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AUTOMORPHISMS OF K(H) WITH RESPECT TO THE STAR PARTIAL ORDER

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Abstract. Let H be a separable infinite dimensional complex Hilbert space, and let $K(H)$ be the set of all compact bounded linear operators on H. In the paper we characterize the bijective, additive, continuous maps on $K(H)$ which preserve the star partial order in both directions.

1. Introduction

Let M_n be the algebra of all $n \times n$ complex matrices. On M_n many different partial orders can be defined. One such order is the rank substractivity order which was introduced by Hartwig [5] in the following way

$$A \ll B \quad \text{if and only if} \quad \text{rank}\,(B-A) = \text{rank}\,B - \text{rank}\,A.$$

Hartwig observed that there exists another equivalent definition of the rank substractivity order, namely

$$A \ll B \quad \text{if and only if} \quad A^-A = A^-B \text{ and } AA^- = BA^-$$

where A^- is a generalized inner inverse of A. The partial order $\ll$ is thus usually called the minus partial order.

Recently Šemrl [11] extended the minus partial order from M_n to $B(H)$, the algebra of all bounded linear operators on an infinite dimensional Hilbert space H. Since $A \in B(H)$ has a generalized inner inverse if and only if its image is closed (see for example [8]) and Šemrl did not want to restrict his attention only to closed range operators, he found an appropriate equivalent definition of the minus partial order on M_n without using inner inverses, and then extended this definition to $B(H)$. More precisely, he proved that for $A, B \in M_n$ we have $A \ll B$ if and only if there exist idempotent matrices $P, Q \in M_n$ such that $\text{Im}\,P = \text{Im}\,A$, $\text{Ker}\,A = \text{Ker}\,Q$, $PA = PB$ and $AQ = BQ$. When extending the concept of the minus partial order from M_n to $B(H)$ Šemrl also replaced $\text{Im}\,A$ in the first of the four equations by its closure, since the image of a bounded idempotent operator is closed.

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Another order on M_n is the star order which was introduced by Drazin [2] in the following way

$$A \underset{*}{\leqslant} B \quad \text{if and only if} \quad A^*A = A^*B \text{ and } AA^* = BA^*, \tag{1}$$

where $A,B \in M_n$ and A^* stands for the conjugate transpose of A.

Motivated by Šemrl's extension of the minus partial order from M_n to $B(H)$ Dolinar and Marovt extended in [3] the star partial order to $B(H)$ in the following way.

DEFINITION 1. Let H be a complex Hilbert space and $B(H)$ the algebra of all bounded linear operators on H. For $A,B \in B(H)$ we write $A \underset{*}{\leqslant} B$ if and only if there exist self-adjoint idempotent operators $P,Q \in B(H)$ such that

(i) $\operatorname{Im} P = \overline{\operatorname{Im} A}$,

(ii) $\operatorname{Ker}\ A = \operatorname{Ker}\ Q$,

(iii) $PA = PB$,

(iv) $AQ = BQ$.

The order $\underset{*}{\leqslant}$ is called the star partial order on $B(H)$.

Dolinar and Marovt [3] proved that the order introduced in the above definition is indeed a partial order and then showed that this definition is equivalent to the usual definition of the star order (1) for $B(H)$.

In [11] Šemrl also described the structure of corresponding automorphisms for the minus partial order. Namely, he characterized the bijective maps form $B(H)$ to $B(H)$ which preserve the minus partial order in both directions. It is the aim of this paper to present a similar result in the case of the star partial order. However, in our paper we restricted ourself to bijective maps from $K(H)$ to $K(H)$, where $K(H) \subset B(H)$ is the set of all compact operators, and we additionally assumed that our maps are additive and continuous. We restricted ourself to the set of all compact operators in $B(H)$ since there exists a Hilbert space H and an operator $A \in B(H)$ such that there is no rank one operator $C \in B(H)$ with $C \underset{*}{\leqslant} A$ (see Example in the next section) and we did not find a proof without the use of rank one operators. The following is our main result.

THEOREM 2. *Let H be a separable infinite dimensional complex Hilbert space. Assume that $\phi : K(H) \to K(H)$ is a bijective, additive and continuous map such that for every pair $A,B \in K(H)$ we have*

$$A \underset{*}{\leqslant} B \text{ if and only if } \phi(A) \underset{*}{\leqslant} \phi(B).$$

Then there exist operators $U,V : H \to H$ which are both unitary or both antiunitary and a nonzero $\alpha \in \mathbb{C}$ such that $\phi(A) = \alpha UAV$ for every $A \in K(H)$ or $\phi(A) = \alpha UA^*V$ for every $A \in K(H)$.

REMARK 3. Example at the end of the paper shows that without additivity assumption the structure of the star order preservers on $K(H)$ can be much more complicated.

2. Proof of the main result

Let us start by presenting some properties of the star partial order on $B(H)$. The following lemma was proved in [3].

LEMMA 4. *If $A,B \in B(H)$, then the following statements are equivalent.*

(i) $A \underset{*}{\leqslant} B$.

(ii) There exist closed subspaces H_1, H_2 of H such that $A,B\colon H_1 \oplus H_1^{\perp} \to H_2 \oplus H_2^{\perp}$ have matrix representations

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} A_1 & 0 \\ 0 & B_1 \end{bmatrix}$$

where $A_1\colon H_1 \to H_2$ and $B_1\colon H_1^{\perp} \to H_2^{\perp}$ are bounded linear operators and A_1 is injective with $\overline{\operatorname{Im} A} = H_2$.

(iii) $\overline{\operatorname{Im} A} \perp \overline{\operatorname{Im}(B-A)}$ *and* $\overline{\operatorname{Im} A^*} \perp \overline{\operatorname{Im}(B^*-A^*)}$.

LEMMA 5. *If $P \in B(H)$ is a self-adjoint idempotent and $A \underset{*}{\leqslant} P$, then A is a self-adjoint idempotent and $AP = PA = A$.*

Proof. Let $P \in B(H)$ be a self-adjoint idempotent and $A \underset{*}{\leqslant} P$. It is known (see for example [3]) that $A \underset{*}{\leqslant} P$ implies $A \ll P$ where $\ll$ denotes the minus partial order on $B(H)$. By [11, Lemma 4] it follows that A is an idempotent and that $AP = PA = A$. It remains to show that $A = A^*$. It is well known (see for example [1]) that if A is an idempotent on H, then A is a self-adjoint operator if and only if A is a normal operator. Since $A \underset{*}{\leqslant} P$, we have $A^*A = A^*P$ and $AA^* = PA^*$. It follows that A^*P and PA^* are self-adjoint operators. So, on the one hand we have

$$A^*A = P^*A = PA = A$$

and on the other hand we have

$$AA^* = AP^* = AP = A.$$

This yields that A is a normal and hence a self-adjoint idempotent. □

Let $x,y \in H$ be nonzero vectors. We denote by $x \otimes y^* \in B(H)$ a rank one operator defined by $(x \otimes y^*)z = \langle z,y \rangle x$, $z \in H$. Note that every rank one operator in $B(H)$ can be written in this form. Let $B_1(H)$ be the set of all rank one operators in $B(H)$.

The proof of the next lemma is the same as the proof of Proposition 2.4 in [7].

LEMMA 6. *Let $x, y \in H$ be nonzero vectors and $A \in B(H)$. The following two statements are equivalent:*

(i) $x \otimes y^* \underset{*}{\leqslant} A$.

(ii) $A^*x = \langle x,x \rangle y$ *and* $Ay = \langle y,y \rangle x$.

LEMMA 7. *Let $A \in B(H)$. The following two statements are equivalent:*

(i) There exists $C \in B_1(H)$ such that $C \underset{*}{\leqslant} A$.

(ii) The operator AA^* has a nonzero eigenvalue.

Proof. Let us first assume that there exists $C \in B_1(H)$ such that $C \underset{*}{\leqslant} A$. Then there exist nonzero $x, y \in H$ such that $x \otimes y^* = C$. From Lemma 6 it follows that $A^*x = \langle x,x \rangle y$ and $Ay = \langle y,y \rangle x$. So

$$AA^*x = \langle x,x \rangle Ay = \langle x,x \rangle \langle y,y \rangle x = \|x\|^2 \|y\|^2 x.$$

We proved that $\|x\|^2 \|y\|^2$ is a nonzero eigenvalue of AA^*.

Conversely, suppose that there exists a nonzero eigenvalue λ of AA^*. So there is a nonzero $x \in H$ such that $AA^*x = \lambda x$. Let $y = \frac{A^*x}{\langle x,x \rangle}$. Hence $A^*x = \langle x,x \rangle y$. Note that $y \neq 0$. In order to show that $x \otimes y^* \underset{*}{\leqslant} A$ we will prove that $Ay = \langle y,y \rangle x$. From

$$\langle y,y \rangle = \left\langle \frac{A^*x}{\langle x,x \rangle}, \frac{A^*x}{\langle x,x \rangle} \right\rangle = \frac{1}{\langle x,x \rangle^2} \langle AA^*x, x \rangle = \frac{1}{\langle x,x \rangle^2} \langle \lambda x, x \rangle$$

we have $\lambda = \langle x,x \rangle \langle y,y \rangle$. We may conclude that

$$Ay = \frac{1}{\langle x,x \rangle} AA^*x = \frac{1}{\langle x,x \rangle} \lambda x = \langle y,y \rangle x. \qquad \square$$

We will now give an example of a Hilbert space H and a positive operator $M \in B(H)$ without nonzero eigenvalues. Then $A = M^{\frac{1}{2}}$ is well defined and by Lemma 7 there is no $C \in B_1(H)$ with $C \underset{*}{\leqslant} A$.

EXAMPLE 8. Let $H = L^2[0,1]$. We define the operator $M \colon H \to H$ in the following way:

$$M(\omega)(x) = x \cdot \omega(x)$$

for every $\omega \in H$ and every $x \in [0,1]$. Note that the spectrum of M lies in $[0,1]$ and that M has no eigenvalues.

Let us now show that this situation is impossible for the space $K(H)$.

LEMMA 9. *Let $A \in K(H)$, $A \neq 0$. Then there exists an operator $C \in B_1(H)$ such that $C \underset{*}{\leqslant} A$.*

Proof. It is known (see for example [1]) that $A \in K(H)$ if and only if $A^* \in K(H)$. Also, $A \in K(H)$ if and only if $A^*A \in K(H)$. Suppose that for some nonzero $A \in K(H)$ there is no such $C \in B_1(H)$ that $C \underset{*}{\leqslant} A$. It follows from Lemma 7 that positive operator AA^* has no nonzero eigenvalues. Since $\|AA^*\|$ is an eigenvalue of AA^*, it follows that $\|AA^*\| = 0$ and therefore $AA^* = 0$. Also, $\|A^*\|^2 = \|AA^*\|$, so $A^* = 0$, hence $A = 0$, a contradiction. □

From now on let H be an infinite dimensional complex Hilbert space and assume that $\phi\colon K(H) \to K(H)$ is a bijective map such that for every pair $A,\ B \in K(H)$ we have

$$A \underset{*}{\leqslant} B \text{ if and only if } \phi(A) \underset{*}{\leqslant} \phi(B).$$

In order to prove that ϕ preserves rank-one operators we will need the following auxiliary result.

LEMMA 10. *The operator $B \in K(H)$ is of rank one if and only if $B \neq 0$ and for every $A \in K(H)$ where $A \underset{*}{\leqslant} B$ it follows that $A = 0$ or $A = B$.*

Proof. Let $B \in B_1(H)$ and suppose $A \underset{*}{\leqslant} B, A \in K(H)$. Clearly, $0 \underset{*}{\leqslant} B$ and $B \underset{*}{\leqslant} B$. By Lemma 4 it follows that A and B have the following matrix representations:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} A_1 & 0 \\ 0 & B_1 \end{bmatrix}.$$

Suppose that $A \neq 0$. If $B_1 \neq 0$, then rank $B \geqslant 2$. So $B_1 = 0$ and hence $A = B$.

Conversely, let $B \neq 0$ and suppose that for every $A \in K(H)$ where $A \underset{*}{\leqslant} B$ we have $A = 0$ or $A = B$. Assume that rank $B \geqslant 2$. Then there exists an operator $C \in B_1(H)$ such that $C \underset{*}{\leqslant} B$. Since $C \neq B$, we obtain a contradiction. □

LEMMA 11. *Let $B \in K(H)$. Then $B \in B_1(H)$ if and only if $\phi(B) \in B_1(H)$.*

Proof. The operator $B = 0$ is the only operator with the property that $A \underset{*}{\leqslant} B$ implies $A = B$. So $\phi(0) = 0$. Let $B \in B_1(H)$. By Lemma 10 and since ϕ preserves the order $\underset{*}{\leqslant}$ it follows that for every $\phi(A) \in K(H)$ where $\phi(A) \underset{*}{\leqslant} \phi(B)$ we have $\phi(A) = 0$ or $\phi(A) = \phi(B)$. Again using Lemma 10 we may conclude that $\phi(B) \in B_1(H)$.

The converse implication follows from the fact that ϕ^{-1} also preserves the order $\underset{*}{\leqslant}$. □

Let us now recall the singular value decomposition for compact operators in $B(H)$, see for example [6, 10].

DEFINITION 12. Let $A \in K(H)$. Then there exist orthonormal sequences $\{v_j\}$ and $\{u_j\}$ in H such that

$$Av_j = \sigma_j u_j, \quad A^* u_j = \sigma_j v_j.$$

Here σ_j are positive real values which are called *singular values* of A.
Given an arbitrary $x \in H$ we have

$$Ax = \sum_j \sigma_j \langle x, v_j \rangle u_j,$$

where the series converges in the norm topology on H. Then

$$A = \sum_j \sigma_j (u_j \otimes v_j^*)$$

is called a *singular value decomposition* of A.

Note that A is of the finite rank k if and only if its singular value decomposition contains exactly k nonzero summands.

With the next lemma we will characterize the rank two operators in $K(H)$.

LEMMA 13. *The operator $A \in K(H)$ is of rank two if and only if $A \notin \{0\} \cup B_1(H)$ and for $C \in K(H)$, $C \neq A$, $C \underset{*}{\leqslant} A$ it follows that $C \in \{0\} \cup B_1(H)$.*

Proof. First let us assume that rank $A = 2$ and let $C \underset{*}{\leqslant} A$ for $C \in K(H)$, $C \neq A$. By Lemma 4, A and C have the following matrix representations:

$$C = \begin{bmatrix} C_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} C_1 & 0 \\ 0 & A_1 \end{bmatrix}.$$

Suppose that rank $C > 1$. Since $C \neq A$, it follows that $A_1 \neq 0$, hence rank $A > 2$, a contradiction.

Conversely, let rank $A \geqslant 2$ and assume that for every $C \in K(H)$, where $C \neq A$ and $C \underset{*}{\leqslant} A$, it follows that $C \in \{0\} \cup B_1(H)$. If rank $A > 2$ then there exist orthonormal sets of vectors $\{u_j\}$ and $\{v_j\}$ such that

$$A = \sum_j \sigma_j (u_j \otimes v_j^*),$$

where $\sigma_j \neq 0$ at least for $j = 1,2,3$. Now, take for example the operator

$$C = \sum_{j=1}^{2} \sigma_j (u_j \otimes v_j^*).$$

We may check that $C^*C = C^*A$ and $CC^* = AC^*$. It follows that $C \underset{*}{\leqslant} A$. Note that $C \neq A$ and rank $C = 2$. This is a contradiction hence rank $A = 2$. $\square$

The following lemma may be proved by induction in the same way as Lemma 13.

LEMMA 14. *The operator $A \in K(H)$ is of rank n if and only if rank $A \geqslant n$ and for $C \in K(H)$, $C \neq A$, $C \underset{*}{\leqslant} A$ it follows that rank $C \leqslant n-1$.*

LEMMA 15. *Let $A \in K(H)$. We have rank $A = n$ if and only if rank $\phi(A) = n$.*

Proof. Let $A \in K(H)$. Then rank $A = 1$ if and only if rank $\phi(A) = 1$. Suppose that the result holds true for every $A \in K(H)$ with rank $A < n$. Suppose rank $A = n$, $n > 1$. First note that then rank $\phi(A) \geqslant n$. Also, by Lemma 14 we may conclude that for every $C \in K(H)$, $C \neq A$, $C \underset{*}{\leqslant} A$ it follows that rank $C \leqslant n-1$. Since ϕ is bijective and preserves the order $\underset{*}{\leqslant}$ in both directions, it follows that for every $\phi(C) \in K(H)$ where $\phi(C) \neq \phi(A)$ and $\phi(C) \underset{*}{\leqslant} \phi(A)$ we have rank $\phi(C) \leqslant n-1$. By Lemma 14 we conclude that rank $\phi(A) = n$.

The inverse implication follows from the fact that ϕ^{-1} also preserves the order $\underset{*}{\leqslant}$. □

LEMMA 16. *Let $A, B \in K(H)$ with rank $A = 1$ and rank $B = 2$. Suppose $B = \alpha_1 u_1 \otimes v_1^* + \alpha_2 u_2 \otimes v_2^*$ is the singular value decomposition of B with singular values α_1, α_2 and $\alpha_1 \neq \alpha_2$. Then $A \underset{*}{\leqslant} B$ if and only if $A = \alpha_1 u_1 \otimes v_1^*$ or $A = \alpha_2 u_2 \otimes v_2^*$.*

Proof. If $A = \alpha_1 u_1 \otimes v_1^*$ or $A = \alpha_2 u_2 \otimes v_2^*$, then we have $A^*A = A^*B$ and $AA^* = BA^*$ and hence $A \underset{*}{\leqslant} B$.

Conversely, let $A \underset{*}{\leqslant} B$. So, $A^*A = A^*B$ and $AA^* = BA^*$. Let $A = \gamma z \otimes w^*$ be the singular value decomposition of A. Hence $\gamma > 0$ and $\|z\| = \|w\| = 1$. From

$$A^*A = (\gamma w \otimes z^*)(\gamma z \otimes w^*) = \gamma^2 w \otimes w^*$$

and

$$A^*B = (\gamma w \otimes z^*)(\alpha_1 u_1 \otimes v_1^* + \alpha_2 u_2 \otimes v_2^*) = \gamma \alpha_1 \langle u_1, z \rangle w \otimes v_1^* + \gamma \alpha_2 \langle u_2, z \rangle w \otimes v_2^*$$

we obtain that

$$\gamma \langle x, w \rangle w = \alpha_1 \langle u_1, z \rangle \langle x, v_1 \rangle w + \alpha_2 \langle u_2, z \rangle \langle x, v_2 \rangle w \tag{2}$$

for every $x \in H$. Suppose $w = \delta_1 v_1 + \delta_2 v_2 + \delta_3 v_3$ where $v_3 \in \{v_1, v_2\}^{\perp}$ is a nonzero vector and $\delta_1, \delta_2, \delta_3 \in \mathbb{C}$ with $\delta_3 \neq 0$. For $x = v_3$ it follows by the equation (2) that $\gamma \delta_3 \langle v_3, v_3 \rangle = 0$ and hence $\delta_3 = 0$, a contradiction. We may conclude that there exist $\delta_1, \delta_2 \in \mathbb{C}$ such that

$$w = \delta_1 v_1 + \delta_2 v_2.$$

Let $x = v_1$. From the equation (2) we get $\gamma \delta_1 = \alpha_1 \langle u_1, z \rangle$ and hence, since γ is nonzero, $\delta_1 = \frac{\alpha_1 \langle u_1, z \rangle}{\gamma}$. Let now $x = v_2$. Then $\delta_2 = \frac{\alpha_2 \langle u_2, z \rangle}{\gamma}$. It follows that

$$\gamma w = \alpha_1 \langle u_1, z \rangle v_1 + \alpha_2 \langle u_2, z \rangle v_2. \tag{3}$$

By using the second equation $AA^* = BA^*$, we obtain the following equation

$$\gamma \langle x,z\rangle z = \alpha_1 \langle w,v_1\rangle \langle x,z\rangle u_1 + \alpha_2 \langle w,v_2\rangle \langle x,z\rangle u_2$$

which holds for every $x \in H$. It follows that

$$\gamma z = \alpha_1 \langle w,v_1\rangle u_1 + \alpha_2 \langle w,v_2\rangle u_2. \tag{4}$$

Denote $\beta_1 = \frac{\alpha_1\langle w,v_1\rangle}{\gamma}$ and $\beta_2 = \frac{\alpha_2\langle w,v_2\rangle}{\gamma}$. So, $z = \beta_1 u_1 + \beta_2 u_2$. From the equation (3) we get $\gamma w = \alpha_1 \overline{\beta_1} v_1 + \alpha_2 \overline{\beta_2} v_2$ and hence

$$w = \frac{\alpha_1}{\gamma}\overline{\beta_1} v_1 + \frac{\alpha_2}{\gamma}\overline{\beta_2} v_2. \tag{5}$$

Using the equation (4) we obtain

$$\gamma z = \frac{\alpha_1^2}{\gamma}\overline{\beta_1} u_1 + \frac{\alpha_2^2}{\gamma}\overline{\beta_2} u_2.$$

Since $z = \beta_1 u_1 + \beta_2 u_2$ and vectors u_1 and u_2 are orthogonal, we obtain $\gamma\beta_1 = \frac{\alpha_1^2}{\gamma}\overline{\beta_1}$ and $\gamma\beta_2 = \frac{\alpha_2^2}{\gamma}\overline{\beta_2}$. The first equation yields that $\beta_1 = c\overline{\beta_1}$ where $c > 0$, so $\beta_1 \in \mathbb{R}$. Similarly, $\beta_2 \in \mathbb{R}$.

Suppose first that $\beta_1 = 0$. Then $z = \beta_2 u_2$ and by $\|z\| = \|u_2\| = 1$ we may conclude that $\beta_2 = 1$ or $\beta_2 = -1$. It follows that $\alpha_2^2 = \gamma^2$ and since $\alpha_2, \gamma > 0$ we have $\alpha_2 = \gamma$. Also, from the equation (5) we get $w = \beta_2 v_2$. We may conclude that

$$A = \gamma z \otimes w^* = \alpha_2 \beta_2^2 u_2 \otimes v_2^* = \alpha_2 u_2 \otimes v_2^*.$$

Suppose now that $\beta_2 = 0$. We may similarly conclude that $A = \alpha_1 u_1 \otimes v_1^*$. Finally, suppose $\beta_1 \neq 0$ and $\beta_2 \neq 0$. It follows that $\alpha_1^2 = \gamma^2 = \alpha_2^2$. Since α_1 and α_2 are positive, we have $\alpha_1 = \alpha_2$, a contradiction. $\square$

From the proof of Lemma 16 we can conclude also the following.

COROLLARY 17. *Let $A,B \in K(H)$ such that rank $A = 1$ and rank $B = 2$. Suppose that $A = \gamma z \otimes w^*$ and $B = \alpha(u_1 \otimes v_1^* + u_2 \otimes v_2^*)$ are the singular value decompositions of A and B. If $A \underset{*}{\leqslant} B$, then $\alpha = \gamma$.*

Now we can tell more about the map ϕ.

LEMMA 18. *Let $P \in K(H)$ be a self-adjoint idempotent operator of rank two and let $\phi(P) = \alpha_1 u_1 \otimes v_1^* + \alpha_2 u_2 \otimes v_2^*$ be the singular value decomposition of $\phi(P)$ with singular values α_1 and α_2. Then $\alpha_1 = \alpha_2$. Moreover, if $R \in K(H)$ is another self-adjoint idempotent operator of rank two where $\phi(R) = \beta(a_1 \otimes b_1^* + a_2 \otimes b_2^*)$ is the singular value decomposition of $\phi(R)$ with singular value β, then $\beta = \alpha_1$.*

Proof. Let P be a self-adjoint idempotent of rank two and let $\phi(P) = \alpha_1 u_1 \otimes v_1^* + \alpha_2 u_2 \otimes v_2^*$ be the singular value decomposition of $\phi(P)$ with $\alpha_1 \neq \alpha_2$. For a rank one operator A in $K(H)$ it follows by Lemma 16 that if $A \overset{*}{\leqslant} \phi(P)$, then $A = \alpha_1 u_1 \otimes v_1^*$ or $A = \alpha_2 u_2 \otimes v_2^*$. Since ϕ preserves the order $\overset{*}{\leqslant}$ in both directions, there exist only two rank one operators $Q_i,\ i \in \{1,2\}$, such that $Q_i \overset{*}{\leqslant} P$. Here $Q_i = \phi^{-1}(\alpha_i u_i \otimes v_i^*)$. This is a contradiction since P is a self-adjoint idempotent of rank 2 and hence for every self-adjoint idempotent Q of rank one with $\operatorname{Im} Q \subset \operatorname{Im} P$ it follows $Q \overset{*}{\leqslant} P$.

Suppose now $\phi(P) = \alpha(u_1 \otimes v_1^* + u_2 \otimes v_2^*)$ is the singular value decomposition of $\phi(P)$ and let R be a self-adjoint idempotent operator of rank two where $\phi(R) = \beta(a_1 \otimes b_1^* + a_2 \otimes b_2^*)$ is the singular value decomposition of $\phi(R)$ with singular value β. Then $P = e_1 \otimes e_1^* + e_2 \otimes e_2^*$ and $R = f_1 \otimes f_1^* + f_2 \otimes f_2^*$ for some orthonormal sets of vectors $\{e_1, e_2\}$ and $\{f_1, f_2\}$. It follows that $e_i \otimes e_i^* \overset{*}{\leqslant} P$ and $f_i \otimes f_i^* \overset{*}{\leqslant} R,\ i \in \{1,2\}$. Let $\phi(e_2 \otimes e_2^*) = \gamma s_1 \otimes s_2^*$ and $\phi(f_1 \otimes f_1^*) = \delta z_1 \otimes z_2^*$ be the singular value decompositions of $\phi(e_2 \otimes e_2^*)$ and $\phi(f_1 \otimes f_1^*)$. By Corollary 17 we have $\alpha = \gamma$ and $\delta = \beta$. There exists an idempotent self-adjoint operator M of rank two such that $\{e_2, f_1\} \subset \operatorname{Im} M$. Since ϕ preserves the order $\overset{*}{\leqslant}$, we have $\phi(e_2 \otimes e_2^*) \overset{*}{\leqslant} \phi(M)$ and $\phi(f_1 \otimes f_1^*) \overset{*}{\leqslant} \phi(M)$. Let $\phi(M) = \theta(m_1 \otimes n_1^* + m_2 \otimes n_2^*)$ be the singular value decomposition of $\phi(M)$ with singular value θ. By Corollary 17 it follows that $\alpha = \theta = \beta$. □

The next result follows directly from the previous two lemmas.

COROLLARY 19. *Let $P, Q \in K(H)$, $P \neq Q$, be self-adjoint idempotent operators of rank one. If $\phi(P) = \alpha s_1 \otimes s_2^*$ and $\phi(Q) = \beta z_1 \otimes z_2^*$ are the singular value decompositions of $\phi(P)$ and $\phi(Q)$, then $\alpha = \beta$.*

LEMMA 20. *Let $P \in K(H)$ be a self-adjoint idempotent operator and let $\phi(P) = \alpha_1 u_1 \otimes v_1^* + \alpha_2 u_2 \otimes v_2^* + \ldots + \alpha_n u_n \otimes v_n^*$ be the singular value decomposition of $\phi(P)$. Then $\alpha_1 = \alpha_2 = \ldots = \alpha_n = \alpha$. If $Q \in K(H)$ is another self-adjoint idempotent operator where $\phi(Q) = \beta(m_1 \otimes n_1^* + m_2 \otimes n_2^* + \ldots + m_k \otimes n_k^*)$ is the singular value decomposition of $\phi(Q)$, then $\alpha = \beta$.*

Proof. Let P be a self-adjoint idempotent operator and let $\phi(P) = \alpha_1 u_1 \otimes v_1^* + \alpha_2 u_2 \otimes v_2^* + \ldots + \alpha_n u_n \otimes v_n^*$ be the singular value decomposition. Suppose there exist $i, j \in \{1, 2, \ldots, n\}$ such that $\alpha_i \neq \alpha_j$. Since $\alpha_i u_i \otimes v_i^* \overset{*}{\leqslant} \phi(P)$ and $\alpha_j u_j \otimes v_j^* \overset{*}{\leqslant} \phi(P)$, we conclude that $\phi^{-1}(\alpha_i u_i \otimes v_i^*) \overset{*}{\leqslant} P$ and $\phi^{-1}(\alpha_j u_j \otimes v_j^*) \overset{*}{\leqslant} P$. By Lemma 5 and since ϕ^{-1} also preserves the rank, it follows that $\phi^{-1}(\alpha_i u_i \otimes v_i^*)$ and $\phi^{-1}(\alpha_j u_j \otimes v_j^*)$ are self-adjoint idempotent operators of rank one. By Corollary 19 we may conclude $\alpha_i = \alpha_j$, a contradiction.

Let $P, Q \in K(H)$ be self-adjoint idempotent operators and let $\phi(P) = \alpha(u_1 \otimes v_1^* + u_2 \otimes v_2^* + \ldots + u_n \otimes v_n^*)$, $\phi(Q) = \beta(m_1 \otimes n_1^* + m_2 \otimes n_2^* + \ldots + m_k \otimes n_k^*)$ be their singular value decompositions, respectively. The proof that $\alpha = \beta$ is similar to the proof of Lemma 18, where P and Q are both of rank two. □

COROLLARY 21. *For every self-adjoint idempotent $P \in K(H)$ we obtain the same scalar α in the singular value decomposition of $\phi(P)$.*

Proof of Theorem. By Corollary 21 we can assume that a scalar in the singular value decomposition of $\phi(P)$ is equal to one for every self-adjoint idempotent $P \in K(H)$. In addition, from now on we will assume that $\phi: K(H) \to K(H)$ also is additive and continuous. Since ϕ is bijective and additive, it follows that ϕ^{-1} is also additive. Hilbert space H is separable, so there exists an orthonormal basis $\{e_1, e_2, \ldots\}$ in H. There also exist $u_i, v_i \in H$, $\|u_i\| = \|v_i\| = 1$, $i \in \mathbb{N}$, such that $\phi(e_i \otimes e_i^*) = u_i \otimes v_i^*$.

Step 1. We will show that u_i, u_j are orthogonal and that v_i, v_j are orthogonal for $i \neq j$. Let $A = u_i \otimes v_i^* + u_j \otimes v_j^*$. Since ϕ is additive, it follows that

$$\phi\left(e_i \otimes e_i^* + e_j \otimes e_j^*\right) = u_i \otimes v_i^* + u_j \otimes v_j^* = A.$$

Recall that ϕ preserves the rank, hence u_i and u_j are linearly independent and also v_i and v_j are linearly independent. By Lemma 20, the singular value decomposition for $\phi\left(e_i \otimes e_i^* + e_j \otimes e_j^*\right)$ is of the form $s_i \otimes z_i^* + s_j \otimes z_j^*$, where s_i, s_j are orthonormal and z_i, z_j are orthonormal. So, $\phi\left(e_i \otimes e_i^* + e_j \otimes e_j^*\right) = A$ is a partial isometry. Note that $\mathrm{Ker} A = (\mathrm{Im} A^*)^{\perp} = (\mathrm{Lin}\{v_i, v_j\})^{\perp}$. A partial isometry is isometric on the orthogonal complement of its kernel so the restriction of A^*A to $\mathrm{Lin}\{v_i, v_j\}$ is the identity operator. Hence, $A^*Av_i = v_i$ and $A^*Av_j = v_j$. Also, $A^* = v_i \otimes u_i^* + v_j \otimes u_j^*$, therefore

$$A^*Av_i = v_i + \langle u_j, u_i\rangle \langle v_i, v_j\rangle v_i + \langle u_i, u_j\rangle v_j + \langle v_i, v_j\rangle v_j$$

and hence

$$0 = \langle u_j, u_i\rangle \langle v_i, v_j\rangle v_i + \left(\langle u_i, u_j\rangle + \langle v_i, v_j\rangle\right) v_j.$$

Since v_i and v_j are linearly independent we may conclude that $\langle u_j, u_i\rangle \langle v_i, v_j\rangle = 0$, $\langle u_i, u_j\rangle + \langle v_i, v_j\rangle = 0$, and hence $\langle u_i, u_j\rangle = \langle v_i, v_j\rangle = 0$.

Step 2. We will show that both sequences $\{u_i\}$ and $\{v_i\}$ are orthonormal bases in H. Suppose first that both $\{u_i\}$ and $\{v_i\}$ are not orthonormal bases in H. So, there exist x_0 and y_0, $\|x_0\| = \|y_0\| = 1$, such that x_0 is orthogonal to $\{u_i\}$ and y_0 is orthogonal to $\{v_i\}$. Let $i \in \mathbb{N}$ be arbitrary and let us denote $A = u_i \otimes v_i^* + 2x_0 \otimes y_0^*$. Then A is a rank two operator with a singular value decomposition $u_i \otimes v_i^* + 2x_0 \otimes y_0^*$. Assume that $B \overset{*}{\leqslant} A$ is a rank one operator. Lemma 16 yields that either $B = u_i \otimes v_i^*$ or $B = 2x_0 \otimes y_0^*$. Also, $\phi^{-1}(A)$ is a rank two operator. Let $\mu_1 a_1 \otimes b_1^* + \mu_2 a_2 \otimes b_2^*$ be a singular value decomposition of $\phi^{-1}(A)$. Since ϕ preserves the order in both directions, there are exactly two rank one operators C such that $C \overset{*}{\leqslant} \phi^{-1}(A)$. Also, since $\phi^{-1}(u_i \otimes v_i^*) = e_i \otimes e_i^*$ and ϕ^{-1} is injective, we may assume without loss of generality that $e_i \otimes e_i^* = \mu_1 a_1 \otimes b_1^*$ and $\phi^{-1}(2x_0 \otimes y_0^*) = \mu_2 a_2 \otimes b_2^*$. We may conclude that a_2 and b_2 are orthogonal to e_i. This holds for every $i \in \mathbb{N}$, a contradiction.

Suppose now that only one of the sequences, for example $\{u_i\}$, is not a basis in H. So, there exist x_0, $\|x_0\| = 1$, such that x_0 is orthogonal to $\{u_i\}$. As before, let us denote $A = u_i \otimes v_i^* + 2x_0 \otimes v_j^*$ where $j \in \mathbb{N}$ and $j \neq i$. We obtain a contradiction in a similar way as before.

Step 3. We may assume without loss of generality that $\phi(e_i \otimes e_i^*) = e_i \otimes e_i^*$. Since sequences $\{u_i\}$ and $\{v_i\}$ are orthonormal bases in H, there exist unitary operators $U, V \in B(H)$ with $U(u_i) = e_i$ and $V^*(v_i) = e_i$, $i \in \mathbb{N}$. If we define $\psi(A) = U\phi(A)V$, then $\psi(e_i \otimes e_i^*) = e_i \otimes e_i^*$. So, we may assume that $\phi(e_i \otimes e_i^*) = e_i \otimes e_i^*$.

Step 4. For any n denote $P_n = \sum_{i=1}^n e_i \otimes e_i^*$. We will show that $\phi(P_nK(H)P_n) = P_nK(H)P_n$. Let $x \otimes y^* \in P_nK(H)P_n$ be a rank one operator with $\|x\| = \|y\| = 1$. Our aim is to show that $\phi(x \otimes y^*) \in P_nK(H)P_n$. Suppose $j \geqslant n+1$. Then $A = x \otimes y^* + 2e_j \otimes e_j^*$ is a rank two operator. Assume that B is a rank one operator and that $B \overset{*}{\leqslant} A$. Then by Lemma 16, B is either $x \otimes y^*$ or $2e_j \otimes e_j^*$. Recall that ϕ is additive. So, $\phi(2e_j \otimes e_j^*) = 2\phi\left(e_j \otimes e_j^*\right) = 2e_j \otimes e_j^*$ and hence $\phi(A) = \phi(x \otimes y^*) + 2e_j \otimes e_j^*$. The operator $\phi(A)$ is of rank two. Let $\mu_1 u_1 \otimes v_1^* + \mu_2 u_2 \otimes v_2^*$ be the singular value decomposition of $\phi(A)$. Then $\mu_i u_i \otimes v_i^* \overset{*}{\leqslant} \phi(A)$, $i \in \{1,2\}$. Since ϕ preserves the order, also $2e_j \otimes e_j^* \overset{*}{\leqslant} \phi(A)$, $\phi(x \otimes y^*) \overset{*}{\leqslant} \phi(A)$ and therefore we may assume without loss of generality that $2e_j \otimes e_j^* = \mu_1 u_1 \otimes v_1^*$. Hence $\phi(x \otimes y^*) = \mu_2 u_2 \otimes v_2^*$. Note that $\langle e_j, u_2\rangle = \langle e_j, v_2\rangle = 0$. This equality holds for every $j \geqslant n+1$, hence $\phi(x \otimes y^*) \in P_nK(H)P_n$.

It is straightforward to show that for $\alpha x \otimes y^* \in P_nK(H)P_n$, where $\alpha > 0$, $\alpha \neq 1$, and $\|x\| = \|y\| = 1$, we have $\phi(\alpha x \otimes y^*) \in P_nK(H)P_n$. By using the fact that ϕ is additive we may conclude that if $A \in P_nK(H)P_n$, it follows $\phi(A) \in P_nK(H)P_n$. We have proved that $\phi(P_nK(H)P_n) \subset P_nK(H)P_n$. Recall that ϕ^{-1} is also additive. Since ϕ preserves the order in both directions, we may conclude that $\phi(P_nK(H)P_n) = P_nK(H)P_n$.

Step 5. We will determine the restrictions of ϕ on finite dimensional spaces $P_nK(H)P_n$. Let $n_0 \in \mathbb{N}, n_0 \geqslant 3$, be fixed. The set $P_{n_0}K(H)P_{n_0}$ can be identified with M_{n_0} according to the basis $\{e_1, \ldots, e_{n_0}\}$. Recall that $\phi(e_i \otimes e_i^*) = e_i \otimes e_i^*$, $i \in \mathbb{N}$. The restriction of ϕ to $P_{n_0}K(H)P_{n_0}$ can be considered as a bijective, additive and continuous map $\phi_{n_0} : M_{n_0} \to M_{n_0}$ which preserves the star order in both directions and sends the identity matrix to itself. To present its form let us first state the following result of Guterman ([4], Theorem 3.1).

An additive map $T : M_{n_0} \to M_{n_0}$ preserves the star order in one direction (i.e., $A \overset{*}{\leqslant} B$ implies $T(A) \overset{*}{\leqslant} T(B)$ for every $A, B \in M_{n_0}$) if and only if either $T \equiv 0$, or there exist unitary matrices $U_{n_0}, V_{n_0} \in M_{n_0}$ and a nonzero $\alpha \in \mathbb{C}$, such that T has one of the following forms:

(i) $T(A) = \alpha U_{n_0} A V_{n_0}$ for all $A \in M_{n_0}$, or

(ii) $T(A) = \alpha U_{n_0} A^t V_{n_0}$ for all $A \in M_{n_0}$, or

(iii) $T(A) = \alpha U_{n_0} A^* V_{n_0}$ for all $A \in M_{n_0}$, or

(iv) $T(A) = \alpha U_{n_0} \overline{A} V_{n_0}$ for all $A \in M_{n_0}$, where A^t denotes the transpose of A, and $\overline{A}$ is the matrix obtained from A by taking complex conjugate values of its entries.

Applying this result to ϕ_{n_0} we will specify the structure of matrices U_{n_0}, V_{n_0} in this particular case. Since ϕ_{n_0} is injective and additive and since $\phi_{n_0}(e_i \otimes e_i^*) = e_i \otimes e_i^*$, we have $\alpha U_{n_0} e_i \otimes e_i^* V_{n_0} = e_i \otimes e_i^*$ for every $i \in \{1,2,...,n_0\}$. It follows that unitary matrices U_{n_0} and V_{n_o} are diagonal and that $|\alpha| = 1$. Since αU_{n_0} is a unitary matrix, we may set $\alpha = 1$ and change U_{n_0} acordingly. Also, $\phi_{n_0}(I) = I$ and hence $U_{n_0} V_{n_0} = I$, i.e., $V_{n_0} = U_{n_0}^*$. We conclude that there exits a diagonal and unitary matrix $U_{n_0} \in M_{n_0}$ such that $\phi_{n_0}(A) = U_{n_0} A U_{n_0}^*$ for every $A \in M_{n_0}$, or $\phi_{n_0}(A) = U_{n_0} A^t U_{n_0}^*$ for every $A \in M_{n_0}$, or $\phi_{n_0}(A) = U_{n_0} A^* U_{n_0}^*$ for every $A \in M_{n_0}$, or $\phi_{n_0}(A) = U_{n_0} \overline{A} U_{n_0}^*$ for every $A \in M_{n_0}$. Let us note that the absolute values of all diagonal elements of matrix U_{n_0} are equal to 1.

Step 6. Let us show that matrices U_{n_0} of different sizes are well related. Let $n_0 \in \mathbb{N}$, $n_0 \geqslant 3$, be fixed and suppose that $\phi_{n_0}(A) = U_{n_0} A U_{n_0}^*$ for every $A \in M_{n_0}$. Since $\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \in M_{n_0+1}$ for every $A \in M_{n_0}$, we may conclude that $\phi_{n_0+1}(B) = U_{n_0+1} B U_{n_0+1}^*$ for every $B \in M_{n_0+1}$. So, the restriction of ϕ_{n_0+1} to M_{n_0} equals ϕ_{n_0}. Let $U_{n_0} = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{n_0})$ and $U_{n_0+1} = \operatorname{diag}(\mu_1, \mu_2, \ldots, \mu_{n_0}, \lambda_{n_0+1})$. Since $|\lambda_1| = |\mu_1| = 1$, we may assume without loss of generality that $\lambda_1 = \mu_1 = 1$. Let

$$P_1 = \begin{bmatrix} \frac{1}{n_0} & \cdots & \frac{1}{n_0} \\ \vdots & \ddots & \vdots \\ \frac{1}{n_0} & \cdots & \frac{1}{n_0} \end{bmatrix} \in M_{n_0} \quad \text{and} \quad P_2 = \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix} \in M_{n_0+1}.$$

The upper left $n_0 \times n_0$ block of $U_{n_0+1} P_2 U_{n_0+1}^*$ equals the matrix $U_{n_0} P_1 U_{n_0}^*$, so $\lambda_i = \mu_i$ for every $i \in \{1,2,...,n_0\}$ and therefore $U_{n_0+1} = \begin{bmatrix} U_{n_0} & 0 \\ 0 & \lambda_{n_0+1} \end{bmatrix}$.

Step 7. We first consider the case when $\phi_3(A) = U_3 A U_3^*$. Let us assume that the restriction ϕ_3 of ϕ to $P_3 K(H) P_3$ is of the following form $\phi_3(A) = U_3 A U_3^*$ for every $A \in P_3 K(H) P_3$. It follows that $\phi_{n_0}(A) = U_{n_0} A U_{n_0}^*$ for every $A \in P_{n_0} K(H) P_{n_0}$ and every $n_0 \in \mathbb{N}$, $n_0 \geqslant 3$. As before, U_{n_0} is a diagonal matrix $\operatorname{diag}(\lambda_1, \ldots, \lambda_{n_0})$, $|\lambda_i| = 1$ for every $i \in \{1,2,...,n_0\}$. We define an operator $U \colon H \to H$ in the following way: $U e_i = \lambda_i e_i$, $i \in \mathbb{N}$. Then U is a unitary operator and $\phi(A) = UAU^*$ for every A for which there exists $n \in \{3,4,5,...\}$ such that $A \in P_n K(H) P_n$. Without loss of generality we may assume that $\phi(A) = A$ for every A for which there exists $n \in \{3,4,5,...\}$ such that $A \in P_n K(H) P_n$.

Step 8. We will show that $\phi(P) = P$ for every self-adjoint idempotent $P \in K(H)$, when ϕ is as in Step 7. Let $Q = x \otimes x^*$ be a rank one self-adjoint idempotent where $x \notin \operatorname{Lin}\{e_j : 1 \leqslant j \leqslant n\}$ for every $n \in \mathbb{N}$. Recall that $\{e_1, e_2, \ldots\}$ is an orthonormal basis in H, therefore it easily follows that $\|Q - P_n Q P_n\| \to 0$ as $n \to \infty$. Since $P_n Q P_n \in P_n K(H) P_n$, we may conclude that $\phi(P_n Q P_n) = P_n Q P_n$ for every $n \in \mathbb{N}$. It follows by the continuity of ϕ that $\phi(Q) = Q$ where $Q = x \otimes x^*$ and $\|x\| = 1$. So, $\phi(P) = P$ for every rank one self-adjoint idempotent P and by the additivity of ϕ we have $\phi(P) = P$ for every self-adjoint idempotent $P \in K(H)$.

Step 9. We consider also the other three cases, when $\phi_3(A) = U_3A^*U_3^*$, $\phi_3(A) = U_3A^tU_3^*$, or $\phi_3(A) = U_3\overline{A}U_3^*$. Assume that the restriction ϕ_3 of ϕ to $P_3K(H)P_3$ is of the form $\phi_3(A) = U_3A^*U_3^*$ for every $A \in P_3K(H)P_3$, then similarly as in Step 7 there is a unitary operator U such that $\phi(A) = UA^*U^*$ for every A for which there exists $n \in \{3,4,5,...\}$ such that $A \in P_nK(H)P_n$, and also that $\phi(P) = P$ for every self-adjoint idempotent $P \in K(H)$. Finally, if we suppose that the restriction ϕ_3 of ϕ to $P_3K(H)P_3$ is of the form $\phi_3(A) = U_3A^tU_3^*$ or of the form $\phi_3(A) = U_3\overline{A}U_3^*$, then similarly as in Step 7 there is an antiunitary operator U such that $\phi(A) = UAU^*$ for every A from $P_nK(H)P_n$ for some $n \geqslant 3$, or $\phi(A) = UA^*U^*$ for every A from $P_nK(H)P_n$ for some $n \geqslant 3$. As in the first two cases we also obtain that $\phi(P) = P$ for every self-adjoint idempotent $P \in K(H)$.

So, it remains to characterize the map $\phi : K(H) \to K(H)$ with the properties stated in the Theorem and with an additional property that $\phi(P) = P$ for every self-adjoint idempotent $P \in K(H)$.

Step 10. We will determine map ϕ on finite rank operators from $K(H)$. Let $A_0 \in K(H)$ be an arbitrary finite rank operator. Then there exists a self-adjoint idempotent $P \in K(H)$ with rank $n \geqslant 3$, such that $A_0 \in PK(H)P$. In the same way as for P_n we can show that $\phi(PK(H)P) = PK(H)P$ and by the result of Guterman ([4], Theorem 3.1) that there exists diagonal unitary matrix U_P from M_n according to an appropriate basis, such that $\phi_P(A) = U_PAU_P^*$ for every $A \in M_n$, or $\phi_P(A) = U_PA^*U_P^*$ for every $A \in M_n$, or $\phi_P(A) = U_P\overline{A}U_P^*$ for every $A \in M_n$, or $\phi_P(A) = U_PA^tU_P^*$ for every $A \in M_n$. Since $\phi_P(Q_1) = Q_1$ for self-adjoint idempotent $Q_1 = \begin{bmatrix} \frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & \ddots & \vdots \\ \frac{1}{n} & \cdots & \frac{1}{n} \end{bmatrix} \in M_n$, it follows that $U_P = \alpha I$, $|\alpha| = 1$. So we can assume without loss of generality that $U_p = I$. From $\phi_P(Q_2) = Q_2$ for self-adjoint idempotent $Q_2 = \begin{bmatrix} \frac{1}{2} & \frac{i}{2} & 0 \\ \frac{-i}{2} & \frac{1}{2} & \\ & 0 & 0 \end{bmatrix} \in M_n$, it follows that $\phi(A_0) = A_0$ or $\phi(A_0) = A_0^*$. Suppose that for a map ϕ it holds $\phi(A_0) = A_0 \neq A_0^*$ and $\phi(B_0) = B_0^* \neq B_0$ for some finite rank operators $A_0, B_0 \in K(H)$. Then $\phi(A_0 + B_0) = \phi(A_0) + \phi(B_0) = A_0 + B_0^*$, a contradiction. So, $\phi(A) = A$ for every finite rank operator $A \in B(H)$, or $\phi(A) = A^*$ for every finite rank operator $A \in B(H)$.

Step 11. We will determine map ϕ on the whole $K(H)$. If Q is an arbitrary operator in $K(H)$, then there is a sequence $\{Q_n\}$ of operators of finite rank such that $\|Q_n - Q\| \to 0$ as $n \to \infty$. By the continuity of ϕ it follows that $\phi(Q) = Q$ for every $Q \in K(H)$ or $\phi(Q) = Q^*$ for every $Q \in K(H)$.

Taking into account assumptions about ϕ in Steps 7 and 9 we obtain that the following implication holds: if $\phi : K(H) \to K(H)$ is a bijective, additive and continuous map which preserves the star partial order in both directions, then there exist operators $U, V : H \to H$, which are both unitary or both antiunitary, and $\alpha \in \mathbb{C}$ such that $\phi(A) = \alpha UAV$ for every $A \in K(H)$ or $\phi(A) = \alpha UA^*V$ for every $A \in K(H)$. The inverse implication follows immediately from the definition of the star partial order. $\square$

3. On non-additive maps

It would be interesting to find the form of the map $\phi: K(H) \to K(H)$ without the assumptions of additivity and/or continuity. Let us present an example of a bijective non-additive map $\phi: K(H) \to K(H)$ which has more involved structure than additive ones. We will first recall the following lemma which follows from the singular value decomposition (see [7] and [9]).

LEMMA 22. *If $A \in M_n$ is nonzero, then there exists a unique decomposition, called Penrose decomposition,*

$$A = \sum_{j=1}^{k} t_j V_j$$

where $t_1 > t_2 > \dots > t_k > 0$ and $V_1, V_2, \dots, V_k$ are mutually orthogonal nonzero partial isometries.

Similarly, we may define Penrose decomposition for operators from $K(H)$. Let $A = \sum_j \sigma_j (u_j \otimes v_j^*)$ be a singular value decomposition of $A \in K(H)$. We reorder this sum, unifying the summands with the same σ_j, and obtain: $A = \sum_{\alpha>0} \alpha U_\alpha$. Here (by the definition of singular value decomposition) U_α is a partial isometry for every α and $U_\alpha U_\beta^* = U_\alpha^* U_\beta = 0$ for $\alpha \neq \beta$. (Note that almost all partial isometries U_α are zero.)

PROPOSITION 23. *Let $A, B \in K(H)$ have Penrose decompositions $A = \sum_{\alpha>0} \alpha U_\alpha$ and $B = \sum_{\beta>0} \beta V_\beta$. Then $A \underset{*}{\leqslant} B$ if and only if for every $\alpha > 0$ it holds that $U_\alpha \underset{*}{\leqslant} V_\alpha$.*

Proof. Let $A, B \in K(H)$ and $A \underset{*}{\leqslant} B$. So, $A^*A = A^*B$ and $AA^* = BA^*$. By using Penrose decomposition of A and multiplying equation $A^*A = A^*B$ from the left by U_α we get

$$\alpha^2 U_\alpha = \alpha U_\alpha U_\alpha^* B. \tag{6}$$

Also, multiplying the operator B from the right by V_β^* we have $BV_\beta^* = \beta V_\beta V_\beta^*$. Therefore, $\alpha U_\alpha V_\beta^* = \beta U_\alpha U_\alpha^* V_\beta V_\beta^*$. Using similarly the equation $AA^* = BA^*$, we get $\alpha U_\alpha^* V_\beta = \beta U_\alpha^* U_\alpha V_\beta^* V_\beta$. It follows that

$$\alpha^2 U_\alpha^* V_\beta = \beta U_\alpha^* \alpha U_\alpha V_\beta^* V_\beta = \beta^2 U_\alpha^* U_\alpha U_\alpha^* V_\beta V_\beta^* V_\beta = \beta^2 U_\alpha^* V_\beta.$$

If $\alpha \neq \beta$, we may conclude that $U_\alpha^* V_\beta = 0$. Similarly, $U_\alpha V_\beta^* = 0$.

Multiplying (6) from the left by U_α^*, $\alpha > 0$, we get

$$U_\alpha^* \alpha U_\alpha = U_\alpha^* U_\alpha U_\alpha^* B = U_\alpha^* B = \alpha U_\alpha^* V_\alpha.$$

Since $\alpha \neq 0$, it follows that $U_\alpha^* U_\alpha = U_\alpha^* V_\alpha$. Similarly, $U_\alpha U_\alpha^* = V_\alpha U_\alpha^*$. Therefore, $U_\alpha \underset{*}{\leqslant} V_\alpha$.

The reverse implication is trivial. □

Now we are in position to present an example of a non-additive, bijective transformation that preserves the star order in both directions (see also Legiša's result in [7]).

EXAMPLE 24. We define a map $T(f,g)\colon K(H) \to K(H)$ as follows. Let $f\colon (0,\infty) \to (0,\infty)$ be a bijective continuous map on the set of positive real numbers and let $g\colon (0,\infty) \to \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. For a zero operator let $T(f,g)(0) = 0$. If $A = \sum_{\alpha>0} \alpha V_\alpha$ is Penrose decomposition of a nonzero operator $A \in K(H)$, let

$$T(f,g)(A) = \sum_{\alpha>0} f(\alpha)g(\alpha)V_\alpha.$$

It easily follows from the previous proposition (see also [7]) that $T(f,g)$ is bijective, non-additive and preserves the star order in both directions.

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