Operators and Matrices
Volume 7, Number 2 (2013), 293–308 doi:10.7153/oam-07-17

REFINED JENSEN'S OPERATOR INEQUALITY WITH CONDITION ON SPECTRA

JADRANKA MIĆIĆ, JOSIP PEČARIĆ AND JURICA PERIĆ

(*Communicated by C.-K. Li*)

Abstract. We give a refinement of Jensen's inequality for n-tuples of self-adjoint operators, unital n-tuples of positive linear mappings and real valued continuous convex functions with condition on the spectra of the operators. The refined Jensen's inequality is used to obtain a refinement of inequalities among quasi-arithmetic means under similar conditions. As an application of these results we give a refinement of inequalities among power means.

1. Introduction

We recall some notations and definitions. Let $\mathscr{B}(H)$ be the C^*-algebra of all bounded linear operators on a Hilbert space H and 1_H stands for the identity operator. We define bounds of a self-adjoint operator $A \in \mathscr{B}(H)$ by

$$m_A = \inf_{\|x\|=1} \langle Ax, x\rangle \quad \text{and} \quad M_A = \sup_{\|x\|=1} \langle Ax, x\rangle$$

for $x \in H$. If $\mathsf{Sp}(A)$ denotes the spectrum of A, then $\mathsf{Sp}(A)$ is real and $\mathsf{Sp}(A) \subseteq [m_A, M_A]$.

For an operator $A \in \mathscr{B}(H)$ we define operators $|A|$, A^+, A^- by

$$|A| = (A^*A)^{1/2}, \qquad A^+ = (|A|+A)/2, \qquad A^- = (|A|-A)/2.$$

Obviously, if A is self-adjoint, then $|A| = (A^2)^{1/2}$ and $A^+, A^- \geqslant 0$ (called positive and negative parts of $A = A^+ - A^-$).

B. Mond and J. Pečarić in [7] proved the following version of Jensen's operator inequality

$$f\left(\sum_{i=1}^n w_i \Phi_i(A_i)\right) \leqslant \sum_{i=1}^n w_i \Phi_i\left(f(A_i)\right), \tag{1.1}$$

for operator convex functions f defined on an interval I, where $\Phi_i : \mathscr{B}(H) \to \mathscr{B}(K)$, $i = 1, \dots, n$, are unital positive linear mappings, $A_1, \dots, A_n$ are self-adjoint operators with the spectra in I and $w_1, \dots, w_n$ are non-negative real numbers with $\sum_{i=1}^n w_i = 1$.

Mathematics subject classification (2010): 47A63, 47A64.
Keywords and phrases: self-adjoint operator, positive linear mapping, convex function, Jensen's operator inequality, quasi-arithmetic mean.

Paper OaM-07-17

F. Hansen, J. Pečarić and I. Perić gave in [1] a generalization of (1.1) for a unital field of positive linear mappings. Recently, J.Mićić, J.Pečarić and Y.Seo in [5] gave a generalization of this results for a not-unital field of positive linear mappings.

Very recently, J. Mićić, Z. Pavić and J. Pečarić gave in [3, Theorem 1] Jensen's operator inequality without operator convexity as follows.

THEOREM A. *Let $(A_1,\dots,A_n)$ be an n-tuple of self-adjoint operators $A_i \in \mathscr{B}(H)$ with bounds m_i and M_i, $m_i \leqslant M_i$, $i=1,\dots,n$. Let $(\Phi_1,\dots,\Phi_n)$ be an n-tuple of positive linear mappings $\Phi_i : \mathscr{B}(H) \to \mathscr{B}(K)$, $i=1,\dots,n$, such that $\sum_{i=1}^n \Phi_i(1_H) = 1_K$. If*

$$(m_A, M_A) \cap [m_i, M_i] = \emptyset \quad \text{for } i=1,\dots,n, \tag{1.2}$$

where m_A and M_A, $m_A \leqslant M_A$, are bounds of the self-adjoint operator $A = \sum_{i=1}^n \Phi_i(A_i)$, then

$$f\left(\sum_{i=1}^n \Phi_i(A_i)\right) \leqslant \sum_{i=1}^n \Phi_i\left(f(A_i)\right) \tag{1.3}$$

holds for every continuous convex function $f : I \to \mathbb{R}$ provided that the interval I contains all m_i, M_i.

If $f : I \to \mathbb{R}$ is concave, then the reverse inequality is valid in (1.3).

In the same paper [3], we study the quasi-arithmetic operator mean

$$\mathscr{M}_\varphi(\mathbf{A}, \mathbf{\Phi}, n) = \varphi^{-1}\left(\sum_{i=1}^n \Phi_i\left(\varphi(A_i)\right)\right), \tag{1.4}$$

where $(A_1,\dots,A_n)$ is an n-tuple of self-adjoint operators in $\mathscr{B}(H)$ with the spectra in I, $(\Phi_1,\dots,\Phi_n)$ is an n-tuple of positive linear mappings $\Phi_i : \mathscr{B}(H) \to \mathscr{B}(K)$ such that $\sum_{i=1}^n \Phi_i(1_H) = 1_K$, and $\varphi : I \to \mathbb{R}$ is a continuous strictly monotone function.

The following results about the monotonicity of this mean is proven in [3, Theorem 3].

THEOREM B. *Let $(A_1,\dots,A_n)$ and $(\Phi_1,\dots,\Phi_n)$ be as in the definition of the quasi-arithmetic mean (1.4). Let m_i and M_i, $m_i \leqslant M_i$ be the bounds of A_i, $i=1,\dots,n$. Let $\varphi, \psi : I \to \mathbb{R}$ be continuous strictly monotone functions on an interval I which contains all m_i, M_i. Let m_φ and M_φ, $m_\varphi \leqslant M_\varphi$, be the bounds of the mean $\mathscr{M}_\varphi(\mathbf{A}, \mathbf{\Phi}, n)$, such that*

$$\left(m_\varphi, M_\varphi\right) \cap [m_i, M_i] = \emptyset \qquad \text{for } i=1,\dots,n. \tag{1.5}$$

If one of the following conditions

(i) $\psi \circ \varphi^{-1}$ is convex and ψ^{-1} is operator monotone,

(i') $\psi \circ \varphi^{-1}$ is concave and $-\psi^{-1}$ is operator monotone,

is satisfied, then

$$\mathscr{M}_\varphi(\mathbf{A}, \mathbf{\Phi}, n) \leqslant \mathscr{M}_\psi(\mathbf{A}, \mathbf{\Phi}, n). \tag{1.6}$$

If one of the following conditions

(ii) $\psi\circ\varphi^{-1}$ *is concave and* ψ^{-1} *is operator monotone,*

(ii') $\psi\circ\varphi^{-1}$ *is convex and* $-\psi^{-1}$ *is operator monotone,*

is satisfied, then the reverse inequality is valid in (1.6).

In this paper we study a refinement of Jensen's inequality given in Theorem A. As an application of this result, we give a refinement of inequalities order among quasi-arithmetic means given in Theorem B and inequalities among power means.

2. Jensen's operator inequality

To obtain our result we need a result given in the following lemma.

LEMMA 1. *Let f be a convex function on an interval I, $x,y\in I$ and $p_1,p_2\in[0,1]$ such that $p_1+p_2=1$. Then*

$$\min\{p_1,p_2\}\left[f(x)+f(y)-2f\left(\frac{x+y}{2}\right)\right] \leqslant p_1 f(x)+p_2 f(y)-f(p_1x+p_2y). \tag{2.1}$$

Proof. This results follows from [6, Theorem 1, p. 717]. □

In Theorem A we prove that Jensen's operator inequality holds for every continuous convex function and for every $n-$tuple of self-adjoint operators $(A_1,\ldots,A_n)$, for every $n-$tuple of positive linear mappings $(\Phi_1,\ldots,\Phi_n)$ in the case when the interval with bounds of the operator $A=\sum_{i=1}^n\Phi_i(A_i)$ has no intersection points with the interval with bounds of the operator A_i for each $i=1,\ldots,n$. It is interesting to consider can we make a refinement of this inequality. To achieve this we need the following result, where we use the idea given in [2, Theorem 12].

LEMMA 2. *Let A be a self-adjoint operator $A\in B(H)$ with $\mathsf{Sp}(A)\subseteq[m,M]$, for some scalars $m<M$. Then*

$$f(A)\leqslant \frac{M1_H-A}{M-m}f(m)+\frac{A-m1_H}{M-m}f(M)-\delta_f\widetilde{A} \tag{2.2}$$

$$\left(\textit{resp.}\qquad f(A)\geqslant \frac{M1_H-A}{M-m}f(m)+\frac{A-m1_H}{M-m}f(M)+\delta_f\widetilde{A}\qquad\right)$$

holds for every continuous convex (*resp. concave*) *function* $f:[m,M]\to\mathbb{R}$, *where*

$$\delta_f=f(m)+f(M)-2f\left(\tfrac{m+M}{2}\right)\quad (\textit{resp. } \delta_f=2f\left(\tfrac{m+M}{2}\right)-f(m)-f(M)),$$
$$\textit{and}\quad \widetilde{A}=\tfrac{1}{2}1_H-\tfrac{1}{M-m}\left|A-\tfrac{m+M}{2}1_H\right|.$$

Proof. We prove only the convex case. Putting $x = m, y = M$ in (2.1) it follows that

$$f\left(p_1 m + p_2 M\right) \leqslant p_1 f(m) + p_2 f(M) \\ - \min\{p_1, p_2\}\left(f(m) + f(M) - 2f\left(\tfrac{m+M}{2}\right)\right) \tag{2.3}$$

holds for every $p_1, p_2 \in [0,1]$ such that $p_1 + p_2 = 1$. For any $t \in [m, M]$ we can write

$$f(t) = f\left(\frac{M-t}{M-m} m + \frac{t-m}{M-m} M\right).$$

Then by using (2.3) for $p_1 = \frac{M-t}{M-m}$ and $p_2 = \frac{t-m}{M-m}$ we get

$$f(t) \leqslant \frac{M-t}{M-m} f(m) + \frac{t-m}{M-m} f(M) \\ - \left(\frac{1}{2} - \frac{1}{M-m}\left|t - \frac{m+M}{2}\right|\right)\left(f(m) + f(M) - 2f\left(\frac{m+M}{2}\right)\right), \tag{2.4}$$

since

$$\min\left\{\frac{M-t}{M-m}, \frac{t-m}{M-m}\right\} = \frac{1}{2} - \frac{1}{M-m}\left|t - \frac{m+M}{2}\right|.$$

Finally we use the continuous functional calculus for a self-adjoint operator A: $f, g \in \mathscr{C}(I), Sp(A) \subseteq I$ and $f \geqslant g$ implies $f(A) \geqslant g(A)$; and $h(t) = |t|$ implies $h(A) = |A|$. Then by using (2.4) we obtain the desired inequality (2.2). $\square$

THEOREM 3. *Let $(A_1, \ldots, A_n)$ be an $n-$tuple of self-adjoint operators $A_i \in B(H)$ with the bounds m_i and M_i, $m_i \leqslant M_i$, $i = 1, \ldots, n$. Let $(\Phi_1, \ldots, \Phi_n)$ be an $n-$tuple of positive linear mappings $\Phi_i : B(H) \to B(K)$, $i = 1, \ldots, n$, such that $\sum_{i=1}^n \Phi_i(1_H) = 1_K$. Let*

$$(m_A, M_A) \cap [m_i, M_i] = \emptyset \quad \text{for } i = 1, \ldots, n, \qquad \text{and} \qquad m < M,$$

where m_A and M_A, $m_A \leqslant M_A$, are the bounds of the operator $A = \sum_{i=1}^n \Phi_i(A_i)$ and

$$m = \max\{M_i \colon M_i \leqslant m_A, i \in \{1, \ldots, n\}\},\ M = \min\{m_i \colon m_i \geqslant M_A, i \in \{1, \ldots, n\}\}.$$

If $f : I \to \mathbb{R}$ is a continuous convex (resp. concave) function provided that the interval I contains all m_i, M_i, then

$$f\left(\sum_{i=1}^n \Phi_i(A_i)\right) \leqslant \sum_{i=1}^n \Phi_i\left(f(A_i)\right) - \delta_f \widetilde{A} \leqslant \sum_{i=1}^n \Phi_i\left(f(A_i)\right) \tag{2.5}$$

(*resp.*

$$f\left(\sum_{i=1}^n \Phi_i(A_i)\right) \geqslant \sum_{i=1}^n \Phi_i\left(f(A_i)\right) + \delta_f \widetilde{A} \geqslant \sum_{i=1}^n \Phi_i\left(f(A_i)\right)) \tag{2.6}$$

holds, where

$$\begin{aligned} \delta_f \equiv \delta_f(\bar{m}, \bar{M}) &= f(\bar{m}) + f(\bar{M}) - 2f\left(\tfrac{\bar{m}+\bar{M}}{2}\right) \\ (\textit{resp.} \quad \delta_f \equiv \delta_f(\bar{m}, \bar{M}) &= 2f\left(\tfrac{\bar{m}+\bar{M}}{2}\right) - f(\bar{m}) - f(\bar{M})\ ), \\ \widetilde{A} \equiv \widetilde{A}_A(\bar{m}, \bar{M}) &= \tfrac{1}{2} 1_K - \tfrac{1}{\bar{M}-\bar{m}}\left|A - \tfrac{\bar{m}+\bar{M}}{2} 1_K\right| \end{aligned} \tag{2.7}$$

and $\bar{m} \in [m, m_A]$, $\bar{M} \in [M_A, M]$, $\bar{m} < \bar{M}$, *are arbitrary numbers.*

Proof. We prove only the convex case.

Since $A = \sum_{i=1}^{n} \Phi_i(A_i) \in B(K)$ is the self-adjoint operator such that $\bar{m}1_K \leqslant m_A 1_K \leqslant \sum_{i=1}^{n} \Phi_i(A_i) \leqslant M_A 1_K \leqslant \bar{M}1_K$ and f is convex on $[\bar{m}, \bar{M}] \subseteq I$, then by Lemma 2 we obtain

$$f\left(\sum_{i=1}^{n} \Phi_i(A_i)\right) \leqslant \frac{\bar{M}1_K - \sum_{i=1}^{n} \Phi_i(A_i)}{\bar{M} - \bar{m}} f(\bar{m}) + \frac{\sum_{i=1}^{n} \Phi_i(A_i) - \bar{m}1_K}{\bar{M} - \bar{m}} f(\bar{M}) - \delta_f \widetilde{A}, \quad (2.8)$$

where δ_f and $\widetilde{A}$ are defined by (2.7).

But since f is convex on $[m_i, M_i]$ and since $(m_A, M_A) \cap [m_i, M_i] = \emptyset$ implies $(\bar{m}, \bar{M}) \cap [m_i, M_i] = \emptyset$, then

$$f(A_i) \geqslant \frac{\bar{M}1_H - A_i}{\bar{M} - \bar{m}} f(\bar{m}) + \frac{A_i - \bar{m}1_H}{\bar{M} - \bar{m}} f(\bar{M}), \qquad i = 1, \ldots, n$$

holds. Applying a positive linear mapping Φ_i, summing and adding $-\delta_f \widetilde{A}$, we obtain

$$\sum_{i=1}^{n} \Phi_i(f(A_i)) - \delta_f \widetilde{A} \geqslant \frac{\bar{M}1_K - \sum_{i=1}^{n} \Phi_i(A_i)}{\bar{M} - \bar{m}} f(\bar{m}) + \frac{\sum_{i=1}^{n} \Phi_i(A_i) - \bar{m}1_K}{\bar{M} - \bar{m}} f(\bar{M}) - \delta_f \widetilde{A}, \quad (2.9)$$

since $\sum_{i=1}^{n} \Phi_i(1_H) = 1_K$. Combining the two inequalities (2.8) and (2.9), we have LHS of (2.5). Since $\delta_f \geqslant 0$ and $\widetilde{A} \geqslant 0$ then we have RHS of (2.5). □

REMARK 4. Specially, if $m_A < M_A$, then Theorem 3 in the convex case gives

$$f\left(\sum_{i=1}^{n} \Phi_i(A_i)\right) \leqslant \sum_{i=1}^{n} \Phi_i(f(A_i)) - \bar{\delta}_f \bar{A} \leqslant \sum_{i=1}^{n} \Phi_i(f(A_i)),$$

where

$$\bar{\delta}_f \equiv \delta_f(m_A, M_A) = f(m_A) + f(M_A) - 2f\left(\tfrac{m_A + M_A}{2}\right)$$

$$\text{and} \qquad \bar{A} \equiv \widetilde{A}_A(m_A, M_A) = \tfrac{1}{2}1_K - \tfrac{1}{M_A - m_A}\left|A - \tfrac{m_A + M_A}{2}1_K\right|.$$

But if $m < M$ and $m_A = M_A$, then the inequality (2.5) holds, but $\bar{\delta}_f \bar{A}$ is not defined. Some examples of this case are given in Example 5 I) and II).

EXAMPLE 5. We give three examples for the matrix cases and $n = 2$. Then we have refined inequalities given in Figure 1.

We put $f(t) = t^4$ which is convex but not operator convex in (2.5). Also, we define mappings $\Phi_1, \Phi_2 : M_3(\mathbb{C}) \to M_2(\mathbb{C})$ as follows: $\Phi_1((a_{ij})_{1 \leqslant i,j \leqslant 3}) = \frac{1}{2}(a_{ij})_{1 \leqslant i,j \leqslant 2}$, $\Phi_2 = \Phi_1$ (then $\Phi_1(I_3) + \Phi_2(I_3) = I_2$).

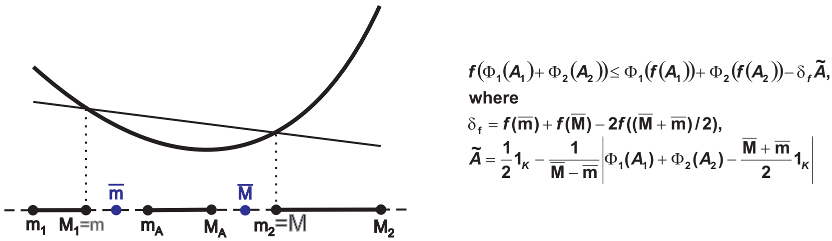

Figure 1: *Refinement for two operators and a convex function f*

I) First, we observe an example when $\delta_f\widetilde{A}$ is equal the difference RHS and LHS of Jensen's inequality. If $A_1=-3I_3$ and $A_2=2I_3$, then $A=\Phi_1(A_1)+\Phi_2(A_2)=-0.5I_2$, so $m=-3$, $M=2$. We put also that $\bar{m}=-3$ and $\bar{M}=2$. We obtain

$$(\Phi_1(A_1)+\Phi_2(A_2))^4=0.0625I_2\leqslant 48.5I_2=\Phi_1\left(A_1^4\right)+\Phi_2\left(A_2^4\right)$$

and its improvement

$$(\Phi_1(A_1)+\Phi_2(A_2))^4=0.0625I_2=\Phi_1\left(A_1^4\right)+\Phi_2\left(A_2^4\right)-48.4375I_2,$$

since $\delta_f=96.875$, $\widetilde{A}=0.5I_2$.

II) Next, we observe an example when $\delta_f\widetilde{A}$ is not equal the difference RHS and LHS of Jensen's inequality. If

$$A_1=\begin{pmatrix}-1&0&0\\0&-2&0\\0&0&-1\end{pmatrix}\quad\text{and}\quad A_2=\begin{pmatrix}2&0&0\\0&3&0\\0&0&4\end{pmatrix}\quad\text{then}\quad A=\frac{1}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix},$$

so $m=-1$, $M=2$. We put also that $\bar{m}=-1$ and $\bar{M}=2$. We obtain

$$(\Phi_1(A_1)+\Phi_2(A_2))^4=\frac{1}{16}\begin{pmatrix}1&0\\0&1\end{pmatrix}\leqslant\begin{pmatrix}\frac{17}{2}&0\\0&\frac{97}{2}\end{pmatrix}=\Phi_1\left(A_1^4\right)+\Phi_2\left(A_2^4\right)$$

and its improvement

$$\begin{aligned}(\Phi_1(A_1)+\Phi_2(A_2))^4&=\frac{1}{16}\begin{pmatrix}1&0\\0&1\end{pmatrix}\leqslant\frac{1}{16}\begin{pmatrix}1&0\\0&641\end{pmatrix}\\&=\Phi_1\left(A_1^4\right)+\Phi_2\left(A_2^4\right)-\frac{135}{16}\begin{pmatrix}1&0\\0&1\end{pmatrix},\end{aligned}$$

since $\delta_f=135/8$, $\widetilde{A}=I_2/2$.

III) Next, we observe an example with matrices that are not special. If

$$A_1=\begin{pmatrix}-4&1&1\\1&-2&-1\\1&-1&-1\end{pmatrix}\quad\text{and}\quad A_2=\begin{pmatrix}5&-1&-1\\-1&2&1\\-1&1&3\end{pmatrix}\quad\text{then}\quad A=\frac{1}{2}\begin{pmatrix}1&0\\0&0\end{pmatrix},$$

so $m_1 = -4.8662$, $M_1 = -0.3446$, $m_2 = 1.3446$, $M_2 = 5.8662$, $m = -0.3446$, $M = 1.3446$ and we put $\bar{m} = m$, $\bar{M} = M$ (rounded to four decimal places). We have

$$(\Phi_1(A_1)+\Phi_2(A_2))^4 = \frac{1}{16}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \leqslant \begin{pmatrix}\frac{1283}{2} & -255\\ -255 & \frac{237}{2}\end{pmatrix} = \Phi_1\left(A_1^4\right)+\Phi_2\left(A_2^4\right)$$

and its improvement

$$\begin{aligned}(\Phi_1(A_1)+\Phi_2(A_2))^4 &= \frac{1}{16}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \leqslant \begin{pmatrix}639.9213 & -255\\ -255 & 117.8559\end{pmatrix}\\ &= \Phi_1\left(A_1^4\right)+\Phi_2\left(A_2^4\right) - \begin{pmatrix}1.5787 & 0\\ 0 & 0.6441\end{pmatrix}\end{aligned}$$

(rounded to four decimal places), since

$$\delta_f = 3.1574, \qquad \widetilde{A} = \begin{pmatrix}0.5 & 0\\ 0 & 0.2040\end{pmatrix}.$$

But, if we put $\bar{m} = m_A = 0$, $\bar{M} = M_A = 0.5$ in the example III), then $\tilde{A} = \mathbf{0}$, so we do not have an improvement of Jensen's inequality. Also, if we put $\bar{m} = 0$, $\bar{M} = 1$, then $\tilde{A} = 0.5\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, $\delta_f = 7/8$ and $\delta_f\tilde{A} = 0.4375\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, which is worse than the above improvement.

We have the following obvious corollary of Theorem 3 with the convex combination of operators A_i, $i = 1,\ldots,n$.

COROLLARY 6. *Let $(A_1,\ldots,A_n)$ be an $n-$tuple of self-adjoint operators $A_i \in B(H)$ with the bounds m_i and M_i, $m_i \leqslant M_i$, $i = 1,\ldots,n$. Let $(\alpha_1,\ldots,\alpha_n)$ be an $n-$tuple of nonnegative real numbers such that $\sum_{i=1}^n \alpha_i = 1$. Let*

$$(m_A, M_A)\cap[m_i, M_i] = \emptyset \quad \text{for } i = 1,\ldots,n, \qquad \text{and} \qquad m < M,$$

where m_A and M_A, $m_A \leqslant M_A$, are the bounds of $A = \sum_{i=1}^n \alpha_i A_i$ and

$$m = \max\{M_i \leqslant m_A, i\in\{1,\ldots,n\}\},\ M = \min\{m_i \geqslant M_A, i\in\{1,\ldots,n\}\}.$$

If $f : I \to \mathbb{R}$ is a continuous convex (resp. concave) function provided that the interval I contains all m_i, M_i, then

$$\begin{aligned} & f\left(\sum_{i=1}^n \alpha_i A_i\right) \leqslant \textstyle\sum_{i=1}^n \alpha_i f(A_i) - \delta_f\tilde{\tilde{A}} \leqslant \sum_{i=1}^n \alpha_i f(A_i)\\ (\textit{resp.} \qquad & f\left(\sum_{i=1}^n \alpha_i A_i\right) \geqslant \textstyle\sum_{i=1}^n \alpha_i f(A_i) + \delta_f\tilde{\tilde{A}} \geqslant \sum_{i=1}^n \alpha_i f(A_i) \qquad)\end{aligned}$$

holds, where δ_f is defined by (2.7), $\tilde{\tilde{A}} = \frac{1}{2}1_H - \frac{1}{\bar{M}-\bar{m}}\left|\sum_{i=1}^n \alpha_i A_i - \frac{\bar{m}+\bar{M}}{2}1_H\right|$ and $\bar{m} \in [m, m_A]$, $\bar{M} \in [M_A, M]$, $\bar{m} < \bar{M}$, are arbitrary numbers.

Proof. We apply Theorem 3 for positive linear mappings $\Phi_i : \mathscr{B}(H) \to \mathscr{B}(H)$ determined by $\Phi_i : B \mapsto \alpha_i B$, $i = 1,\ldots,n$. $\square$

3. Quasi-arithmetic means

In this section we will study a refinement of inequalities among quasi-arithmetic mean defined by (1.4).

For convenience we introduce the following denotations

$$\begin{aligned} \delta_{\varphi,\psi}(m,M) &= \psi(m)+\psi(M)-2\psi\circ\varphi^{-1}\left(\tfrac{\varphi(m)+\varphi(M)}{2}\right), \\ \widetilde{A}_{\varphi}(m,M) &= \tfrac{1}{2}1_K - \tfrac{1}{|\varphi(M)-\varphi(m)|}\left|\textstyle\sum_{i=1}^n \Phi_i(\varphi(A_i)) - \tfrac{\varphi(M)+\varphi(m)}{2}1_K\right|, \end{aligned} \tag{3.1}$$

where $(A_1,\ldots,A_n)$ is an n-tuple of self-adjoint operators in $\mathscr{B}(H)$ with the spectra in I, $(\Phi_1,\ldots,\Phi_n)$ is an n-tuple of positive linear mappings $\Phi_i:\mathscr{B}(H)\to\mathscr{B}(K)$ such that $\sum_{i=1}^n \Phi_i(1_H)=1_K$, $\varphi,\psi: I\to\mathbb{R}$ are continuous strictly monotone functions and $m,M\in I$, $m<M$. Of course, we include implicitly that $\widetilde{A}_{\varphi}(m,M)\equiv\widetilde{A}_{\varphi,A}(m,M)$, where $A=\sum_{i=1}^n \Phi_i(\varphi(A_i))$.

In the next theorem we give a refinement of results given in Theorem B.

THEOREM 7. *Let $(A_1,\ldots,A_n)$ and $(\Phi_1,\ldots,\Phi_n)$ be as in the definition of the quasi-arithmetic mean (1.4). Let $\varphi,\psi: I\to\mathbb{R}$ be continuous strictly monotone functions on an interval I which contains all m_i,M_i. Let*

$$\left(m_{\varphi},M_{\varphi}\right)\cap[m_i,M_i]=\emptyset \qquad \textit{for } i=1,\ldots,n, \qquad \textit{and} \qquad m<M,$$

where m_{φ} and M_{φ}, $m_{\varphi}\leqslant M_{\varphi}$, are the bounds of the mean $\mathscr{M}_{\varphi}(\mathbf{A},\mathbf{\Phi},n)$ and $m=\max\left\{M_i\colon M_i\leqslant m_{\varphi}, i\in\{1,\ldots,n\}\right\}$, $M=\min\left\{m_i\colon m_i\geqslant M_{\varphi}, i\in\{1,\ldots,n\}\right\}$.

(i) *If $\psi\circ\varphi^{-1}$ is convex and ψ^{-1} is operator monotone, then*

$$\mathscr{M}_{\varphi}(\mathbf{A},\mathbf{\Phi},n)\leqslant\psi^{-1}\left(\sum_{i=1}^n \Phi_i\left(\psi(A_i)\right)-\delta_{\varphi,\psi}\widetilde{A}_{\varphi}\right)\leqslant\mathscr{M}_{\psi}(\mathbf{A},\mathbf{\Phi},n) \tag{3.2}$$

holds, where $\delta_{\varphi,\psi}\geqslant 0$ and $\widetilde{A}_{\varphi}\geqslant 0$.

(i′) *If $\psi\circ\varphi^{-1}$ is convex and $-\psi^{-1}$ is operator monotone, then the reverse inequality is valid in (3.2), where $\delta_{\varphi,\psi}\geqslant 0$ and $\widetilde{A}_{\varphi}\geqslant 0$.*

(ii) *If $\psi\circ\varphi^{-1}$ is concave and $-\psi^{-1}$ is operator monotone, then (3.2) holds, where $\delta_{\varphi,\psi}\leqslant 0$ and $\widetilde{A}_{\varphi}\geqslant 0$.*

(ii′) *If $\psi\circ\varphi^{-1}$ is concave and ψ^{-1} is operator monotone, then the reverse inequality is valid in (3.2), where $\delta_{\varphi,\psi}\leqslant 0$ and $\widetilde{A}_{\varphi}\geqslant 0$.*

In all the above cases, we assume that $\delta_{\varphi,\psi}\equiv\delta_{\varphi,\psi}(\bar{m},\bar{M})$, $\widetilde{A}_{\varphi}\equiv\widetilde{A}_{\varphi}(\bar{m},\bar{M})$ are defined by (3.1) and $\bar{m}\in[m,m_{\varphi}]$, $\bar{M}\in[M_{\varphi},M]$, $\bar{m}<\bar{M}$, are arbitrary numbers.

Proof. We only prove the case (i). Suppose that φ is a strictly increasing function. Since $m_i 1_H \leqslant A_i \leqslant M_i 1_H$, $i=1,\ldots,n$, and $m_\varphi 1_K \leqslant \mathscr{M}_\varphi(\mathbf{A},\mathbf{\Phi},n) \leqslant M_\varphi 1_K$, then

$$\varphi(m_i)1_H \leqslant \varphi(A_i) \leqslant \varphi(M_i)1_H, \quad i=1,\ldots,n,$$
$$\varphi(m_\varphi)1_K \leqslant \textstyle\sum_{i=1}^n \Phi_i(\varphi(A_i)) \leqslant \varphi(M_\varphi)1_K.$$

Also

$$\left(m_\varphi, M_\varphi\right) \cap [m_i, M_i] = \emptyset \quad \text{ for } i=1,\ldots,n$$

implies

$$\left(\varphi(m_\varphi), \varphi(M_\varphi)\right) \cap [\varphi(m_i), \varphi(M_i)] = \emptyset \quad \text{ for } i=1,\ldots,n. \tag{3.3}$$

Replacing A_i by $\varphi(A_i)$ in (2.5) and taking into account (3.3), we obtain that

$$f\left(\sum_{i=1}^n \Phi_i(\varphi(A_i))\right) \leqslant \sum_{i=1}^n \Phi_i\left(f(\varphi(A_i))\right) - \delta_f \widetilde{A}_\varphi \leqslant \sum_{i=1}^n \Phi_i\left(f(\varphi(A_i))\right) \tag{3.4}$$

holds for every convex function $f: J \to \mathbb{R}$ on an interval J which contains all

$$[\varphi(m_i), \varphi(M_i)] = \varphi([m_i, M_i]),$$

where

$$\delta_f = f(\varphi(\bar{m})) + f(\varphi(\bar{M})) - 2f\left(\frac{\varphi(\bar{m}) + \varphi(\bar{M})}{2}\right) \geqslant 0 \tag{3.5}$$

and $\widetilde{A}_\varphi = \frac{1}{2}1_K - \frac{1}{\varphi(\bar{M})-\varphi(\bar{m})}\left|\sum_{i=1}^n \Phi_i(\varphi(A_i)) - \frac{\varphi(\bar{M})+\varphi(\bar{m})}{2}1_K\right| \geqslant 0$.

Also, if φ is strictly decreasing, then we check that (3.4) holds for convex $f: J \to \mathbb{R}$ on J which contains all $[\varphi(M_i), \varphi(m_i)] = \varphi([m_i, M_i])$, where δ_f is defined by (3.5) and $\widetilde{A}_\varphi = \frac{1}{2}1_K - \frac{1}{\varphi(\bar{m})-\varphi(\bar{M})}\left|\sum_{i=1}^n \Phi_i(\varphi(A_i)) - \frac{\varphi(\bar{M})+\varphi(\bar{m})}{2}1_K\right| \geqslant 0$.

Putting $f = \psi \circ \varphi^{-1}$ in (3.4) and then applying an operator monotone function ψ^{-1}, we obtain (3.2).

The proof of the case (ii) is similar to the above case with the inequality (2.6) instead of (2.5). □

Now, we give a special case of the above theorem. It is a refinement of [3, Corollary 5].

COROLLARY 8. *Let $(A_1,\ldots,A_n)$ and $(\Phi_1,\ldots,\Phi_n)$ be as in the definition of the quasi-arithmetic mean* (1.4)*. Let m_i and M_i, $m_i \leqslant M_i$ be the bounds of A_i, $i=1,\ldots,n$. Let $\varphi,\psi: I \to \mathbb{R}$ be continuous strictly monotone functions on an interval I which contains all m_i, M_i and $\mathscr{I}$ be the identity function on I.*

(i) *If φ^{-1} is convex and*

$$\left(m_\varphi, M_\varphi\right) \cap [m_i, M_i] = \emptyset \quad \textit{for } i=1,\ldots,n, \quad \textit{and} \quad m_{[\varphi]} < M_{[\varphi]} \tag{3.6}$$

is valid, where m_φ *and* M_φ, $m_\varphi \leqslant M_\varphi$ *are the bounds of* $M_\varphi(\mathbf{A},\mathbf{\Phi},n)$ *and* $m_{[\varphi]} = \max\{M_i\colon M_i \leqslant m_\varphi, i \in \{1,\dots,n\}\}$, $M_{[\varphi]} = \min\{m_i\colon m_i \geqslant M_\varphi, i \in \{1,\dots,n\}\}$, *then*

$$M_\varphi(\mathbf{A},\mathbf{\Phi},n) \leqslant M_{\mathscr{I}}(\mathbf{A},\mathbf{\Phi},n) - \delta_{\varphi,\mathscr{I}}(\bar{m},\bar{M})\widetilde{A}_\varphi(\bar{m},\bar{M}) \leqslant M_{\mathscr{I}}(\mathbf{A},\mathbf{\Phi},n) \tag{3.7}$$

holds for every $\bar{m} \in [m_{[\varphi]}, m_\varphi]$, $\bar{M} \in [M_\varphi, M_{[\varphi]}]$, $\bar{m} < \bar{M}$, *where* $\delta_{\varphi,\mathscr{I}}(\bar{m},\bar{M}) \geqslant 0$ *and* $\widetilde{A}_\varphi(\bar{m},\bar{M}) \geqslant 0$ *are defined by* (3.1).

(ii) *If* φ^{-1} *is concave and* (3.6) *is valid, then*

$$M_\varphi(\mathbf{A},\mathbf{\Phi},n) \geqslant M_{\mathscr{I}}(\mathbf{A},\mathbf{\Phi},n) - \delta_{\varphi,\mathscr{I}}(\bar{m},\bar{M})\widetilde{A}_\varphi(\bar{m},\bar{M}) \geqslant M_{\mathscr{I}}(\mathbf{A},\mathbf{\Phi},n), \tag{3.8}$$

holds for every $\bar{m} \in [m_{[\varphi]}, m_\varphi]$, $\bar{M} \in [M_\varphi, M_{[\varphi]}]$, $\bar{m} < \bar{M}$, *where* $\delta_{\varphi,\mathscr{I}}(\bar{m},\bar{M}) \leqslant 0$ *and* $\widetilde{A}_\varphi(\bar{m},\bar{M}) \geqslant 0$ *are defined by* (3.1).

(iii) *If* φ^{-1} *is convex and* (3.6) *is valid and if* ψ^{-1} *is concave, and*

$$(m_\psi, M_\psi) \cap [m_i, M_i] = \emptyset \qquad \text{for } i = 1,\dots,n, \qquad \text{and} \qquad m_{[\psi]} < M_{[\psi]}$$

is valid, where m_ψ *and* M_ψ, $m_\psi \leqslant M_\psi$ *are the bounds of* $M_\psi(\mathbf{A},\mathbf{\Phi},n)$ *and* $m_{[\psi]} = \max\{M_i\colon M_i \leqslant m_\psi, i \in \{1,\dots,n\}\}$, $M_{[\psi]} = \min\{m_i\colon m_i \geqslant M_\psi, i \in \{1,\dots,n\}\}$, *then*

$$\begin{aligned} M_\varphi(\mathbf{A},\mathbf{\Phi},n) &\leqslant M_{\mathscr{I}}(\mathbf{A},\mathbf{\Phi},n) - \delta_{\varphi,\mathscr{I}}(\bar{m},\bar{M})\widetilde{A}_\varphi(\bar{m},\bar{M}) \leqslant M_{\mathscr{I}}(\mathbf{A},\mathbf{\Phi},n) \\ &\leqslant M_{\mathscr{I}}(\mathbf{A},\mathbf{\Phi},n) - \delta_{\psi,\mathscr{I}}(\bar{\bar{m}},\bar{\bar{M}})\widetilde{A}_\psi(\bar{\bar{m}},\bar{\bar{M}}) \leqslant M_\psi(\mathbf{A},\mathbf{\Phi},n) \end{aligned} \tag{3.9}$$

holds for every $\bar{m} \in [m_{[\varphi]}, m_\varphi]$, $\bar{M} \in [M_\varphi, M_{[\varphi]}]$, $\bar{m} < \bar{M}$ *and every* $\bar{\bar{m}} \in [m_{[\psi]}, m_\psi]$, $\bar{\bar{M}} \in [M_\psi, M_{[\psi]}]$, $\bar{\bar{m}} < \bar{\bar{M}}$, *where* $\delta_{\varphi,\mathscr{I}}(\bar{m},\bar{M}) \geqslant 0$, $\widetilde{A}_\varphi(\bar{m},\bar{M}) \geqslant 0$ *and* $\delta_{\psi,\mathscr{I}}(\bar{\bar{m}},\bar{\bar{M}}) \leqslant 0$, $\widetilde{A}_\psi(\bar{\bar{m}},\bar{\bar{M}}) \geqslant 0$ *are defined by* (3.1).

Proof. (i)-(ii): Putting $\psi = \mathscr{I}$ in Theorem 7 (i) and (ii'), we obtain (3.7) and (3.8), respectively.

(iii): Replacing ψ by φ in (ii) and combining this with (i), we obtain the desired inequality (3.9). □

REMARK 9. Let the assumptions of Corollary 8 (iii) be valid. We get the following refinement of inequalities quasi-arithmetic means

$$M_\varphi(\mathbf{A},\mathbf{\Phi},n) \leqslant M_\varphi(\mathbf{A},\mathbf{\Phi},n) + \Delta_{\varphi,\psi}(\bar{m},\bar{M},\bar{\bar{m}},\bar{\bar{M}}) \leqslant M_\psi(\mathbf{A},\mathbf{\Phi},n),$$

where

$$\Delta_{\varphi,\psi}(\bar{m},\bar{M},\bar{\bar{m}},\bar{\bar{M}}) = \delta_{\varphi,\mathscr{I}}(\bar{m},\bar{M})\widetilde{A}_\varphi(\bar{m},\bar{M}) - \delta_{\psi,\mathscr{I}}(\bar{\bar{m}},\bar{\bar{M}})\widetilde{A}_\psi(\bar{\bar{m}},\bar{\bar{M}}) \geqslant 0.$$

Especially,

$$\begin{aligned} M_\varphi(\mathbf{A},\mathbf{\Phi},n) &\leqslant M_\varphi(\mathbf{A},\mathbf{\Phi},n) + \bar{\delta}_\varphi(\bar{m},\bar{M})\widetilde{A}_\varphi(\bar{m},\bar{M}) + \bar{\delta}_\psi(\bar{m},\bar{M})\widetilde{A}_\psi(\bar{m},\bar{M}) \\ &\leqslant M_\psi(\mathbf{A},\mathbf{\Phi},n), \end{aligned}$$

where

$$\bar{\delta}_\varphi(\bar{m},\bar{M}) = \bar{m}+\bar{M}-2\varphi^{-1}\left(\tfrac{\varphi(\bar{m})+\varphi(\bar{M})}{2}\right) \geqslant 0,$$
$$\bar{\delta}_\psi(\bar{m},\bar{M}) = 2\psi^{-1}\left(\tfrac{\psi(\bar{m})+\psi(\bar{M})}{2}\right)-\bar{m}-\bar{M} \geqslant 0.$$

It is interesting to study a refinement of (1.6) under the condition placed only on the bounds of operators whose means we are considering. We study it in the following corollary. It is a refinement of the result given in [4, Theorem 2.1].

COROLLARY 10. *Let A_i, Φ_i, m_i, M_i, $i=1,\dots,n$, and $\varphi,\psi,\mathscr{I}$ as in the assumptions of Corollary 8.*

Let

$$(m_A,M_A)\cap[m_i,M_i]=\emptyset \quad \textit{for } i=1,\dots,n, \qquad \textit{and} \qquad m<M$$

be valid, where m_A and M_A, $m_A\leqslant M_A$, are the bounds of $A=\sum_{i=1}^n\Phi_i(A_i)$ and

$$m=\max\{M_i\colon M_i\leqslant m_A, i\in\{1,\dots,n\}\},\ M=\min\{m_i\colon m_i\geqslant M_A, i\in\{1,\dots,n\}\}.$$

If ψ is convex, ψ^{-1} is operator monotone, φ is concave, φ^{-1} is operator monotone, then

$$\begin{aligned}\mathscr{M}_\varphi(\mathbf{A},\mathbf{\Phi},n) &\leqslant \varphi^{-1}\left(\textstyle\sum_{i=1}^n\Phi_i\left(\varphi(A_i)\right)+\delta_\varphi\widetilde{A}\right)\leqslant M_{\mathscr{I}}(\mathbf{A},\mathbf{\Phi},n)\\ &\leqslant \psi^{-1}\left(\textstyle\sum_{i=1}^n\Phi_i\left(\psi(A_i)\right)-\delta_\psi\bar{A}\right)\leqslant \mathscr{M}_\psi(\mathbf{A},\mathbf{\Phi},n)\end{aligned} \tag{3.10}$$

holds, where

$$\delta_\varphi = 2\varphi\left(\tfrac{\bar{m}+\bar{M}}{2}\right)-\varphi(\bar{m})-\varphi(\bar{M})\geqslant 0, \quad \delta_\psi=\psi(\bar{\bar{m}})+\psi(\bar{\bar{M}})-2\psi\left(\tfrac{\bar{\bar{m}}+\bar{\bar{M}}}{2}\right)\geqslant 0,$$

$$\widetilde{A}=\tfrac{1}{2}1_K-\tfrac{1}{\bar{M}-\bar{m}}\left|A-\tfrac{\bar{m}+\bar{M}}{2}1_K\right|, \quad \bar{A}=\tfrac{1}{2}1_K-\tfrac{1}{\bar{\bar{M}}-\bar{\bar{m}}}\left|A-\tfrac{\bar{\bar{m}}+\bar{\bar{M}}}{2}1_K\right|$$

and $\bar{m},\bar{\bar{m}}\in[m,m_A]$, $\bar{M},\bar{\bar{M}}\in[M_A,M]$, $\bar{m}<\bar{M}$, $\bar{\bar{m}}<\bar{\bar{M}}$ *are arbitrary numbers.*

If ψ is convex, $-\psi^{-1}$ is operator monotone, φ is concave, $-\varphi^{-1}$ is operator monotone, then the reverse inequality is valid in (3.10).

Proof. We only prove (3.10). By replacing φ by $\mathscr{I}$ and next ψ by φ in Theorem 7 (ii') we obtain LHS of (3.10). Also, by replacing φ by $\mathscr{I}$ in Theorem 7 (i) we obtain RHS of (3.10). $\square$

4. Application to the power mean

As an application of results given in the above section we study a refinement of inequalities among power means.

As a special case of the quasi-arithmetic mean (1.4) we can study the operator power mean

$$\mathscr{M}_n^{[r]}(\mathbf{A},\mathbf{\Phi}) = \begin{cases} \left(\sum_{i=1}^n \Phi_i\left(A_i^r\right)\right)^{1/r}, & r \in \mathbb{R}\backslash\{0\}, \\ \exp\left(\sum_{i=1}^n \Phi_i\left(\ln\left(A_i\right)\right)\right), & r=0, \end{cases} \tag{4.1}$$

where $(A_1,\ldots,A_n)$ is an $n-$tuple of strictly positive operators in $\mathscr{B}(H)$ and $(\Phi_1,\ldots,\Phi_n)$ is an $n-$tuple of positive linear mappings $\Phi_i : \mathscr{B}(H) \to \mathscr{B}(K)$ such that $\sum_{i=1}^n \Phi_i(1_H) = 1_K$.

For convenience we introduce denotations as a special case of (3.1) as follows

$$\begin{aligned} \delta_{r,s}(m,M) &= \begin{cases} m^s+M^s-2\left(\frac{m^r+M^r}{2}\right)^{s/r}, & r\neq 0, \\ m^s+M^s-2\left(mM\right)^{s/2}, & r=0, \end{cases} \\ \widetilde{A}_r(m,M) &= \begin{cases} \frac{1}{2}1_K - \frac{1}{|M^r-m^r|}\left|\sum_{i=1}^n \Phi_i(A_i^r) - \frac{M^r+m^r}{2}1_K\right|, & r\neq 0, \\ \frac{1}{2}1_K - \left|\ln\left(\frac{M}{m}\right)\right|^{-1}\left|\sum_{i=1}^n \Phi_i(\ln A_i) - \ln\sqrt{Mm}1_K\right|, & r=0, \end{cases} \end{aligned} \tag{4.2}$$

where $m,M \in \mathbb{R}$, $0<m<M$ and $r,s \in \mathbb{R}$, $r \leqslant s$. Of course, we include implicitly that $\widetilde{A}_r(m,M) \equiv \widetilde{A}_{r,A}(m,M)$, where $A=\sum_{i=1}^n \Phi_i(A_i^r)$ for $r \neq 0$ and $A=\sum_{i=1}^n \Phi_i(\ln A_i)$ for $r=0$.

Applying Theorem 7 on the operator power means we obtain the following refinement of inequalities among power means given in [3, Corollary 7].

COROLLARY 11. *Let $(A_1,\ldots,A_n)$ and $(\Phi_1,\ldots,\Phi_n)$ be as in the definition of the power mean (4.1). Let m_i and M_i, $0<m_i \leqslant M_i$ be the bounds of A_i, $i=1,\ldots,n$.*

(i) *If $r \leqslant s$, $s \geqslant 1$ or $r \leqslant s \leqslant -1$,*

$$\left(m^{[r]},M^{[r]}\right) \cap [m_i,M_i] = \emptyset, \quad i=1,\ldots,n, \quad \text{and} \quad m<M,$$

where $m^{[r]}$ and $M^{[r]}$, $m^{[r]} \leqslant M^{[r]}$ are the bounds of $\mathscr{M}_n^{[r]}(\mathbf{A},\mathbf{\Phi})$ and

$$m=\max\left\{M_i \colon M_i \leqslant m^{[r]}, i\in\{1,\ldots,n\}\right\}, \qquad M=\min\left\{m_i \colon m_i \geqslant M^{[r]}, i\in\{1,\ldots,n\}\right\},$$

then

$$\mathscr{M}_n^{[r]}(\mathbf{A},\mathbf{\Phi}) \leqslant \left(\sum_{i=1}^n \Phi_i\left(A_i^s\right) - \delta_{r,s}\widetilde{A}_r\right)^{1/s} \leqslant \mathscr{M}_n^{[s]}(\mathbf{A},\mathbf{\Phi}), \tag{4.3}$$

holds, where $\delta_{r,s} \geqslant 0$ *for* $s \geqslant 1$, $\delta_{r,s} \leqslant 0$ *for* $s \leqslant -1$ *and* $\widetilde{A}_r \geqslant 0$. *Here we assume that* $\delta_{r,s} \equiv \delta_{r,s}(\bar{m},\bar{M})$, $\widetilde{A}_r \equiv \widetilde{A}_r(\bar{m},\bar{M})$ *are defined by* (4.2) *and* $\bar{m} \in [m, m^{[r]}]$, $\bar{M} \in [M^{[r]}, M]$, $\bar{m} < \bar{M}$, *are arbitrary numbers.*

(ii) *If* $r \leqslant s$, $r \leqslant -1$ *or* $1 \leqslant r \leqslant s$,

$$\left(m^{[s]}, M^{[s]}\right) \cap [m_i, M_i] = \emptyset, \quad i = 1, \ldots, n, \qquad \textit{and} \qquad m < M,$$

where $m^{[s]}$ *and* $M^{[s]}$, $m^{[s]} \leqslant M^{[s]}$ *are the bounds of* $\mathscr{M}_n^{[s]}(\mathbf{A}, \mathbf{\Phi})$ *and*

$$m = \max\left\{M_i \colon M_i \leqslant m^{[s]}, i \in \{1, \ldots, n\}\right\}, \qquad M = \min\left\{m_i \colon m_i \geqslant M^{[s]}, i \in \{1, \ldots, n\}\right\},$$

then

$$\mathscr{M}_n^{[r]}(\mathbf{A}, \mathbf{\Phi}) \leqslant \left(\sum_{i=1}^{n} \Phi_i\left(A_i^r\right) - \delta_{s,r}\widetilde{A}_s\right)^{1/r} \leqslant \mathscr{M}_n^{[s]}(\mathbf{A}, \mathbf{\Phi}),$$

holds, where $\delta_{s,r} \geqslant 0$ *for* $r \leqslant -1$, $\delta_{s,r} \leqslant 0$ *for* $r \geqslant 1$ *and* $\widetilde{A}_s \geqslant 0$. *Here we assume that* $\delta_{s,r} \equiv \delta_{s,r}(\bar{m},\bar{M})$, $\widetilde{A}_s \equiv \widetilde{A}_s(\bar{m},\bar{M})$ *are defined by* (4.2) *and* $\bar{m} \in [m, m^{[s]}]$, $\bar{M} \in [M^{[s]}, M]$, $\bar{m} < \bar{M}$, *are arbitrary numbers.*

Proof. We prove only the case (i). We put $\varphi(t) = t^r$ and $\psi(t) = t^s$ for $t > 0$.

Then $\psi \circ \varphi^{-1}(t) = t^{s/r}$ is concave for $r \leqslant s$, $s \leqslant 0$ and $r \neq 0$. Since $-\psi^{-1}(t) = -t^{1/s}$ is operator monotone for $s \leqslant -1$ and $\left(m^{[r]}, M^{[r]}\right) \cap [m_i, M_i] = \emptyset$ is satisfied, then by applying Theorem 7 (ii) we obtain (4.3) for $r \leqslant s \leqslant -1$.

But, $\psi \circ \varphi^{-1}(t) = t^{s/r}$ is convex for $r \leqslant s$, $s \geqslant 0$ and $r \neq 0$. Since $\psi^{-1}(t) = t^{1/s}$ is operator monotone for $s \geqslant 1$, then by applying Theorem 7 (i) we obtain (4.3) for $r \leqslant s$, $s \geqslant 1$, $r \neq 0$.

If $r = 0$ and $s \geqslant 1$, we put $\varphi(t) = \ln t$ and $\psi(t) = t^s$, $t > 0$. Since $\psi \circ \varphi^{-1}(t) = \exp(st)$ is convex, then similarly as above we obtain the desired inequality.

In the case (ii) we put $\varphi(t) = t^s$ and $\psi(t) = t^r$ for $t > 0$ and we use the same technique as in the case (i). $\square$

Figure 2 shows regions (1), (2), (4), (6), (7) in where the monotonicity of the power mean holds true [3, Corollary 6], also Figure 2 shows regions (1)–(7) which this holds true with condition on spectra [3, Corollary 7]. We show in [3, Example 2] that the order among power means does not hold generally without a condition on spectra in regions (3), (5). Now, by using Corollary 11 we give a refinement of inequalities among power means in the regions (2)–(6) (see Remark 13).

COROLLARY 12. *Let* $(A_1, \ldots, A_n)$ *and* $(\Phi_1, \ldots, \Phi_n)$ *be as in the definition of the power mean* (4.1). *Let* m_i *and* M_i, $0 < m_i \leqslant M_i$ *be the bounds of* A_i, $i = 1, \ldots, n$. *Let*

$$\left(m^{[r]}, M^{[r]}\right) \cap [m_i, M_i] = \emptyset, \quad i = 1, \ldots, n, \qquad m_{[r]} < M_{[r]},$$
$$\left(m^{[s]}, M^{[s]}\right) \cap [m_i, M_i] = \emptyset, \quad i = 1, \ldots, n, \qquad m_{[s]} < M_{[s]},$$

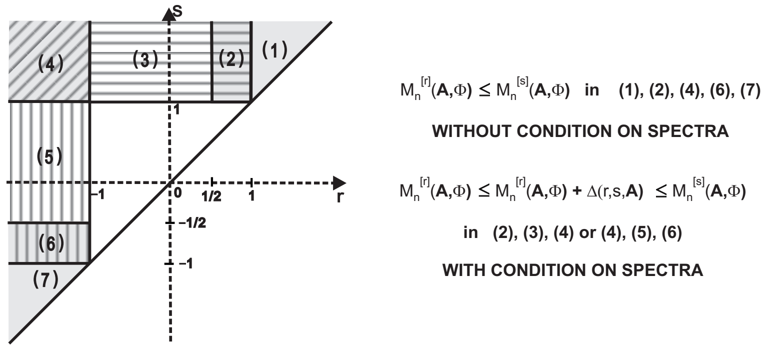

Figure 2: *Regions describing inequalities among power means*

where $m^{[r]}$, $M^{[r]}$, $m^{[r]} \leqslant M^{[r]}$ *and* $m^{[s]}$, $M^{[s]}$, $m^{[s]} \leqslant M^{[s]}$ *are the bounds of* $\mathscr{M}_n^{[r]}(\mathbf{A},\mathbf{\Phi})$ *and* $\mathscr{M}_n^{[s]}(\mathbf{A},\mathbf{\Phi})$, *respectively, and*

$$\begin{aligned} m_{[r]} &= \max\left\{M_i \leqslant m^{[r]}, i \in \{1,\ldots,n\}\right\},\ M_{[r]} = \min\left\{m_i \geqslant M^{[r]}, i \in \{1,\ldots,n\}\right\} \\ m_{[s]} &= \max\left\{M_i \leqslant m^{[s]}, i \in \{1,\ldots,n\}\right\},\ M_{[s]} = \min\left\{m_i \geqslant M^{[s]}, i \in \{1,\ldots,n\}\right\}. \end{aligned}$$

Let $\bar{m} \in [m_{[r]}, m^{[r]}]$, $\bar{M} \in [M^{[r]}, M_{[r]}]$, $\bar{m} < \bar{M}$, *and* $\bar{\bar{m}} \in [m_{[s]}, m^{[s]}]$, $\bar{\bar{M}} \in [M^{[s]}, M_{[s]}]$, $\bar{\bar{m}} < \bar{\bar{M}}$ *be arbitrary numbers.*

(**i**) *If* $r \leqslant 1 \leqslant s$, *then*

$$\begin{aligned} \mathscr{M}_n^{[r]}(\mathbf{A},\mathbf{\Phi}) &\leqslant \textstyle\sum_{i=1}^n \Phi_i(A_i) - \delta_{r,1}(\bar{m},\bar{M})\widetilde{A}_r(\bar{m},\bar{M}) \leqslant \mathscr{M}_n^{[1]}(\mathbf{A},\mathbf{\Phi}) \\ &\leqslant \textstyle\sum_{i=1}^n \Phi_i(A_i) - \delta_{s,1}(\bar{\bar{m}},\bar{\bar{M}})\widetilde{A}_s(\bar{\bar{m}},\bar{\bar{M}}) \leqslant \mathscr{M}_n^{[s]}(\mathbf{A},\mathbf{\Phi}) \end{aligned} \tag{4.4}$$

holds, where $\delta_{r,1}(\bar{m},\bar{M}) \geqslant 0$, $\widetilde{A}_r(\bar{m},\bar{M}) \geqslant 0$, $\delta_{s,1}(\bar{\bar{m}},\bar{\bar{M}}) \leqslant 0$ *and* $\widetilde{A}_s(\bar{\bar{m}},\bar{\bar{M}}) \geqslant 0$ *are defined by* (4.2).

(**ii**) *Furthermore if* $r \leqslant -1 \leqslant s$, *then*

$$\begin{aligned} \mathscr{M}_n^{[r]}(\mathbf{A},\mathbf{\Phi}) &\leqslant \left(\textstyle\sum_{i=1}^n \Phi_i\left(A_i^{-1}\right) - \delta_{r,-1}(\bar{m},\bar{M})\widetilde{A}_r(\bar{m},\bar{M})\right)^{-1} \leqslant \mathscr{M}_n^{[-1]}(\mathbf{A},\mathbf{\Phi}) \\ &\leqslant \left(\textstyle\sum_{i=1}^n \Phi_i\left(A_i^{-1}\right) - \delta_{s,-1}(\bar{\bar{m}},\bar{\bar{M}})\widetilde{A}_s(\bar{\bar{m}},\bar{\bar{M}})\right)^{-1} \leqslant \mathscr{M}_n^{[s]}(\mathbf{A},\mathbf{\Phi}) \end{aligned} \tag{4.5}$$

holds, where $\delta_{r,-1}(\bar{m},\bar{M}) \leqslant 0$, $\widetilde{A}_r(\bar{m},\bar{M}) \geqslant 0$, $\delta_{s,-1}(\bar{\bar{m}},\bar{\bar{M}}) \geqslant 0$ *and* $\widetilde{A}_s(\bar{\bar{m}},\bar{\bar{M}}) \geqslant 0$ *are defined by* (4.2).

(iii) *Furthermore if* $r \leqslant -1$, $s \geqslant 1$, *then*

$$\begin{aligned}\mathscr{M}_n^{[r]}(\mathbf{A},\boldsymbol{\Phi}) &\leqslant \left(\sum_{i=1}^n \Phi_i\left(A_i^{-1}\right) - \delta_{r,-1}(\bar{m},\bar{M})\widetilde{A}_r(\bar{m},\bar{M})\right)^{-1} \leqslant \mathscr{M}_n^{[-1]}(\mathbf{A},\boldsymbol{\Phi}) \\ &\leqslant \mathscr{M}_n^{[1]}(\mathbf{A},\boldsymbol{\Phi}) \leqslant \sum_{i=1}^n \Phi_i\left(A_i\right) - \delta_{s,1}(\bar{\bar{m}},\bar{\bar{M}})\widetilde{A}_s(\bar{\bar{m}},\bar{\bar{M}}) \\ &\leqslant \mathscr{M}_n^{[s]}(\mathbf{A},\boldsymbol{\Phi})\end{aligned} \tag{4.6}$$

holds, where $\delta_{r,-1}(\bar{m},\bar{M}) \leqslant 0$, $\widetilde{A}_r(\bar{m},\bar{M}) \geqslant 0$, $\delta_{s,1}(\bar{\bar{m}},\bar{\bar{M}}) \leqslant 0$, $\widetilde{A}_s(\bar{\bar{m}},\bar{\bar{M}}) \geqslant 0$ *are defined by* (4.2).

Proof. We prove only (4.4). If $r \leqslant 1$, then putting $s = 1$ in Corollary 11 (i) we get LHS of (4.4). Also, if $s \geqslant 1$, then putting $r = 1$ in Corollary 11 (ii) we get RHS of (4.4). $\square$

REMARK 13. Let the assumptions of Corollary 12 be valid. We get refinement of inequalities among power means as follows.

If $r \leqslant 1 \leqslant s$, then

$$\begin{aligned}\mathscr{M}_n^{[r]}(\mathbf{A},\boldsymbol{\Phi}) &\leqslant \mathscr{M}_n^{[r]}(\mathbf{A},\boldsymbol{\Phi}) + \delta_{r,1}(\bar{m},\bar{M})\widetilde{A}_r(\bar{m},\bar{M}) - \delta_{s,1}(\bar{\bar{m}},\bar{\bar{M}})\widetilde{A}_s(\bar{\bar{m}},\bar{\bar{M}}) \\ &\leqslant \mathscr{M}_n^{[s]}(\mathbf{A},\boldsymbol{\Phi}).\end{aligned}$$

If $r \leqslant -1 \leqslant s$, then

$$\begin{aligned}\mathscr{M}_n^{[r]}(\mathbf{A},\boldsymbol{\Phi}) &\leqslant \mathscr{M}_n^{[r]}(\mathbf{A},\boldsymbol{\Phi}) + \left(\sum_{i=1}^n \Phi_i\left(A_i^{-1}\right) - \delta_{s,-1}(\bar{\bar{m}},\bar{\bar{M}})\widetilde{A}_s(\bar{\bar{m}},\bar{\bar{M}})\right)^{-1} \\ &\quad - \left(\sum_{i=1}^n \Phi_i\left(A_i^{-1}\right) - \delta_{r,-1}(\bar{m},\bar{M})\widetilde{A}_r(\bar{m},\bar{M})\right)^{-1} \\ &\leqslant \mathscr{M}_n^{[s]}(\mathbf{A},\boldsymbol{\Phi}).\end{aligned}$$

If $r \leqslant -1$, $s \geqslant 1$, then

$$\begin{aligned}\mathscr{M}_n^{[r]}(\mathbf{A},\boldsymbol{\Phi}) &\leqslant \mathscr{M}_n^{[r]}(\mathbf{A},\boldsymbol{\Phi}) + \mathscr{M}_n^{[1]}(\mathbf{A},\boldsymbol{\Phi}) - \delta_{s,1}(\bar{\bar{m}},\bar{\bar{M}})\widetilde{A}_s(\bar{\bar{m}},\bar{\bar{M}}) \\ &\quad - \left(\sum_{i=1}^n \Phi_i\left(A_i^{-1}\right) - \delta_{r,-1}(\bar{m},\bar{M})\widetilde{A}_r(\bar{m},\bar{M})\right)^{-1} \\ &\leqslant \mathscr{M}_n^{[s]}(\mathbf{A},\boldsymbol{\Phi}).\end{aligned}$$

REFERENCES

[1] F. Hansen, J. Pečarić and I. Perić, *Jensen's operator inequality and it's converses*, Math. Scand. **100** (2007), 61–73.
[2] M. Klaričić Bakula, J. Pečarić and J. Perić, *On the converse Jensen inequality*, manuscript
[3] J. Mićić, Z. Pavić and J. Pečarić, *Jensen's inequality for operators without operator convexity*, Linear Algebra Appl. **434** (2011), 1228–1237.
[4] J. Mićić, Z. Pavić and J. Pečarić, *Jensen type inequalities on quasi-arithmetic operator means*, Sci. Math. Japon. **73** (2011), 183–192.
[5] J. Mićić, J. Pečarić and Y. Seo, *Converses of Jensen's operator inequality*, Oper. Matrices **4** (2010), 385–403.
[6] D. S. Mitrinović, J. E. Pečarić and A. M. Fink, Classical and New Inequalities in Analysis, Kluwer Acad. Publ., Dordrecht-Boston-London, 1993.
[7] B. Mond and J. Pečarić, *Converses of Jensen's inequality for several operators*, Revue d'analyse numer. et de théorie de l'approxim. **23** (1994) 179–183.

(Received November 2, 2011)

Jadranka Mićić
Faculty of Mechanical Engineering and Naval Architecture
University of Zagreb
Ivana Lučića 5
10000 Zagreb, Croatia
e-mail: jmicic@fsb.hr

Josip Pečarić
Faculty of Textile Technology
University of Zagreb
Prilaz baruna Filipovića 30
10000 Zagreb, Croatia
e-mail: pecaric@hazu.hr

Jurica Perić
Faculty of Science, Department of Mathematics
University of Split
Teslina 12
21000 Split, Croatia
e-mail: jperic@pmfst.hr