Operators and Matrices
Volume 7, Number 1 (2013), 219–224 doi:10.7153/oam-07-13

B–WEAK COMPACTNESS OF WEAK DUNFORD–PETTIS OPERATORS

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(*Communicated by D. Bakić*)

Abstract. We characterize Banach lattices on which each weak Dunford-Pettis operators is b-weakly compact and we derive some characterizations of KB-spaces.

1. Introduction and notation

An operator from a Banach lattice E into a Banach space X is said to be b-weakly compact if it carries each b-order bounded subset of E into a relatively weakly compact subset of X. This class of operators is introduced and studied by Alpay, Altin and Tonyali in [2]. Note that the definition of b-weakly compact operators is based on the notion of b-order bounded subsets which is defined on vector lattices in [2].

Recall that a subset A of a Banach lattice E is called b-order bounded if it is order bounded in the topological bidual E''. It is clear that every order bounded subset of E is b-order bounded. However, the converse is not true in general. In fact, the subset $A = \{e_n : n \in \mathbb{N}\}$ is b-order bounded in the Banach lattice c_0, but it is not order bounded in c_0, where e_n is the sequence of reals numbers with all terms zero except for the n'th which is 1. But a Banach lattice E is said to have the (b)-property if $A \subset E$ is order bounded in E whenever it is order bounded in its topological bidual E.

On the other hand, let us recall from ([1], p. 349) that an operator $T : X \longrightarrow Y$ between two Banach spaces is called weak Dunford-Pettis whenever (x_n) converges weakly to 0 in X and (y'_n) converges weakly to 0 in Y' imply $\lim \langle T(x_n), y'_n \rangle = 0$. There exists an operator which is weak Dunford-Pettis but not b-weakly compact. In fact, the identity operator of the Banach lattice c_0 is weak Dunford-Pettis but it is not b-weakly compact. Conversely, there exists an operator which is b-weakly compact but not weak Dunford-Pettis. In fact, the identity operator of the Banach lattice l^2 is b-weakly compact but it is not weak Dunford-Pettis.

In [6] the authors studied the b-weak compactness of semi-compact operators. They proved that if E and F are Banach lattices such that the norm of E is order continuous or F is Dedekind σ-complete, then each semi-compact operator $T : E \longrightarrow F$ is b-weakly compact if and only if, E is a KB-space or the norm of F is order continuous. Also, in [5] we studied the b-weak compactness of order weakly compact

Mathematics subject classification (2010): 46A40, 46B40, 46B42.
Keywords and phrases: b-weakly compact operator, weak Dunford-Pettis operator, Banach lattice, (b)-property, order continuous norm, KB-space.

(resp. AM-compact) operators. It is established that if E and F are two Banach lattices such that the norm of E is order continuous, then each order weakly compact (resp. AM-compact) operator $T:E\longrightarrow F$ is b-weakly compact if, and only if, E or F is a KB-space.

The main goal of this paper is to study the b-weak compactness of weak Dunford-Pettis operators. In fact, we will prove that if E and F are two Banach lattices such that the norm of E is order continuous or F is Dedekind complete with the (b)-property, then each weak Dunford-Pettis operator $T:E\to F$ is b-weakly compact if and only if E or F is a KB-space. As consequences, we will obtain some characterizations for KB-spaces.

To state our results, we need to fix some notations and recall some definitions. A Banach lattice is a Banach space $(E,\|.\|)$ such that E is a vector lattice and its norm satisfies the following property: for each $x,y\in E$ such that $|x|\leqslant|y|$, we have $\|x\|\leqslant\|y\|$. If E is a Banach lattice, its topological dual E', endowed with the dual norm, is also a Banach lattice. A norm $\|.\|$ of a Banach lattice E is order continuous if for each generalized sequence (x_α) such that $x_\alpha\downarrow 0$ in E, the sequence (x_α) converges to 0 for the norm $\|.\|$ where the notation $x_\alpha\downarrow 0$ means that the sequence (x_α) is decreasing, its infimum exists and $\inf(x_\alpha)=0$. We refer the reader to [1] for unexplained terminology on Banach lattice theory.

2. Main results

We will use the term operator $T:E\longrightarrow F$ between two Banach lattices to mean a bounded linear mapping. It is positive if $T(x)\geqslant 0$ in F whenever $x\geqslant 0$ in E. An operator $T:E\longrightarrow F$ is regular if $T=T_1-T_2$ where T_1 and T_2 are positive operators from E into F. It is well known that each positive linear mapping on a Banach lattice is continuous. For terminology concerning positive operators, we refer the reader to the excellent book of Aliprantis–Burkinshaw [1].

Recall from Aliprantis-Burkinshaw ([1], p. 222) that a Banach lattice E is said to be lattice embeddable into another Banach lattice F whenever there exists a lattice homomorphism $T:E\to F$ and there exist two positive constants K and M satisfying

$$K\|x\|\leqslant\|T(x)\|\leqslant M\|x\| \text{ for all } x\in E.$$

T is called a lattice embedding from E into F. In this case $T(E)$ is a closed sublattice of F which can be identified with E.

Let us recall that a Banach lattice E is called a KB-space whenever every increasing norm bounded sequence of E^+ is norm convergent. As an example, each reflexive Banach lattice is a KB-space.

Each KB-space has the (b)-property, but a Banach lattice with the (b)-property is not necessary a KB-space. In fact, the Banach lattice l^∞ has the (b)-property but it is not a KB-space. However, by Proposition 2.1 of [2], a Banach lattice E is a KB-space if and only if it has the (b)-property and its norm is order continuous.

On the other hand, we note the existence of a Banach lattice with an order continuous norm without the (b)-property. In fact, the norm of c_0 is order continuous but c_0 does not have the (b)-property. Also, the norm of l^∞ is not order continuous and l^∞ has the (b)-property, but does not contain a complemented copy of c_0.

Also, there exists a Banach lattice with the (b)-property without being Dedekind σ-complete. In fact, the Banach lattice c of all convergent sequences has the (b)-property but is not Dedekind σ-complete. And there exists a Banach lattice which is Dedekind σ-complete without having the (b)-property. In fact, the Banach lattice c_0 is Dedekind σ-complete but does not have the (b)-property.

We will need the following Lemma, which is established in [5].

LEMMA 2.1. *Let E be a Banach lattice with an order continuous norm. Then E has the (b)-property if and only if E does not contain a complemented copy of c_0.*

Note that there exists an operator on a Banach lattice E which is not weak Dunford-Pettis, however the norm of E is order continuous. As an example, the identity operator of the Banach lattice l^2 is not weak Dunford-Pettis, however the norm of l^2 is order continuous.

Also, the class of weak Dunford-Pettis operators is a two sided ideal in the space of all operators on a Banach lattice.

THEOREM 2.2. *Let E and F be two Banach lattices such that the norm of E is order continuous. Then the following assertions are equivalent:*

(1). Each operator $T : E \longrightarrow F$ is b-weakly compact.

(2). Each weak Dunford-Pettis operator $T : E \longrightarrow F$ is b-weakly compact.

(3). Each positive weak Dunford-Pettis operator $T : E \longrightarrow F$ is b-weakly compact.

(4). One of the following assertions holds:

a. E is a KB-space,

b. F is a KB-space.

Proof. (1)$\Longrightarrow$(2) Obvious.

(2)$\Longrightarrow$(3) Obvious.

(3)$\Longrightarrow$(4) Suppose that E and F are not KB-spaces. Since E has an order continuous norm, it follows from Proposition 2.1 of [2] that E does not have the (b)-property and hence by Lemma 2.1, E contains a complemented copy of c_0 and there exists a positive projection $P : E \longrightarrow c_0$.

On the other hand, since F is not a KB-space, it follows from Theorem 4.61 of [1] that c_0 is lattice embeddable in F. And hence there exists a lattice embedding S from c_0 into F and a constant $M > 0$ such that $\|S((\alpha_n))\| \geqslant M\|(\alpha_n)\|_\infty$ for all $(\alpha_n) \in c_0$.

In the first time, observe that the embedding $S : c_0 \longrightarrow F$ is not a b-weakly compact operator. In fact, the canonical basis (e_n) of c_0 is a disjoint b-order bounded sequence but $\|S((e_n))\| \geqslant M\|(e_n)\|_\infty = M$ for each n. Hence Proposition 2.8 of [2] implies that the embedding $T : c_0 \longrightarrow F$ is not b-weakly compact.

Now, we consider the operator $T = S \circ P : E \longrightarrow c_0 \longrightarrow F$. Since $T = S \circ Id_{c_0} \circ P$ and the identity operator $Id_{c_0} : c_0 \longrightarrow c_0$ is weak Dunford-Pettis, then T is weak Dunford-Pettis. But it is not a b-weakly compact operator. Otherwise, the composed operator $T \circ i$, which is exactly the embedding $S : c_0 \longrightarrow F$, is b-weakly compact, where $i : c_0 \longrightarrow E$ is the canonical injection of c_0 into E. This presentees a contradiction.

(4)$\Longrightarrow$(1) Follows from Proposition 2.1 of [3] and Corollary 2.3 of [4]. □

REMARK. The assumption "E with an order continuous norm" is essential in Theorem 2.2. In fact, each positive operator T from l^∞ into c_0 is b-weakly compact, but neither l^∞ nor c_0 is a KB-space.

As consequences, we obtain the following characterizations of KB-spaces.

COROLLARY 2.3. *Let E be a Banach lattice with an order continous norm. Then the following assertions are equivalent:*

(1). Each operator $T : E \longrightarrow E$ is b-weakly compact.

(2). Each weak Dunford-Pettis operator $T : E \longrightarrow E$ is b-weakly compact.

(3). Each positive weak Dunford-Pettis operator $T : E \longrightarrow E$ is b-weakly compact.

(4). E is a KB-space.

Note that there exists an operator which is weak Dunford-Pettis but its second power is not b-weakly compact. In fact, the identity operator of the Banach lattice c_0 is weak Dunford-Pettis, but its second power, which is also the identity operator of c_0, is not b-weakly compact.

Another consequence of Theorem 2.2 is the following result.

COROLLARY 2.4. *Let E be a Banach lattice with an order continuous norm. Then the following assertions are equivalent:*

1 – For all positive operators S and T from E into E with $0 \leqslant S \leqslant T$ and T is weak Dunford-Pettis, S is b-weakly compact.

2 – Each positive weak Dunford-Pettis operator $T : E \longrightarrow E$ is b-weakly compact.

3 – For each positive weak Dunford-Pettis operator $T : E \longrightarrow E$, the second power T^2 is b-weakly compact.

4 – E is a KB-space.

Whenever the Banach lattice F is Dedekind σ-complete, we obtain the following result.

THEOREM 2.5. *Let E and F be two Banach lattices such that F is Dedekind σ-complete. If each positive weak Dunford-Pettis operator $T : E \longrightarrow F$ is b-weakly compact, then one of the following statements is valid:*

(1). E is a KB-space,

(2). the norm of F is order continuous.

Proof. By way of contradiction, we suppose that neither E is a KB-space nor the norm of F is order continuous, and we show that there exists a positive weak Dunford-Pettis $T : E \longrightarrow F$ which is not b-weakly compact.

If E is not a KB-space, then it follows from the Lemma 2.1 and Lemma 3.4 of [4] the existence of a b-order bounded disjoint sequence (x_n) of E^+ with $\|x_n\| = 1$ for all n, and there exists a positive disjoint sequence (g_n) of E' with $\|g_n\| \leqslant 1$ such that $g_n(x_n) = 1$ for all n and $g_n(x_m) = 0$ for $n \neq m$.

We consider the operator $S: E \longrightarrow l^\infty$ defined by $S(x) = (g_k(x))_{k=1}^\infty$ for all $x \in E$.

Clearly the operator S is positive and weak Dunford-Pettis (because $S = Id_{l^\infty} \circ S$ and the identity operator Id_{l^∞} is weak Dunford-Pettis), but S is not b-weakly compact. In fact, since (x_n) is a b-order bounded disjoint sequence of E^+ and $\|S(x_n)\| = \|(g_k(x_n))_{k=1}^\infty\| = \|(e_n)\| = 1$ for each n, then it follows from Proposition 2.8 of [2] that the operator S is not b-weakly compact.

On the other hand, since the norm of F is not order continuous and F is Dedekind σ-complete, it follows from Corollary 2.4.3 of [7] that F contains a complemented copy of l^∞ and there exists a positive projection $P: F \longrightarrow l^\infty$.

Now, we consider the composed operator $T = i \circ S : E \longrightarrow l^\infty \longrightarrow F$, where i is the canonical injection of l^∞ into F. This operator is weak Dunford-Pettis (because $T = i \circ Id_{l^\infty} \circ S$ and Id_{l^∞} is weak Dunford-Pettis), but not b-weakly compact. Otherwise, the composed operator $P \circ T$ which is exactly the operator S would be b-weakly compact, and this is a contradiction. $\square$

If we suppose in addition that the Banach lattice F has the (b)-property, then we deduce the following characterizations which is the same as in Theorem 2.2, but here we have the assumptions on the Banach lattice F.

COROLLARY 2.6. *Let E and F be two Banach lattices such that F is Dedekind σ-complete and has the (b)-property. Then the following assertions are equivalent:*

(1). Each operator $T: E \longrightarrow F$ is b-weakly compact.

(2). Each weak Dunford-Pettis operator $T: E \longrightarrow F$ is b-weakly compact.

(3). Each positive weak Dunford-Pettis operator $T: E \longrightarrow F$ is b-weakly compact.

(4). One of the following assertions holds

a. E is a KB-space.

b. F is a KB-space.

REMARK. In Corollary 2.6, the assumption "F is Dedekind σ-complete and has the (b)-property" is essential. In fact, it follows from the proof of Proposition 1 of [8] that each operator T from l^∞ into c is weakly compact and hence is b-weakly compact, but neither l^∞ nor c is a KB-space. On the other hand, each operator T from l^∞ into c_0 is weakly compact and hence is b-weakly compact, but neither l^∞ nor c_0 is a KB-space.

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(Received July 28, 2011)

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