_{k,n}(t)$ are greater than or equal to 1 which are expressed as $\cosh\theta$. It can be proved in the same way that for $1 \leqslant k \leqslant n-1$,

$$q_{k,n}(\cosh\theta) = \frac{\sinh(n+1)\theta + (1-r^2)\sum_{j=1}^{k}\sinh(n+1-2j)\theta}{2^n\sinh\theta}.$$

Hence, $q_{k,n}(\cosh\theta) = 0$ if and only if (2.2) holds, and $w(A_k) = \cosh\theta_k$ where θ_k is the maximum root of (2.2). $\square$

It is shown in [4] that $W(A(a_1, a_2, \ldots, a_{n-1})) = W(A(a_{n-1}, a_{n-2}, \ldots, a_1))$, it suffices to consider $k \leqslant [n/2]$ for the numerical range of $A_k(1, \ldots, 1, r, 1, \ldots, 1) \in M_n$. We compare the numerical radii of the matrices $A_k(1, \ldots, 1, r, 1, \ldots, 1) \in M_n$, $k = 1, 2, \ldots, [n/2]$.

THEOREM 2.2. *Let $1 \leqslant k \leqslant [n/2]-1$ and A_k be the weighted shift matrices defined in (1.3).*

(i) If $0 < r < 1$ then $w(A_k) > w(A_{k+1})$.

(ii) If $r \geqslant 2$ then $w(A_k) < w(A_{k+1})$.

Proof. Assume $0 < r < 1$. Consider the trigonometric polynomial obtained in (i) of Theorem 2.1,

$$f_k(\theta) = \sin(n+1)\theta + (1-r^2)\sum_{j=1}^{k}\sin(n+1-2j)\theta.$$

It is clear that $f_k(\theta) > 0$ for all $\theta \in (0, \pi/(n+1))$. On the other hand,

$$f_k(\pi/(n-(k-1))) = -\sin(k/(n-(k-1)))\pi + (1-r^2)\sin(k/(n-(k-1)))\pi < 0.$$

Since $n-(k-1)>2$, we have $\pi/(n-(k-1))<\pi/2$. Hence, there exists the smallest $\theta_k \in (\pi/(n+1), \pi/(n-(k-1)))$ such that $f_k(\theta_k)=0$. Observe that

$$\begin{aligned} f_{k+1}(\theta) &= \sin(n+1)\theta + (1-r^2)\sum_{j=1}^{k+1}\sin(n+1-2j)\theta \\ &= f_k(\theta) + (1-r^2)\sin(n-(2k+1))\theta. \end{aligned} \tag{2.9}$$

Since both $f_k(\theta)$ and $(1-r^2)\sin(n-(2k+1))\theta$ are positive for $\theta \in (0,\theta_k)$, and

$$f_{k+1}(\theta_k) = 0 + (1-r^2)\sin(n-(2k+1))\theta_k > 0,$$

it follows that $f_{k+1}(\theta)>0$ for all $\theta \in (0,\theta_k]$. Further, we find that

$$f_{k+1}(\pi/(n-k)) = -\sin((k+1)/(n-k))\pi + (1-r^2)\sin((k+1)/(n-k))\pi < 0.$$

Hence, there exists the smallest $\theta_{k+1} \in (\theta_k, \pi/(n-k))$ such that $f_{k+1}(\theta_{k+1})=0$, we obtain that $\cos\theta_k > \cos\theta_{k+1}$. This proves part (i).

Assume $r \geqslant 2$. Consider the hyperbolic trigonometric polynomial obtained in (ii) of Theorem 2.1,

$$g_k(\theta) = \sinh(n+1)\theta + (1-r^2)\sum_{j=1}^{k}\sinh(n+1-2j)\theta.$$

Substituting $\sinh\theta = (e^{\theta}-e^{-\theta})/2$, we have that

$$2e^{(n+1)\theta}g_k(\theta) = \left(e^{2(n+1)\theta} - (r^2-1)\sum_{j=1}^{k}e^{2(n-j+1)\theta}\right) + \left((r^2-1)\sum_{j=1}^{k}e^{2j\theta} - 1\right). \tag{2.10}$$

The second term in the right-hand side of (2.10) is always positive. Concerning the first term, we have

$$e^{2(n+1)\theta} - (r^2-1)\sum_{j=1}^{k}e^{2(n-j+1)\theta} > e^{2(n+1)\theta} - k(r^2-1)e^{2n\theta} = e^{2n\theta}(e^{2\theta} - k(r^2-1)).$$

Hence

$$g_k(\theta) > 0 \text{ for all } \theta \geqslant (\ln(k(r^2-1)))/2. \tag{2.11}$$

Substituting $\theta = (\ln(r^2-1))/2$ into (2.10), we obtain that

$$\begin{aligned} &2e^{(n+1)(\ln(r^2-1))/2}g_k((\ln(r^2-1))/2) \\ &= -(r^2-1)^{n+2-k}\frac{(r^2-1)^{k-1}-1}{(r^2-1)-1} + (r^2-1)^2\frac{(r^2-1)^k-1}{(r^2-1)-1} - 1 < 0, \end{aligned}$$

and thus $g_k((\ln(r^2-1))/2) < 0$. Then there exists the largest $\theta_k \in ((\ln(r^2-1))/2, (\ln(k(r^2-1)))/2)$ such that $g_k(\theta_k)=0$.

Since $g_{k+1}(\theta) = g_k(\theta) + (1-r^2)\sinh((n+1)-2(k+1))\theta$, it follows that

$$2e^{(n+1)\theta}g_{k+1}(\theta) = 2e^{(n+1)\theta}g_k(\theta) - (r^2-1)\Big(e^{2((n+1)-(k+1))\theta} - e^{2(k+1)\theta}\Big). \quad (2.12)$$

By the hypothesis that $k \leqslant [n/2]-1$, then $2k < n-1$, and thus $e^{2((n+1)-(k+1))\theta} - e^{2(k+1)\theta} > 0$. Then, by(2.12), $g_{k+1}(\theta_k) < 0$, while by (2.11), $g_{k+1}(\theta) > 0$ for all $\theta \geqslant (\ln((k+1)(r^2-1)))/2$. Hence, there exists the largest $\theta_{k+1} \in (\theta_k, (\ln((k+1)(r^2-1)))/2)$ such that $f_{k+1}(\theta_{k+1}) = 0$. The assertion $w(A_{k+1}) = \cosh\theta_{k+1} > \cosh\theta_k = w(A_k)$ follows. $\square$

REMARK. The result of Theorem 2.2 is restricted to the case $0 < r < 1$ or $r \geqslant 2$ for the matrix $A_k = A_k(1,\ldots,1,r,1,\ldots,1)$. At present, we have no analogous results if $1 < r < 2$. However, the following example proposes a conjecture that for $1 < r < 2$, the inequality $w(A_k) < w(A_{k+1})$ holds.

We consider the 4×4 weighted shift matrices $A_k = A_k(1,\ldots,1,r,1,\ldots,1)$. Direct computation finds that

$$w(A_1(r,1,1)) = \Big(\frac{(1/2+r^2/4)+((1/2+r^2/4)^2-r^2/4)^{1/2}}{2}\Big)^{1/2}$$

and

$$w(A_2(1,r,1)) = \Big(\frac{(1/2+r^2/4)+((1/2+r^2/4)^2-1/4)^{1/2}}{2}\Big)^{1/2}.$$

It is clear that for $1 < r < 2$, $w(A_1) < w(A_2)$.

3. Weighted shift operators

Let $A = A(a_1, a_2, \ldots)$ be a weighted shift operator with weights $(a_1, a_2, \ldots)$ defined in (1.1). The numerical range $W(A(a_1,a_2,\ldots))$ is a circular disc about the origin. In particular, when $a_n = 1$ for all n, $W(A)$ is an open unit disc. Berger and Stampfli [1] showed that

$$w(A) = \frac{1}{2}\Big(((1+h)^2-1)^{\frac{1}{2}} + ((1+h)^2-1)^{-\frac{1}{2}}\Big)$$

if $a_1 = (1+h) > \sqrt{2}$, $a_2 = a_3 = \cdots = 1$. We compute the numerical radius in the case $a_2 = 1+h$, $a_1 = a_3 = a_4 = \cdots = 1$.

THEOREM 3.1. *Let $A = A(1,1+h,1,1,\ldots)$ be a weighted shift operator with weights $(1,1+h,1,1,\ldots)$, and $1+h > \sqrt{6}/2$. Then*

$$w(A) = \frac{1}{2}\Big(\Big((h(2+h)+\sqrt{(h(2+h))^2+4h(2+h)})/2\Big)^{\frac{1}{2}} + \Big((h(2+h)+\sqrt{(h(2+h))^2+4h(2+h)})/2\Big)^{-\frac{1}{2}}\Big).$$

Proof. The weighted shift operator A on H^2 satisfies

$$Af(z) = zf(z) + hf'(0)z^2$$

for $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots \in H^2$. Suppose that $\|\Re(A)\| = \alpha > 1$ with $\Re(A)f = \alpha f$. Then

$$\left(zf(z) + \frac{f(z) - f(0)}{z}\right) + h\left(f'(0)z^2 + \frac{f''(0)}{2}z\right) = 2\alpha f(z) \tag{3.1}$$

Compare coordinates-wise of the equation $\Re(A)f = \alpha f$, we have

$$f'(0) = 2\alpha f(0) \tag{3.2}$$

and

$$f(0) + (1+h)f''(0)/2 = 2\alpha f'(0) \tag{3.3}$$

Substitute $f'(0)$ and $f''(0)$ of equations (3.2) and (3.3) into (3.1), we have

$$(z^2 - 2\alpha z + 1)f(z) = \left(1 - 2\alpha h z^3 - ((4\alpha^2 - 1)/(1+h))hz^2\right)f(0) \tag{3.4}$$

Setting $\alpha = \cosh x$ for $x > 0$, the equation (3.4) yields

$$(z - e^x)(z - e^{-z})f(z) = \left(1 - (e^x + e^{-x})hz^3 - ((e^{2x} + e^{-2x} + 1)/(1+h))hz^2\right)f(0). \tag{3.5}$$

Taking $z = e^{-x}$ in (3.5), we obtain

$$1 - (e^x + e^{-x})he^{-3x} - ((e^{2x} + e^{-2x} + 1)/(1+h))he^{-2x} = 0, \tag{3.6}$$

Simplify equation (3.6), we have

$$e^{4x} - h(2+h)e^{2x} - h(2+h) = 0. \tag{3.7}$$

If $1 + h > \sqrt{6}/2$, equation (3.7) is solvable by

$$e^{2x} = \left(h(2+h) + \sqrt{(h(2+h))^2 + 4h(2+h)}\right)/2, \tag{3.8}$$

and thus

$$\begin{aligned} w(A) = \cosh x = \frac{1}{2}&\left(\left((h(2+h) + \sqrt{(h(2+h))^2 + 4h(2+h)})/2\right)^{\frac{1}{2}}\right. \\ &\left. + \left((h(2+h) + \sqrt{(h(2+h))^2 + 4h(2+h)})/2\right)^{-\frac{1}{2}}\right). \quad \square \end{aligned}$$

In the following, we compare the numerical radii of two weighted shift operators $A(1+h, 1, 1, \ldots)$ and $A(1, 1+h, 1, \ldots)$.

THEOREM 3.2. *Let* $A_1 = A_1(1+h, 1, 1, \ldots)$ *and* $A_2 = A_2(1, 1+h, 1, \ldots)$ *be two weighted shift operators. If* $(1+h) > \sqrt{2}$ *then* $w(A_1) < w(A_2)$.

Proof. It is shown in [1],

$$w(A_1) = \cosh x_1 = \left(((1+h)^2 - 1)^{\frac{1}{2}} + ((1+h)^2 - 1)^{-\frac{1}{2}}\right)/2,$$

where

$$e^{2x_1} = (1+h)^2 - 1 = h(2+h). \tag{3.9}$$

By Theorem 3.1, $w(A_2) = \cosh x_2$, where

$$e^{2x_2} = \left(h(2+h) + \sqrt{(h(2+h))^2 + 4h(2+h)}\right)/2. \tag{3.10}$$

Comparing (3.9) with (3.10), we have $x_1 < x_2$, and thus $w(A_1) = \cosh x_1 < \cosh x_2 = w(A_2)$. $\square$

REFERENCES

[1] C. A. BERGER AND J. G. STAMPFLI, *Mapping theorems for the numerical range*, American Journal of Mathematics, **89** (1967), 1047–1055.

[2] M. T. CHIEN AND B. S. TAM, *Circularity of the numerical range*, Linear algebra and Its Applications, **201** (1994), 113–133.

[3] M. T. CHIEN, *On the numerical range of tridiagonal operators*, Linear Algebra and Its Applications, **246** (1996), 203–214.

[4] M. T. CHIEN AND H. NAKAZATO, *The q-numerical radius of weighted shift operators with periodic weights*, Linear Algebra and Its Applications, **422** (2007), 198–218.

[5] M. T. CHIEN AND H. NAKAZATO, *The numerical radius of a weighted shift operator with geometric weights*, Electronic Journal of Linear Algebra, **18** (2009), 58–63.

[6] K. E. GUSTAFSON AND D. K. M. RAO, *Numerical range, the Field of Values of Linear Operators and Matrices*, Springer, New York, 1997.

[7] M. MARCUS AND B. N. SHURE, *The numerical range of certain 0,1-matrices*, Linear and Multilinear Algebra, **7** (1979), 111–120.

[8] A. L. SHEILDS, *Weighted shift operators and analytic function theory*, Mathematical Surveys. Vol. 13, American Mathematical Society, Providence, 1974.

[9] Q. F. STOUT, *The numerical range of a weighted shift*, Proceedings of the American Mathematical Society, **88** (1983), 495–502.

(Received June 27, 2011)

Mao-Ting Chien
Department of Mathematics
Soochow University
Taipei 11102, Taiwan
e-mail: mtchien@scu.edu.tw

Hue-An Sheu
Department of Mathematics
Soochow University
Taipei 11102, Taiwan
e-mail: wayne0826@yahoo.com.tw