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NUMERICAL RADIUS AND DISTANCE FROM UNITARY OPERATORS

CATALIN BADEA AND MICHEL CROUZEIX

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Abstract. Denote by $w(A)$ the numerical radius of a bounded linear operator A acting on Hilbert space. Suppose that A is invertible and that $w(A) \leqslant 1+\varepsilon$ and $w(A^{-1}) \leqslant 1+\varepsilon$ for some $\varepsilon \geqslant 0$. It is shown that $\inf\{\|A-U\| : U \text{ unitary}\} \leqslant c\varepsilon^{1/4}$ for some constant $c > 0$. This generalizes a result due to J.G. Stampfli, which is obtained for $\varepsilon = 0$. An example is given showing that the exponent $1/4$ is optimal. The more general case of the operator ρ-radius $w_\rho(\cdot)$ is discussed for $1 \leqslant \rho \leqslant 2$.

1. Introduction and statement of the results

Let H be a complex Hilbert space endowed with the inner product $\langle \cdot, \cdot \rangle$ and the associated norm $\|\cdot\|$. We denote by $\mathscr{B}(H)$ the C*-algebra of all bounded linear operators on H equipped with the operator norm

$$\|A\| = \sup\{\|Ah\| : h \in H, \|h\| = 1\}.$$

It is easy to see that unitary operators can be characterized as invertible contractions with contractive inverses, i.e. as operators A with $\|A\| \leqslant 1$ and $\|A^{-1}\| \leqslant 1$. More generally, if $A \in \mathscr{B}(H)$ is invertible then

$$\inf\{\|A-U\| : U \text{ unitary }\} = \max\left(\|A\| - 1, 1 - \frac{1}{\|A^{-1}\|}\right).$$

We refer to [6, Theorem 1.3] and [9, Theorem 1] for a proof of this equality using the polar decomposition of bounded operators. It also follows from this proof that if $A \in \mathscr{B}(H)$ is an invertible operator satisfying $\|A\| \leqslant r$ and $\|A^{-1}\| \leqslant r$ for some $r \geqslant 1$, then there exists a unitary operator $U \in \mathscr{B}(H)$ such that $\|A-U\| \leqslant r-1$.

The numerical radius of the operator A is defined by

$$w(A) = \sup\{|\langle Ah, h\rangle| : h \in H, \|h\| = 1\}.$$

Stampfli has proved in [8] that numerical radius contractivity of A and of its inverse A^{-1}, that is $w(A) \leqslant 1$ and $w(A^{-1}) \leqslant 1$, imply that A is unitary. We define a function $\psi(r)$ for $r \geqslant 1$ by

$$\psi(r) = \sup\{\|A\| : A \in \mathscr{B}(H), w(A) \leqslant r,\ w(A^{-1}) \leqslant r\},$$

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the supremum being also considered over all Hilbert spaces H. Then the conditions $w(A) \leqslant r$ and $w(A^{-1}) \leqslant r$ imply $\max(\|A\|-1, 1-\|A^{-1}\|^{-1}) \leqslant \max(\|A\|-1, \|A^{-1}\|-1) \leqslant \psi(r)-1$, hence the existence of a unitary operator U such that $\|A-U\| \leqslant \psi(r)-1$. We have the two-sided estimate

$$r+\sqrt{r^2-1} \leqslant \psi(r) \leqslant 2r.$$

The upper bound follows from the well-known inequalities $w(A) \leqslant \|A\| \leqslant 2w(A)$, while the lower bound is obtained by choosing $H = \mathbb{C}^2$ and

$$A = \begin{pmatrix} 1 & 2y \\ 0 & -1 \end{pmatrix} \quad \text{with } y = \sqrt{r^2-1},$$

in the definition of ψ. Indeed, we have $A = A^{-1}$, $w(A) = \sqrt{1+y^2} = r$, and $\|A\| = y+\sqrt{1+y^2} = r+\sqrt{r^2-1}$.

Our first aim is to improve the upper estimate.

THEOREM 1.1. *Let $r \geqslant 1$. Then*

$$\psi(r) \leqslant X(r) + \sqrt{X(r)^2-1}, \quad \text{with} \quad X(r) = r+\sqrt{r^2-1}. \tag{1}$$

The estimate given in Theorem 1.1 is more accurate than $\psi(r) \leqslant 2r$ for r close to 1, more precisely for $1 \leqslant r \leqslant 1.0290855\ldots$. It also gives $\psi(1) = 1$ (leading to Stampfli's result) and the following asymptotic estimate.

COROLLARY 1.2. *We have*

$$\psi(1+\varepsilon) \leqslant 1+\sqrt[4]{8\varepsilon}+O(\varepsilon^{1/2}), \quad \varepsilon \to 0.$$

Our second aim is to prove that the exponent $1/4$ in Corollary 1.2 is optimal. This is a consequence of the following result.

THEOREM 1.3. *Let n be a positive integer of the form $n = 8k+4$. There exists a $n \times n$ invertible matrix A_n with complex entries such that*

$$w(A_n) \leqslant \frac{1}{\cos\frac{\pi}{n}}, \quad w(A_n^{-1}) \leqslant \frac{1}{\cos\frac{\pi}{n}}, \quad \|A_n\| = 1+\frac{1}{8\sqrt{n}}.$$

Indeed, Theorem 1.3 implies that

$$\psi\left(\frac{1}{\cos\frac{\pi}{n}}\right) \geqslant \|A_n\| = 1+\frac{1}{8\sqrt{n}}.$$

Taking $1+\varepsilon = 1/\cos\frac{\pi}{n} = 1+\frac{\pi^2}{2n^2}+O(\frac{1}{n^4})$, we see that the exponent $\frac{1}{4}$ cannot be improved.

More generally, we can consider for $\rho \geqslant 1$ the ρ-radius $w_\rho(A)$ introduced by Sz.-Nagy and Foiaş (see [5, Chapter 1] and the references therein). Consider the class $\mathscr{C}_\rho$ of operators $T \in \mathscr{B}(H)$ which admit unitary ρ-dilations, i.e. there exist a super-space $\mathscr{H} \supset H$ and a unitary operator $U \in \mathscr{B}(\mathscr{H})$ such that

$$T^n = \rho P U^n P^*, \qquad \text{for } n = 1, 2, \ldots.$$

Here P denotes the orthogonal projection from $\mathscr{H}$ onto H. Then the operator ρ-radius is defined by

$$w_\rho(A) = \inf\{\lambda > 0 ; \lambda^{-1} A \in \mathscr{C}_\rho\}.$$

From this definition it is easily seen that $r(A) \leqslant w_\rho(A) \leqslant \rho\|A\|$, where $r(A)$ denotes the spectral radius of A. Also, $w_\rho(A)$ is a non-increasing function of ρ. Another equivalent definition follows from [5, Theorem 11.1]:

$$w_\rho(A) = \sup_{h \in \mathscr{E}_\rho} \left\{ (1-\tfrac{1}{\rho}) |\langle Ah, h\rangle| + \sqrt{(1-\tfrac{1}{\rho})^2 |\langle Ah, h\rangle|^2 + (\tfrac{2}{\rho}-1) \|Ah\|^2} \right\}, \quad \text{with}$$

$$\mathscr{E}_\rho = \{h \in H ; \|h\| = 1 \text{ and} (1-\tfrac{1}{\rho})^2 |\langle Ah, h\rangle|^2 - (1-\tfrac{2}{\rho}) \|Ah\|^2 \geqslant 0\}.$$

Notice that $\mathscr{E}_\rho = \{h \in H ; \|h\| = 1\}$ whenever $1 \leqslant \rho \leqslant 2$. This shows that $w_1(A) = \|A\|$, $w_2(A) = w(A)$ and $w_\rho(A)$ is a convex function of A if $1 \leqslant \rho \leqslant 2$.

We now define a function $\psi_\rho(r)$ for $r \geqslant 1$ by

$$\psi_\rho(r) = \sup\{\|A\| ; A \in \mathscr{B}(H), w_\rho(A) \leqslant r,\ w_\rho(A^{-1}) \leqslant r\}.$$

As before, the conditions $w_\rho(A) \leqslant r$ and $w_\rho(A^{-1}) \leqslant r$ imply the existence of a unitary operator U such that $\|A - U\| \leqslant \psi_\rho(r) - 1$, and we have $\psi_\rho(r) \leqslant \rho r$. We will generalize the estimate (1) from Theorem 1.1 by proving, for $1 \leqslant \rho \leqslant 2$, the following result.

THEOREM 1.4. *For $1 \leqslant \rho \leqslant 2$ we have*

$$\psi_\rho(r) \leqslant X_\rho(r) + \sqrt{X_\rho(r)^2 - 1}, \tag{2}$$

$$\textit{with} \quad X_\rho(r) = \frac{2 + \rho r^2 - \rho + \sqrt{(2 + \rho r^2 - \rho)^2 - 4r^2}}{2r}.$$

COROLLARY 1.5. *For $1 \leqslant \rho \leqslant 2$ we have*

$$\psi_\rho(1+\varepsilon) \leqslant 1 + \sqrt[4]{8(\rho-1)\varepsilon} + O(\varepsilon^{1/2}), \quad \varepsilon \to 0.$$

We recover in this way for $1 \leqslant \rho \leqslant 2$ the recent result of Ando and Li [2, Theorem 2.3], namely that $w_\rho(A) \leqslant 1$ and $w_\rho(A^{-1}) \leqslant 1$ imply A is unitary. The range $1 \leqslant \rho \leqslant 2$ coincides with the range of those $\rho \geqslant 1$ for which $w_\rho(\cdot)$ is a norm. Contrarily to [2], we have not been able to treat the case $\rho > 2$.

The organization of the paper is as follows. In Section 2 we prove Theorem 1.4, which reduces to Theorem 1.1 in the case $\rho = 2$. The proof of Theorem 1.3 which shows the optimality of the exponent $1/4$ in Corollary 1.2 is given in Section 3.

As a concluding remark, we would like to mention that the present developments have been influenced by the recent work of Sano/Uchiyama [7] and Ando/Li [2]. In [3], inspired by the paper of Stampfli [8], we have developed another (more complicated) approach in the case $\rho = 2$.

2. Proof of Theorem 1.4 about ψ_ρ

Let us consider $M = \frac{1}{2}(A + (A^*)^{-1})$; then

$$M^*M - 1 = \tfrac{1}{4}(A^*A + (A^*A)^{-1} - 2) \geqslant 0.$$

This implies $\|M^{-1}\| \leqslant 1$. In what follows $C^{1/2}$ will denote the positive square root of the self-adjoint positive operator C. From $(A^*A - 2M^*M + 1)^2 = 4M^*M(M^*M - 1)$ we infer

$$A^*A - 2M^*M + 1 \leqslant 2(M^*M)^{1/2}(M^*M - 1)^{1/2},$$
$$\text{whence} \quad A^*A \leqslant ((M^*M)^{1/2} + (M^*M - 1)^{1/2})^2.$$

Therefore $\|A\| \leqslant \|M\| + \sqrt{\|M\|^2 - 1}$.

We now assume $1 \leqslant \rho \leqslant 2$. Then $w_\rho(.)$ is a norm and the two conditions $w_\rho(A) \leqslant r$ and $w_\rho(A^{-1}) \leqslant r$ imply $w_\rho(M) \leqslant r$. The desired estimate of $\psi_\rho(r)$ will follow from the following auxiliary result.

LEMMA 2.1. *Assume $\rho \geqslant 1$. Then the assumptions $w_\rho(M) \leqslant r$ and $\|M^{-1}\| \leqslant 1$ imply $\|M\| \leqslant X_\rho(r)$.*

Proof. The contractivity of M^{-1} implies

$$\|u\| \leqslant \|Mu\|, \quad (\forall u \in H). \tag{3}$$

As $w_\rho(M) \leqslant r$, it follows from a generalization by Durszt [4] of a decomposition due to Ando [1], that the operator M can be decomposed as

$$M = \rho r B^{1/2} U C^{1/2},$$

with U unitary, C selfadjoint satisfying $0 < C < 1$, and $B = f(C)$ with f defined by $f(x) = (1-x)/(1-\rho(2-\rho)x)^{-1}$. Notice that f is a decreasing function on the segment $[0,1]$ and an involution: $f(f(x)) = x$. Let $[\alpha, \beta]$ be the smallest segment containing the spectrum of C. Then $[\sqrt{\alpha}, \sqrt{\beta}]$ is the smallest segment containing the spectrum of $C^{1/2}$ and $[\sqrt{f(\beta)}, \sqrt{f(\alpha)}]$ is the smallest segment containing the spectrum of $B^{1/2}$. We have

$$\|u\| \leqslant \|Mu\| \leqslant \rho r \sqrt{f(\alpha)} \|C^{1/2}u\|, \quad (\forall u \in H).$$

Choosing a sequence u_n of norm-one vectors ($\|u_n\| = 1$) such that $\|C^{1/2}u_n\|$ tends to $\sqrt{\alpha}$, we first get
$1 \leqslant \rho r\sqrt{\alpha f(\alpha)}$, i.e. $1-(2+\rho r^2-\rho)\rho\alpha+\rho^2r^2\alpha^2 \leqslant 0$. Consequently we have

$$\frac{2+\rho r^2-\rho-\sqrt{(2+\rho r^2-\rho)^2-4r^2}}{2\rho r^2} \leqslant \alpha \leqslant \frac{2+\rho r^2-\rho+\sqrt{(2+\rho r^2-\rho)^2-4r^2}}{2\rho r^2},$$

and, using $\alpha = f(f(\alpha))$,

$$\frac{2+\rho r^2-\rho-\sqrt{(2+\rho r^2-\rho)^2-4r^2}}{2\rho r^2} \leqslant f(\alpha) \leqslant \frac{2+\rho r^2-\rho+\sqrt{(2+\rho r^2-\rho)^2-4r^2}}{2\rho r^2}.$$

Similarly, noticing that $\|(M^*)^{-1}\| \leqslant 1$, $M^* = \rho rC^{1/2}U^*B^{1/2}$ and $C = f(B)$, we obtain

$$\frac{2+\rho r^2-\rho-\sqrt{(2+\rho r^2-\rho)^2-4r^2}}{2\rho r^2} \leqslant \beta \leqslant \frac{2+\rho r^2-\rho+\sqrt{(2+\rho r^2-\rho)^2-4r^2}}{2\rho r^2}.$$

Therefore

$$\|M\| \leqslant \rho r\|B^{1/2}\|\,\|C^{1/2}\| = \rho r\sqrt{f(\alpha)\beta} \leqslant \frac{2+\rho r^2-\rho+\sqrt{(2+\rho r^2-\rho)^2-4r^2}}{2r}.$$

This shows that $\|M\| \leqslant X_\rho(r)$. $\square$

3. The exponent $1/4$ is optimal (Proof of Theorem 1.3)

Consider the family of $n\times n$ matrices $A = DBD$, defined for $n = 8k+4$, by

$$D = \mathrm{diag}(e^{i\pi/2n},\ldots,e^{(2\ell-1)i\pi/2n},\ldots,e^{(2n-1)i\pi/2n}),$$

$B = I+\frac{1}{2n^{3/2}}E$, where E is a matrix whose entries are defined as

$$e_{ij} = 1 \text{ if } 3k+2 \leqslant |i-j| \leqslant 5k+2, \qquad e_{ij} = 0 \text{ otherwise.}$$

We first remark that $\|A\| = \|B\| = 1+\frac{1}{8\sqrt{n}}$. Indeed, B is a symmetric matrix with non negative entries, $Be = (1+\frac{1}{8\sqrt{n}})e$ with $e^T = (1,1,1\ldots,1)$. Thus $\|B\| = r(B) = 1+\frac{1}{8\sqrt{n}}$ by the Perron-Frobenius theorem.

Consider now the permutation matrix P defined by $p_{ij} = 1$ if $i = j+1$ modulo n and $p_{ij} = 0$ otherwise and the diagonal matrix $\Delta = \mathrm{diag}(1,\ldots,1,-1)$. Then $P^{-1}DP = e^{i\pi/n}\Delta D$ and $P^{-1}EP = E$, whence $(P\Delta)^{-1}AP\Delta = e^{2i\pi/n}A$. Since $P\Delta$ is a unitary matrix, the numerical range $W(A) = \{\langle Au,u\rangle, ; \|u\| = 1\}$ of A satisfies $W(A) = W((P\Delta)^{-1}AP\Delta) = e^{2i\pi/n}W(A)$. This shows that the numerical range of A is invariant by the rotation of angle $2\pi/n$ centered in 0, and the same property also holds for the numerical range of A^{-1}.

We postpone the proof of the estimates $\left\|\frac{1}{2}(A+A^*)\right\| \leqslant 1$ and $\left\|\frac{1}{2}(A^{-1}+(A^{-1})^*)\right\| \leqslant 1$ to later sections. Using these estimates, we obtain that the numerical range $W(A)$ is contained in the half-plane $\{z;\mathrm{Re}\,z \leqslant 1\}$, whence in the regular n-sided polygon given by the intersection of the half-planes $\{z;\mathrm{Re}(e^{2i\pi k/n}z) \leqslant 1\}$, $k = 1,\ldots,n$. Consequently $w(A) \leqslant 1/\cos(\pi/n)$. The proof of $w(A^{-1}) \leqslant 1/\cos(\pi/n)$ is similar.

3.1. Proof of $\left\|\frac{1}{2}(A+A^*)\right\| \leqslant 1$.

Since the (ℓ, j)-entry of A is $e^{(\ell+j-1)i\frac{\pi}{n}}\Big(\delta_{\ell,j}+\frac{e_{\ell,j}}{2n^{3/2}}\Big)$, the matrix $\frac{1}{2}(A+A^*)$ is a real symmetric matrix whose $(i,j)-$entry is $\cos\big((i+j-1)\frac{\pi}{n}\big)\Big(\delta_{i,j}+\frac{e_{i,j}}{2n^{3/2}}\Big)$. It suffices to show that, for every $u=(u_1,\cdots,u_n)^T\in\mathbb{R}^n$, we have $\|u\|^2-\mathrm{Re}\langle Au,u\rangle\geqslant 0$. Let $\mathscr{E}=\{(i,j)\,;1\leqslant i,j\leqslant n, 3k+2\leqslant |i-j|\leqslant 5k+2\}$. The inequality which has to be proved is equivalent to

$$\sum_{i=1}^n 2\sin^2((i-\tfrac{1}{2})\tfrac{\pi}{n})\,u_i^2-\frac{1}{2n^{3/2}}\sum_{i,j\in\mathscr{E}}\cos((i+j-1)\tfrac{\pi}{n})u_iu_j\geqslant 0.$$

Setting $v_j=u_j\sin((j-\frac{1}{2})\frac{\pi}{n})$, this may be also written as follows

$$2\|v\|^2-\langle Mv,v\rangle+\frac{1}{2n^{3/2}}\langle Ev,v\rangle\geqslant 0,\qquad (v\in\mathbb{R}^n). \tag{4}$$

Here M is the matrix whose entries are defined by

$$m_{ij}=\frac{1}{2n^{3/2}}\cot((i-\tfrac{1}{2})\tfrac{\pi}{n})\cot((j-\tfrac{1}{2})\tfrac{\pi}{n}),\quad \text{if } (i,j)\in\mathscr{E},\quad m_{ij}=0\quad\text{otherwise.}$$

We will see that the Frobenius (or Hilbert-Schmidt) norm of M satisfies $\|M\|_F\leqslant\sqrt{9/32}<3/4$. A fortiori, the operator norm of M satisfies $\|M\|\leqslant\frac{3}{4}$. Together with $\|E\|=n/4$, this shows that $\|M\|+\frac{1}{2n^{3/2}}\|E\|\leqslant\frac{7}{8}$. Property (4) is now verified.

It remains to show that $\|M\|_F^2\leqslant\frac{9}{32}$. First we notice that $m_{ij}=m_{ji}=m_{n+1-i,n+1-j}$, and $m_{ii}=0$. Hence, with $\mathscr{E}'=\{(i,j)\in\mathscr{E}\,;i<j \text{ and } i+j\leqslant n+1\}$,

$$\|M\|_F^2=2\sum_{i<j}|m_{ij}|^2\leqslant 4\sum_{(i,j)\in\mathscr{E}'}|m_{ij}|^2.$$

We have, for $(i,j)\in\mathscr{E}'$,

$$\begin{aligned}&2j\leqslant i+j+5k+2\leqslant n+5k+3=13k+7,\quad\text{thus}\quad 3k+3\leqslant j\leqslant\tfrac{13k+7}{2},\\ &2i\leqslant i+j-3k-2\leqslant n-3k-1=5k+3,\quad\text{thus}\quad 1\leqslant i\leqslant\tfrac{5k+3}{2}.\end{aligned}$$

This shows that

$$\tfrac{3\pi}{16}\leqslant\tfrac{3k+2}{16k+8}\pi\leqslant(j-\tfrac{1}{2})\tfrac{\pi}{n}\leqslant\tfrac{13k+6}{16k+8}\pi\leqslant\pi-\tfrac{3\pi}{16},\quad\text{hence}\quad|\cot((j-\tfrac{1}{2})\tfrac{\pi}{n})|\leqslant\cot\tfrac{3\pi}{16}\leqslant\tfrac{3}{2}.$$

We also use the estimate $\cot((i-\frac{1}{2})\frac{\pi}{n})\leqslant n/(\pi(i-\frac{1}{2}))$ and the relation $\sum_{i\geqslant 1}(i-1/2)^{-2}=\pi^2/2$ to obtain

$$\|M\|_F^2\leqslant 4\sum_{(i,j)\in\mathscr{E}'}|m_{ij}|^2\leqslant\frac{4}{4n^3}\,\frac{n^2}{\pi^2}\sum_{i\geqslant 1}\frac{1}{(i-1/2)^2}\,(2k+1)\,\frac{9}{4}=\frac{9}{32}.$$

3.2. Proof of $\left\|\frac{1}{2}(A^{-1}+(A^{-1})^*)\right\| \leqslant 1$.

We start from

$$\begin{aligned}(A^{-1})^* &= D(1+\tfrac{1}{2n^{3/2}}E)^{-1}D\\ &= D^2-\tfrac{1}{2n^{3/2}}DED+\tfrac{1}{4n^3}DE^2(1+\tfrac{1}{2n^{3/2}}E)^{-1}D,\end{aligned}$$

and we want to show that $\|u\|^2-\mathrm{Re}\langle A^{-1}u,u\rangle \geqslant 0$. We set $v_j = u_j\sin((j-\frac{1}{2})\frac{\pi}{n})$. The inequality $\left\|\frac{1}{2}(A^{-1}+(A^{-1})^*)\right\| \leqslant 1$ is equivalent to

$$2\|v\|^2-\langle (M_1+M_2+M_3+M_4)v,v\rangle \geqslant 0, \qquad (v\in\mathbb{R}^n).$$

Here the entries of the matrices M_p, $1\leqslant p\leqslant 4$, are given by

$$\begin{aligned}(m_1)_{ij} &= -\tfrac{1}{2n^{3/2}}\cot((i-\tfrac{1}{2})\tfrac{\pi}{n})\cot((j-\tfrac{1}{2})\tfrac{\pi}{n})e_{ij},\\ (m_2)_{ij} &= \tfrac{1}{2n^{3/2}}\,e_{ij},\\ (m_3)_{ij} &= \tfrac{1}{4n^3}\cot((i-\tfrac{1}{2})\tfrac{\pi}{n})\cot((j-\tfrac{1}{2})\tfrac{\pi}{n})f_{ij},\\ (m_4)_{ij} &= -\tfrac{1}{4n^3}\,f_{ij},\end{aligned}$$

e_{ij} and f_{ij} respectively denoting the entries of the matrices E and $F=E^2(1+\frac{1}{2n^{3/2}}E)^{-1}$. Noticing that $M_1=-M$, we have $\|M_1\|\leqslant\frac{3}{4}$, $\|M_2\|=\frac{1}{8\sqrt{n}}$, $\|F\|\leqslant\frac{n^2/16}{1-1/(8\sqrt{n})}\leqslant\frac{n^2}{14}$ and $\|M_4\|=\frac{1}{4n^3}\|F\|$. Now we use

$$\|M_3\|^2\leqslant\|M_3\|_F^2\leqslant\tfrac{1}{16n^6}\max_{ij}|f_{ij}|^2\sum_{i,j}|\cot((i-\tfrac{1}{2})\tfrac{\pi}{n})|^2\,|\cot((j-\tfrac{1}{2})\tfrac{\pi}{n})|^2,$$

together with

$$\begin{aligned}\sum_{i,j}|\cot((i-\tfrac{1}{2})\tfrac{\pi}{n})|^2\,|\cot((j-\tfrac{1}{2})\tfrac{\pi}{n})|^2 &= \Big(\sum_{i=1}^{n}|\cot((i-\tfrac{1}{2})\tfrac{\pi}{n})|^2\Big)^2\\ &\leqslant 4\Big(\sum_{i=1}^{n/2}|\cot((i-\tfrac{1}{2})\tfrac{\pi}{n})|^2\Big)^2\leqslant n^4,\end{aligned}$$

to obtain

$$\|M_3\|\leqslant\tfrac{1}{4n}\max_{ij}|f_{ij}|.$$

Using the notation $\|E\|_\infty := \max\{\|Eu\|_\infty ; u\in\mathbb{C}^n, \|u\|_\infty\leqslant 1\}$ for the operator norm induced by the maximum norm in $\mathbb{C}^d$, we have $\|E\|_\infty = n/4$, whence $\|\frac{1}{2n^{3/2}}E\|_\infty\leqslant 1/8$ and thus $\|(1+\frac{1}{2n^{3/2}}E)^{-1}\|_\infty\leqslant\frac{1}{1-1/8}=\frac{8}{7}$. This shows that

$$\max_{ij}|f_{ij}|\leqslant\|(1+\tfrac{1}{2n^{3/2}}E)^{-1}\|_\infty\max_{ij}|e^2_{ij}|\leqslant\frac{2n}{7},$$

by denoting e^2_{ij} the entries of the matrix E^2 and noticing that $\max_{i,j}|e^2_{ij}|=n/4$. Finally, we obtain $\|M_3\|\leqslant\frac{1}{14}$ and $\|M_1+M_2+M_3+M_4\|\leqslant\frac{3}{4}+\frac{1}{8}+\frac{1}{14}+\frac{1}{56}<1$.

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Catalin Badea
Laboratoire Paul Painlevé
UMR no. 8524, Bât. M2
Université Lille 1
59655 Villeneuve d'Ascq Cedex
France
e-mail: badea@math.univ-lille1.fr

Michel Crouzeix
Institut de Recherche Mathématique de Rennes
UMR no. 6625
Université Rennes 1
Campus de Beaulieu
35042 Rennes Cedex
France
e-mail: michel.crouzeix@univ-rennes1.fr