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WEAK ASYMPTOTIC HOMOMORPHISM PROPERTY FOR MASAS IN SEMIFINITE FACTORS

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For reasons too many to enumerate this little piece is dedicated to Kaulacharya Bibhabananda Brahmachari

(*Communicated by Z.-J. Ruan*)

Abstract. The notion of weak asymptotic homomorphism property for masas in semifinite factors is defined and is shown to be equivalent to singularity. The analysis shows that weak asymptotic homomorphism property is a 'spectral phenomenon'.

1. Introduction

This article tries to establish that the techniques developed in [9] for masas in finite factors can be extended without much difficulty to the semifinite case. It heavily relies on results in [9]. The idea is to understand singular masas. It is known to experts that singularity can be very difficult to verify. But as it turned out in [9], both notions singularity and regularity go hand in hand. The notion of weak asymptotic homomorphism property (WAHP hereafter) for finite factors, a property equivalent to singularity for masas, was discovered in [14]. This property is easily seen to be stable under tensor products. In [9], WAHP was explained through measure theoretic ideas and one of the many reasons for 'a relative version of WAHP' being stable under tensor products was automatic fall out of the techniques developed therein. Moving beyond finite von Neumann algebras and masas, it was proved in [6] that the (groupoid) normalizing algebras in many cases are well behaved with respect to tensor products. Very recently, relative WAHP (in the finite case) for arbitrary subalgebras was studied in [5] and many more questions related to normalizers were settled. Keeping the results of [6] in mind and the relation between (relative) WAHP and normalizers in [9], it is natural to ask that how far WAHP extends beyond the finite case. The definition of WAHP (in the finite case) involves the conditional expectation. Thus, we define WAHP for masas in the semifinite case assuming the existence of conditional expectation and show that WAHP is equivalent to singularity. The analysis required to prove this equivalence will indeed show that WAHP and its opposite i.e., regularity are spectral phenomena.

This article is organized as follows. §2 contains all the preliminary preparation that is required to address the problem. In §3, we study normalizers of masas and in §4 we prove the equivalence of WAHP and singularity.

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2. Preliminaries

Let M be an infinite dimensional semifinite von Neumann algebra equipped with a fixed faithful, normal, semifinite, tracial weight τ. Let $M_\tau = \{x \in M : \tau(x^*x) < \infty\}$. Then M_τ is a self-adjoint two sided ideal in M and τ defines a definite inner product on M_τ by $\langle x,y\rangle_\tau = \tau(y^*x)$, $x,y \in M_\tau$. This inner product induces a Hilbert norm on M_τ which is denoted by $\|\cdot\|_\tau$. The completion of M_τ in $\|\cdot\|_\tau$ will be denoted by $L^2(M,\tau)$. Also note that τ can be extended uniquely to a faithful positive functional on $M_\tau^* M_\tau =$ span $\{y^*x : x,y \in M_\tau\}$. We consider the representation of M in $L^2(M,\tau)$ by left multiplication, which is faithful and normal. We only consider separable semifinite von Neumann algebras which is equivalent to the separability of $L^2(M,\tau)$. Furthermore, we will assume M is a factor. If p is a projection in M_τ, then τ is a faithful, normal, finite trace on the corner pMp. We have $\|y\|_\tau = \|y^*\|_\tau$ and $\|x_1 y x_2\|_\tau \leqslant \|x_1\| \, \|x_2\| \, \|y\|_\tau$, for $x_1, x_2 \in M$ and $y \in M_\tau$. The Tomita's modular operator associated to τ is the extension of $J : M_\tau \mapsto M_\tau$ given by $Jx = x^*$ and is an anti-linear unitary. The image of a vector $\zeta \in L^2(M,\tau)$ under J will be denoted by ζ^*. Let $N \subseteq M$ be a σ-weakly closed $*$-subalgebra. Write $N_\tau = N \cap M_\tau$. Let e_N denote the orthogonal projection onto $\overline{N_\tau}^{\|\cdot\|_\tau} \overset{\text{defn.}}{=} L^2(N,\tau)$. If the support projection p of N is such that $p \in M_\tau$, then there is an unique normal τ- preserving conditional expectation from M onto N. This conditional expectation is obtained by restricting the orthogonal projection e_N to $pM_\tau p$.

Let $A \subset M$ be a masa and suppose that there exists a normal conditional expectation $\mathbb{E}_A$ from M onto A. Then $\mathbb{E}_A$ is automatically faithful and unique (see Thm. 2.1 [1]). If $M = \mathbf{B}(\mathscr{H})$, then there is one such masa up to unitary conjugacy, namely, $\ell_\infty \subset \mathbf{B}(\ell_2)$, which is Cartan and very well understood. Thus we assume M is a II_∞ factor. So A is generated by finite projections [1]. The modular automorphism group associated to τ is trivial and hence fixes A pointwise. The semifinite trace τ restricted to the lattice of projections of A takes every value in $[0,\infty]$. This just follows because A is diffuse and if $E = \{\tau(p) : p \in A \text{ is a projection}\}$, then E is closed connected and $\{0,\infty\} \in E$. Thus by a theorem of Takesaki (cf. [15], Chap. 2) $\tau \circ \mathbb{E}_A = \tau$. Since every conditional expectation is a Schwarz mapping (i.e., $\mathbb{E}_A(x)^*\mathbb{E}_A(x) \leqslant \mathbb{E}_A(x^*x)$ for all x, see p. 117 [15]), so $\mathbb{E}_A$ maps M_τ onto A_τ. Thus there exists a sequence of projections $p_i \in A_\tau$ such that Ap_i is maximal abelian in $p_i M p_i$ and $p_i \uparrow 1$.

The assumption of normal conditional expectation seems necessary to define WAHP. Note that it is possible to have a masa in any II_∞ factor which contains no finite projections of the factor [8]. For such a masa very little is known.

PROPOSITION 1. *Let $A \subset M$ be a masa in a II_∞ factor M such that there exists a normal conditional expectation from M onto A. Then there is a $*$-isomorphism θ from A onto $L^\infty([0,1],\mu)$, where μ is a σ-finite, semifinite, nonatomic Borel measure such that $\tau(a) = \int_0^1 \theta(a)d\mu$ for all $a \in A$.*

Proof. Follow the proof of Thm. 3.5.2 [13]. □

Henceforth, we will assume $A = L^\infty([0,1],\mu)$ and $\tau_{|A} = \mu$. Note that $e_A \in \mathscr{A} = A \vee JAJ$. Indeed, note that $p_n J p_n J \overset{s.o.t}{\to} 1$ and $p_n J p_n J e_A = e_{Ap_n}$. But $p_n M p_n$ is a finite factor and Ap_n being a masa in p_nMp_n, the result follows immediately, as $e_{Ap_n} \in \mathscr{A}(p_nJp_nJ)$ by a result of [10]. Also note that $\mathscr{A}$ is a diffuse abelian von Neumann algebra; this is true as there are no one dimensional normal representations of $L^\infty([0,1],\mu)$. Thus e_A is a central projection of $\mathscr{A}'$. Let $N(A)$ denote the group of unitaries of M that conjugates A to itself. Dixmier was the first to study this group [3]. He defined A to be regular if $N(A)'' = M$, semiregular if $N(A)$ acts ergodically on A, and, singular if $N(A) \subset A$. A regular masa is Cartan if there exists a normal conditional expectation from M onto A.

LEMMA 1. *(i)* $e_{N(A)''} \in \mathscr{A}$,
(ii) $\mathscr{A}e_{N(A)''} = \mathscr{A}'e_{N(A)''}$.

Proof. Note that $e_A \in \mathscr{A}$. The arguments are then essentially mimicking the original proof of the finite case [10]. By Prop. 1, the left action of A on $L^2(A,\tau)$ is unitarily equivalent to $L^\infty(\mu)$ acting on $L^2(\mu)$ by multiplication, the latter having a cyclic vector. Thus $L^2(A,\tau)$ has a cyclic vector for the left action of A and hence the same vector is cyclic for $\mathscr{A}e_A$. Thus Ae_A and hence $\mathscr{A}e_A$ are maximal abelian in $\mathbf{B}(L^2(A,\tau))$. Consequently $Ae_A = \mathscr{A}e_A = \mathscr{A}'e_A$. Now it follows that for all $u \in N(A)$, $\mathscr{A}ue_Au^* = u\mathscr{A}e_Au^* = u\mathscr{A}'e_Au^* = \mathscr{A}'ue_Au^*$ (since $Ad(u) \in Aut(\mathscr{A})$). But $ue_Au^* = e_{uA} = e_{Au}$, e_{Au} denoting the orthogonal projection onto the closed subspace $\overline{A_\tau u}^{\|\cdot\|_\tau}$. Note that $ue_Au^* \in \mathscr{A}$. Observe that $e_{N(A)''} = \bigvee_{u \in N(A)} e_{Au}$. This completes the proof. $\square$

In other words, if A is Cartan then $\mathscr{A}$ has a cyclic vector. Following [14] define:

DEFINITION 1. Let $A \subset M$ be a masa such that there is a normal conditional expectation from M onto A. Then A has WAHP, if given $\varepsilon > 0$ and any finite number of elements $x_i \in M_\tau$, $1 \leqslant i \leqslant n$ with $\mathbb{E}_A(x_i) = 0$ for all i, there is a unitary $u \in A$ such that $\left\|\mathbb{E}_A(x_iux_j^*)\right\|_\tau < \varepsilon$ for all i,j.

REMARK 1. Note that in Defn. 1, $x_iux_j^* \in M_\tau$, so $\mathbb{E}_A(x_iux_j^*) \in A_\tau$ and this fact will be used repeatedly in the subsequent sections. It is to be noted that just obtaining a desired unitary to satisfy Defn. 1 is not always illuminating. The best choice of the unitary in Defn. 1 will in fact make much deeper understanding of the situation and will show that WAHP is a spectral phenomenon. This will be clear in §4.

3. Normalizers

This section is devoted to the structure of the normalizing algebra of a masa. Most results in this section require measure theory for which a detailed analysis has been done in [9]. Since the arguments required to establish the desired structure will be borrowed from [9] and many arguments remain exactly the same, we will present only those proofs which require amending arguments of [9].

Following [9], consider the conjugacy invariant of $A \subset M$ derived from writing the direct integral decomposition of its left-right action. Note that $C[0,1]\subset A$ is unital, separable and *w.o.t* dense in A. Using factoriality of M and making arguments similar to §2.3 [9], one finds that the map $C[0,1]\otimes_{alg} C[0,1]\ni a\otimes b\mapsto aJb^*J\in\mathbf{B}(L^2(M,\tau))$ extends to an injective representation of $C[0,1]\otimes C[0,1]$ in $L^2(M,\tau)$. Thus, there exists a complete regular Borel measure η on $[0,1]\times[0,1]$ such that $L^2(M,\tau)$ admits a direct integral decomposition $\int_{[0,1]\times[0,1]}^{\oplus}\mathscr{H}_{t,s}d\eta$, for a η-measurable field of Hilbert spaces $(t,s)\mapsto\mathscr{H}_{t,s}$, and, $\mathscr{A}\cong L^\infty([0,1]\times[0,1],\eta)$ is the algebra of diagonalizable operators with respect to this decomposition. Note that η is nonatomic, since $\mathscr{A}$ is diffuse. We name $[\eta]$ (the class of measures that have same null sets as η) to be the left-right measure of A.

For $\zeta_1,\zeta_2\in L^2(M,\tau)$, let $\kappa_{\zeta_1,\zeta_2}:C[0,1]\otimes C[0,1]\mapsto\mathbb{C}$ be the linear functional defined by, $\kappa_{\zeta_1,\zeta_2}(a\otimes b)=\langle a\zeta_1 b,\zeta_2\rangle_\tau$, $a,b\in C[0,1]$. Then κ_{ζ_1,ζ_2} induces an unique complex Radon measure η_{ζ_1,ζ_2} on $[0,1]\times[0,1]$ given by,

$$\kappa_{\zeta_1,\zeta_2}(a\otimes b)=\int_{[0,1]\times[0,1]}a(t)b(s)d\eta_{\zeta_1,\zeta_2}(t,s). \tag{1}$$

We will write $\eta_{\zeta,\zeta}=\eta_\zeta$. Strictly speaking the null sets of η count and not the measure itself. For an algorithm to compute η from η_ζ, $\zeta\in L^2(M,\tau)$ see §2.3 [9]. Given $\zeta\in L^2(M,\tau)$, using Lemma 5.7 [4] and an appropriate member from $[\eta]$, one can always assume $\eta=\eta_\zeta+\nu$ with $\nu\perp\eta_\zeta$. All statements we make will be independent of the choice of a particular member from $[\eta]$. This is precisely because the structure of $L^2(M,\tau)$ as a A–A bimodule remains unaltered (via an unitary equivalence) in switching over members of $[\eta]$. However, for demonstration we will often fix one measure from the measure class such that $\eta([0,1]\times[0,1])<\infty$. We denote by $\Delta([0,1])$ the diagonal of $[0,1]\times[0,1]$.

The left-right measure enjoys some nice properties. We present here some properties for the sake of convenience.

PROPOSITION 2. (i) $[\eta]$ *is invariant with respect to the flip of coordinates.*
(ii) *If* $\pi_i:[0,1]\times[0,1]\to[0,1]$, $i=1,2$, *denote the coordinate projections then* $[\pi_{i*}\eta]=[\mu]$.
(iii) *The subspace* $\int_{\Delta([0,1])}^{\oplus}\mathscr{H}_{t,t}d\eta(t,t)$ *is identified with* $L^2(A,\tau)$.
(iv) *If* E,F *are measurable subsets of* $[0,1]$ *such that* $\mu(E)>0,\mu(F)>0$, *then* $\eta(E\times F)>0$.

Proof. The proof of these facts are well known. For example, see §2 of [9] and the references therein. So we just sketch the arguments. For (i) observe that $Ad\,J$ implements an automorphism of $\mathscr{A}$. For (ii) note that $A,JAJ\subset\mathscr{A}$ are faithfully represented in $L^2(M,\tau)$. If $p\in A$ is a finite projection, then p corresponds to a measurable set $E\subset[0,1]$ of finite μ measure. Working as in the type II_1 case it is easy to see that the subspace $\int_E^{\oplus}\mathscr{H}_{t,s}d\eta(t,s)$ is identified with $L^2(Ap,\tau)$. Thus an easy approximation argument proves (iii). For (iv) note that the sets E and F correspond to nonzero projections p and q respectively in A. If $\eta(E\times F)=0$, then $p\zeta q=0$ for all

$\zeta \in L^2(M,\tau)$. Thus $pxq = 0$ for all $x \in M_\tau$. Thus we have two nonzero projections in A whose central carriers are orthogonal. This violates that M is a factor. $\square$

In fact, working with self-adjoint vectors ($\zeta = \zeta^*$) one can always produce a member in $[\eta]$ which is symmetric. Prop. 2 allows one to disintegrate η and hence η_ζ, $\zeta \in L^2(M,\tau)$ with respect to (π_i,μ), $i = 1,2$. For more details on disintegration see §3 [9], [2]. We denote these (π_1,μ)-disintegrations by η^t, η^t_ζ and the (π_2,μ)-disintegrations by η^s, η^s_ζ; $t,s \in [0,1]$. Note that these disintegrations are uniquely defined up to an almost sure equivalence. Thus changing the fibres on a set of μ measure zero does not change the measure η.

Fix a member η from the left-right measure of A and fix a pair of disintegrations $[0,1] \ni t \mapsto \eta^t$ and $[0,1] \ni s \mapsto \eta^s$. We assume η is finite. Let $S = [0,1]\times[0,1]$. Let

$$S_{\eta,1} = \{(t,s) \mid \exists\, s \in [0,1] : \eta^t(t,s) > 0\},$$
$$S_{\eta,2} = \{(t,s) \mid \exists\, t \in [0,1] : \eta^s(t,s) > 0\}.$$

From Prop. 3.3 [9] these sets are η-measurable. These sets are not well defined as we have fixed a disintegration. However, finally our results will be independent of the above choice. This will be clear soon, as the structural results will involve operator algebraic statements. Write $S_{\eta,a} = S_{\eta,1} \cap S_{\eta,2}$. Then $\eta_a = \eta_{|S_{\eta,a}}$ has completely atomic disintegration along both coordinates.

It is not at all obvious that these sets $S_{\eta,i}$, $i = 1,2$, or $S_{\eta,a}$ are measurable equivalence realtions. In fact, transitivity is missing and hence it does not directly fit to the setup of Feldman-Moore theory. Thus one has to argue otherwise.

The normalizing groupoid of A denoted by $\mathscr{GN}_M(A)$ is the collection of those partial isometries $v \in M$ such that $vAv^* \subseteq A$ and $v^*Av \subseteq A$. As A is a masa, so $v \in \mathscr{GN}_M(A)$ if and only if $v^*v, vv^* \in A$ and $vAv^* = Avv^* = vv^*A$. The next result is proved exactly along the same lines of proof of Prop. 3.11 [9].

PROPOSITION 3. *Let $A \subset M$ be a masa generated by finite projections of M. Let $\zeta \in L^2(N(A)'',\tau)$. Then η^t_ζ (as well η^s_ζ) is completely atomic μ almost all t (and almost all s).*

Each $v \in \mathscr{GN}_M(A)$ such that v^*v is finite, implements a partial measure preserving automorphism T_v of $L^\infty(\mu)$. Such an automorphism is an invertible, bimeasure preserving, bimeasurable transformation between two measurable subsets $E,F \subset [0,1]$ such that $\mu(E) = \mu(F) < \infty$. Let $\Gamma(v)$ denote the graph of T_v. Since the measure μ is σ-finite and semifinite, working with sets of finite μ measure all arguments of Lemma 5.7, 5.8, 5.9 [9] (the ones which involve the measurable selection principle) are still valid and their proofs go without any change. This is because each time we work on a cut down of A by a finite projection from A, we are back to the setup of finite von Neumann algebras. Thus we deduce the following theorem.

THEOREM 4. *There is an index set Λ (could be empty but atmost countable) and a family $\{v_0\} \cup \{v_\lambda\}_{\lambda\in\Lambda} \subset \mathscr{GN}_M(A)$ of nonzero partial isometries with $v_0 = 1$ and*

$\tau(v_\lambda^* v_\lambda) < \infty$ for all $\lambda \in \Lambda$ such that,

(i) $\Gamma(v_\lambda) \cap \Gamma(v_{\lambda'}) = \emptyset$ *for* $\lambda \neq \lambda'$,
(ii) $\Gamma(v_\lambda) \cap \Gamma(v_0) = \emptyset$ *for all* λ,
(iii) $\eta_a([0,1] \times [0,1] \setminus \Delta([0,1])) = \eta_a(\cup_{\lambda \in \Lambda} \Gamma(v_\lambda))$
(iv) $\displaystyle \bigoplus_{\lambda \in \Lambda} \overline{Av_\lambda}^{\|\cdot\|_\tau} \oplus \overline{A_\tau v_0}^{\|\cdot\|_\tau} \cong \int_{\cup_{\lambda \in \Lambda} \Gamma(v_\lambda) \cup \Delta([0,1])}^{\oplus} \mathbb{C}_{t,s} d\eta_a(t,s) \cong L^2(N(A)'', \tau),\ \mathbb{C}_{t,s} = \mathbb{C},$

and $\mathscr{A}$ restricted to $\oplus_{\lambda \in \Lambda} \overline{Av_\lambda}^{\|\cdot\|_\tau} \oplus \overline{A_\tau v_0}^{\|\cdot\|_\tau}$ is diagonalizable with respect to the decomposition in (iv).

REMARK 2. Notice that the first and the third expressions in (iv) of Thm. 4 forces that the choice of η and its disintegrations are no obstacle at all. Off course, we do not claim uniqueness of the partial isometries involved. In changing η or its disintegrations on sets of measure zero (possibly) affects the measurable selection principle involved in the proof (of the finite case) and the partial isometries might change. Nevertheless, $L^2(N(A)'', \tau)$ will ever decompose as a direct sum of 'discrete bimodules' (see §5 [9] for definition of discrete bimodules, also see [11]).

Proof of Thm. 4. Since many details remain exactly similar to the II_1 case, we only sketch the proof. Write $S_0 = S \setminus \Delta([0,1])$. If $\eta_a(S_0) = 0$, choose the indexing set to be empty. Otherwise argue as follows. Let $\eta_a(S_0) > 0$. Arguing as in Lemma 5.7, 5.8, 5.9 of [9], choose a maximal family of partial isometries $v_\lambda \in \mathscr{G}\mathscr{N}_M(A)$, $\lambda \in \Lambda$, for some index set Λ such that $\tau(v_\lambda^* v_\lambda) < \infty$, $\Gamma(v_\lambda) \subseteq S_{\eta,a} \setminus \Delta([0,1])$ and $\Gamma(v_\lambda) \cap \Gamma(v_{\lambda'}) = \emptyset$ for $\lambda \neq \lambda'$. This forces $\overline{Av_\lambda}^{\|\cdot\|_\tau} \perp \overline{Av_{\lambda'}}^{\|\cdot\|_\tau}$ for all $\lambda \neq \lambda'$. Thus Λ must be atmost countable by the separability assumption of $L^2(M, \tau)$. It is clear that

$$\int_{\Gamma(v_\lambda)}^{\oplus} \mathbb{C}_{t,s} d\eta(t,s) \cong \int_{\Gamma(v_\lambda)}^{\oplus} \mathbb{C}_{t,s} d\eta_a(t,s) \cong \int_{\Gamma(v_\lambda)}^{\oplus} \mathbb{C}_{t,s} d\overline{\eta}_{v_\lambda}(t,s) = \overline{Av_\lambda}^{\|\cdot\|_\tau},\ \mathbb{C}_{t,s} = \mathbb{C},$$

$\overline{\eta}_{v_\lambda}$ denoting the completion of η_{v_λ}.

By maximality, $\eta_a(S_0 \setminus \cup_{\lambda \in \Lambda} \Gamma(v_\lambda)) = 0$. Indeed, if this were not the case, then $\eta_a(S_0 \setminus \cup_{\lambda \in \Lambda} \Gamma(v_\lambda)) > 0$. Write $N = S_0 \setminus \cup_{\lambda \in \Lambda} \Gamma(v_\lambda)$. We can assume that N is Borel. Then $\mu(\pi_i(N)) > 0$ for $i = 1,2$. If for every pair of sets (E_1, E_2) such that $E_i \subseteq \pi_i(N)$, $0 < \mu(E_i) < \infty$, one has $\eta_a((E_1 \times E_2) \cap N) = 0$, then $\eta_a(N) = 0$, which is not the case. Thus choose a pair of measurable sets (E_1, E_2) such that $0 < \mu(E_i) < \infty$ and $\eta_a((E_1 \times E_2) \cap N) > 0$. The sets E_i correspond to finite projections $q_i \in A$, $i = 1,2$.

Considering $A(q_1 \vee q_2) \subset (q_1 \vee q_2) M (q_1 \vee q_2)$ we are back to the finite case. Note that the left-right measure of $A(q_1 \vee q_2) \subset (q_1 \vee q_2) M (q_1 \vee q_2)$ is the restricition of η to $(E_1 \cup E_2) \times (E_1 \cup E_2)$. Applying Lemma 5.7, 5.8, 5.9 of [9] to the set $N \cap (E_1 \times E_2)$, find a nonzero $v \in \mathscr{G}\mathscr{N}_M(A)$ such that $\Gamma(v) \subseteq N \cap (E_1 \times E_2)$. Note that $\tau(v^* v) < \infty$. Then $\{v_\lambda\}_{\lambda \in \Lambda} \cup \{v\}$ violates the maximality of $\{v_\lambda\}_{\lambda \in \Lambda}$.

Let $v_0 = 1$. Then $(i), (ii)$ and (iii) follows. Note that

$$\oplus_{\lambda \in \Lambda} \overline{Av_\lambda}^{\|\cdot\|_\tau} \oplus \overline{A_\tau v_0}^{\|\cdot\|_\tau} \cong \int_{\cup_{\lambda \in \Lambda} \Gamma(v_\lambda) \cup \Delta([0,1])}^{\oplus} \mathbb{C}_{t,s} d\eta_a(t,s) \subseteq L^2(N(A)'', \tau).$$

Finally, let $0\neq\zeta\in L^2(N(A)'',\tau)\ominus\left(\oplus_{\lambda\in\Lambda}\overline{Av_\lambda}^{\|\cdot\|_\tau}\oplus\overline{A_\tau v_0}^{\|\cdot\|_\tau}\right)$. So η_ζ will admit completely atomic disintegrations along both coordinates (Prop. 3). Then by Lemma 5.7 [4], there is finite measure ν singular to η_ζ such that $[\eta_\zeta+\nu]$ is the left-right measure of A. By uniqueness of direct integrals, there is an isomorphism T of measure spaces (S,η) and $(S,\eta_\zeta+\nu)$ which implements a unitary between $L^2(M,\tau)$ to itself that preserves the structure of $L^2(M,\tau)$ as the natural $A-A$ bimodule. Via this unitary one can assume that $\eta_\zeta\ll\eta$. But then $L^2(N(A)'',\tau)$ will admit a direct integral with respect to η_a with multiplicity strictly bigger than one on some set of positive η_a measure (see Lemma 5.7 [4]). This violates $\mathscr{A}e_{N(A)''}$ is maximal abelian in $\mathbf{B}(L^2(N(A)'',\tau))$ (Lemma 1). This completes the argument. □

REMARK 3. The above proof and the statement shows that the choice of disintegrations by changing the fibres on sets of zero measure makes no difference. If $[\eta]=[\eta']$, then there exist a unitary $U:L^2(M,\tau)\mapsto L^2(M,\tau)$ which intertwines $\mathscr{A}$ to itself and preserves the structure of $L^2(M,\tau)$ as the natural $A-A$ bimodule. This unitary is the one that sends $\int_{[0,1]\times[0,1]}^{\oplus}\xi_{t,s}d\eta(t,s)$ to $\int_{[0,1]\times[0,1]}^{\oplus}\xi_{t,s}\sqrt{\frac{d\eta}{d\eta'}}d\eta'(t,s)$. Thus the conclusions of Thm. 4 remain unaltered by choosing different members from $[\eta]$.

THEOREM 5. *If $\zeta\in L^2(M,\tau)\ominus L^2(N(A)'',\tau)$, then the (π_i,μ)-disintegration of η_ζ is completely non atomic μ almost everywhere for $i=1,2$.*

Proof. The proof of Thm. 4 and the discussion preceeding it show that $L^2(N(A)'',\tau)$ is the maximal $A-A$ bimodule in $C_d(A)$ (see Defn. 5.1 of [9] and the discussions following it, also see [11]), when we restrict ourselves to $A-A$ sub bimodules of $L^2(M,\tau)$. Suppose to the contrary the (π_i,μ)-disintegration of η_ζ is not completely non atomic μ almost everywhere for $i=1,2$. Then using the measurable selection principle (see §5 of [9] for details), one finds that $\overline{A\zeta A}^{\|\cdot\|_\tau}$ contains a sub bimodule which belongs to $C_d(A)$. Note that $\overline{A\zeta A}^{\|\cdot\|_\tau}$ is orthogonal to $L^2(N(A)'',\tau)$. But then maximality of $L^2(N(A)'',\tau)$ is violated. □

There are several instances where Thm. 4, 5 are useful. For instance when one tensors two masas, analyze Dye-type theorems on normalizing groupoids etc. Since these facts have been proved using different arguments and such results are known in greater generality, we do not prove them here. Note that the ball (in $\|\cdot\|$) of M_τ is closed in $\|\cdot\|_\tau$. This is because if $x_n\in M_\tau$ with $\|x_n\|\leqslant 1$ and $x_n\to\xi\in L^2(M,\tau)$, then ξ is a bounded vector (i.e., $\|\xi x\|_\tau\leqslant\|x\|_\tau$ for all $x\in M_\tau$), forcing $\xi\in M_\tau$. Also if $M_\tau\ni x_n\overset{\|\cdot\|_\tau}{\to}x\in M_\tau$ and x_n's are bounded, then $x_n\overset{s.o.t}{\to}x$. If $v\in\mathscr{GN}_M(A)$, then $\overline{A_\tau v}^{\|\cdot\|_\tau}=L^2(A,\tau)v$. Following Thm. 4, let $\Lambda\cup\{0\}=\Lambda'$. Let

$$\sum_{\lambda\in\Lambda'}Av_\lambda=\left\{\sum_{\lambda\in\Lambda'}a_\lambda v_\lambda\in M:a_\lambda\in A,\ \sum_{\lambda\in\Lambda'}\|a_\lambda v_\lambda\|_\tau^2<\infty\right\},$$

where $v_\lambda \in \mathscr{GN}_M(A)$ are as in Thm. 4. Then

$$\sum_{\lambda\in\Lambda'} Av_\lambda = \left\{ \sum_{\lambda\in\Lambda'} a_\lambda v_\lambda \in M : a_\lambda \in A_\tau,\ \sum_{\lambda\in\Lambda'} \|a_\lambda v_\lambda\|_\tau^2 < \infty \right\}.$$

PROPOSITION 6. *Under the above setup*

$$\overline{\sum_{\lambda\in\Lambda'} Av_\lambda}^{\|\cdot\|_\tau} \cap M_\tau = \sum_{\lambda\in\Lambda'} Av_\lambda.$$

Proof. Indeed, if $x \in \overline{\sum_{\lambda\in\Lambda'} Av_\lambda}^{\|\cdot\|_\tau} \cap M_\tau$, then from Thm. 4 we have

$$x = \sum_{\lambda\in\Lambda'} \xi_\lambda v_\lambda,\ \xi_\lambda \in L^2(A,\tau),$$

where this series converges in $\|\cdot\|_\tau$. But for $\nu \in \Lambda'$,

$$xv_\nu^* = \sum_{\lambda\in\Lambda'} \xi_\lambda v_\lambda v_\nu^* \ \text{ forces that}$$
$$e_A(xv_\nu^*) = \sum_{\lambda\in\Lambda'} \xi_\lambda e_A(v_\lambda v_\nu^*) = \xi_\nu v_\nu v_\nu^*.$$

Consequently, $x = \sum_{\lambda\in\Lambda'} e_A(xv_\lambda^*)v_\lambda = \sum_{\lambda\in\Lambda'} \mathbb{E}_A(xv_\lambda^*)v_\lambda \in \sum_{\lambda\in\Lambda'} Av_\lambda$ as $xv_\lambda^* \in M_\tau$. The reverse inclusion is obvious. □

COROLLARY 1. (i) $N(A)'' \cap M_\tau = \sum_{\lambda\in\Lambda'} Av_\lambda$.
(ii) *There exists an unique faithful normal conditional expectation from M onto $N(A)''$ preserving τ.*

Proof. (i) follows from Thm. 4 and Prop. 6. From (i) τ restricted to $N(A)''_+$ is semifinite, as the trace restricted to A_+ is semifinite. The modular automorphism group with respect to τ is trivial. Thus by a well known theorem of Takesaki Chap. 2 [15], there is an unique faithful normal τ-preserving conditional expectation onto $N(A)''$. It is easy to see that, for $x \in M_\tau$ one has $\mathbb{E}_{N(A)''}(x) = \sum_{\lambda\in\Lambda'} \mathbb{E}_A(xv_\lambda^*)v_\lambda \in N(A)''_\tau$, where v_λ's are as in Thm. 4. □

REMARK 4. If $A \subset B \subset N(A)''$ is an intermediate von Neumann subalgebra, then there exists an unique faithful normal τ-preserving conditional expectation onto B. This will follow because $L^2(B,\tau)$ will have a decomposition as in Thm. 4, with possibly different partial isometries. However, it is not true that if there exists a normal conditonal expectation onto $N(A)''$, then there is a normal conditonal expectation onto A. In the I_∞ case this is clear, as the continuous masa $L^\infty([0,1],\lambda) \subset \mathbf{B}(L^2[0,1],\lambda)$ is regular but not Cartan (i.e., $N(L^\infty([0,1],\lambda))'' = \mathbf{B}(L^2[0,1],\lambda)$ but there are no normal conditional expectations onto $L^\infty([0,1],\lambda)$), where λ is Lebesgue measure. In the II_∞ setting, write $M = N \overline{\otimes} \mathbf{B}(L^2[0,1],\lambda)$, where N is a type II_1 factor with a Cartan masa B. Consider $A = B \overline{\otimes} L^\infty([0,1],\lambda) \subset M$. Then B is a regular masa in M. However, there are no normal conditional expectations from M onto A.

COROLLARY 2. *If $A \subset M$ is a Cartan masa, then there is an unique normal τ-preserving conditional expectation from M onto any von Neumann subalgebra containing A.*

4. Singularity

In this section we prove the equivalence of WAHP and singularity. We need some preparation.

LEMMA 2. (i) *Let $x \in M_\tau$. Then η_x admits (π_i,μ)-disintegrations $t \mapsto \eta_x^t$ and $s \mapsto \eta_x^s$, $i = 1,2$. Moreover, $\eta_x^t([0,1]\times[0,1]) = \mathbb{E}_A(xx^*)(t)$ and $\eta_x^s([0,1]\times[0,1]) = \mathbb{E}_A(x^*x)(s)$ almost everywhere.*
(ii) *Let $x \in M_\tau$ and $a \in C[0,1] \subset A$. Then the functions $t \mapsto \eta_x^t(1\otimes a)$ and $s \mapsto \eta_x^s(a\otimes 1)$ are in A_τ.*
(iii) *If $x \in M_\tau$ and $a \in C[0,1] \subset A$, then $\eta_x^t(1\otimes a) = \mathbb{E}_A(xax^*)(t)$ almost every t and $\eta_x^s(a\otimes 1) = \mathbb{E}_A(x^*ax)(s)$ almost every s.*

Proof. (i) The associated disintegrations exist because one can assume $\eta_x \ll \eta$ (Lemma 5.7 [4]) and η admits the disintegrations (see Lemma 3.6 [9]). Let $a \in A_\tau \cap C[0,1]$. Then using Prop. 8.5.1 [7] we have

$$\eta_x(a\otimes 1) = \langle ax, x\rangle_\tau = \tau(x^*ax) = \tau(axx^*) = \langle \mathbb{E}_A(xx^*), a^*\rangle_\tau = \tau(a\mathbb{E}_A(xx^*)).$$

Since $t \mapsto \eta_x^t([0,1]\times[0,1])$ is μ-measurable, we have

$$\int_0^1 a(t)\eta_x^t([0,1]\times[0,1])d\mu(t) = \int_0^1 a(t)\mathbb{E}_A(xx^*)(t)d\mu(t).$$

Thus for any μ-measurable set E one has

$$\int_E \eta_x^t([0,1]\times[0,1])d\mu(t) = \int_E \mathbb{E}_A(xx^*)(t)d\mu(t). \tag{2}$$

Indeed, first choose E to be a compact set of finite positive measure. Pointwise approximate χ_E from above by a monotone decreasing sequence of continuous functions f_n such that $0 \leqslant f_n \leqslant 1$ and the support of f_n for all n is contained in a larger compact set of finite measure. Note that $t \mapsto \eta_x^t([0,1]\times[0,1])$ is μ-integrable. Use dominated convergence to conclude that Eq. (2) holds for all compact sets of finite measure. Thus Eq. (2) holds when E is a measurable set of finite measure (by regularity of $\mu_{|E}$). Finally use σ-finiteness. The arguments for the (π_2,μ)-disintegration are similar.
(ii) Note that the stated functions are measurable. Now for almost all t we have $|\eta_x^t(1\otimes a)| \leqslant \|a\|\,\eta_x^t([0,1]\times[0,1]) = \|a\|\,\mathbb{E}_A(xx^*)(t)$. Note that $\mathbb{E}_A(xx^*) \in A_\tau$. Similarly argue for the (π_2,μ)-disintegration.
(iii) For $b \in C[0,1] \cap A_\tau$, we have

$$\eta_x(b\otimes a) = \langle bxa, x\rangle_\tau = \tau(x^*bxa) = \tau(bxax^*) = \langle xax^*, b^*\rangle_\tau = \langle \mathbb{E}_A(xax^*), b^*\rangle_\tau.$$

Thus taking disintegrations we have

$$\int_0^1 b(t)\eta_x^t(1\otimes a)d\mu(t) = \int_0^1 b(t)\mathbb{E}_A(xax^*)(t)d\mu(t),\ (\mathbb{E}_A(xax^*)\in A_\tau).$$

Consequently, the result follows by standard density arguments. The arguments for the (π_2,μ)-disintegration are similar. $\square$

THEOREM 7. *Let $A\subset M$ be a masa generated by finite projections. Let $x,y\in M_\tau$ be such that $\mathbb{E}_{N(A)''}(x)=0$ and $\mathbb{E}_{N(A)''}(y)=0$. Then*

$$\frac{1}{N+1}\sum_{k=0}^{N}\left\|\mathbb{E}_A(xu^ky^*)\right\|_\tau^2 \to 0 \text{ as } N\to\infty,$$

where $u\in A$ corresponds to the unitary $[0,1]\ni t\mapsto e^{2\pi it}$.

Proof. Note that by Cor. 1, $\mathbb{E}_{N(A)''}$ is defined. By Lemma 5.7 [4], the left-right measure of A is $[\eta_x+\nu]$ where ν is singular to η_x. Since $\mathbb{E}_{N(A)''}(x)=0$, so by Thm. 5 the disintegrations $t\mapsto\eta_x^t$ and $s\mapsto\eta_x^s$ are non atomic for almost all t and almost all s respectively. Choose a Borel μ null set F so that for $t\in F^c$ the measure η_x^t is nonatomic. Enlarging this null set by another Borel μ null set and renaming it to F again, we can assume that η_x^t is finite for all $t\in F^c$ (Lemma 2). By a result of Wiener on Fourier coefficients of finite measures one has

$$\frac{1}{N+1}\sum_{k=0}^{N}\left|\eta_x^t(1\otimes u^k)\right|^2 \to 0 \text{ as } N\to\infty \text{ for each } t\in F^c. \tag{3}$$

Fix $t\in F^c$. Then from Lemma 2 we have

$$\left|\eta_x^t(1\otimes u^k)\right|^2 \leqslant \eta_x^t([0,1]\times[0,1])^2 = |\mathbb{E}_A(xx^*)(t)|^2$$

for all $k\in\mathbb{N}$. Thus $\frac{1}{N+1}\sum_{k=0}^{N}\left|\eta_x^t(1\otimes u^k)\right|^2 \leqslant |\mathbb{E}_A(xx^*)(t)|^2$. As $x\in M_\tau$, so $\mathbb{E}_A(xx^*)\in A_\tau$. Noting that the semifinite trace τ restricted to $C[0,1]\subset A$ is the measure μ, use dominated convergence theorem to conclude that

$$\lim_N \int_0^1 \frac{1}{N+1}\sum_{k=0}^{N}\left|\eta_x^t(1\otimes u^k)\right|^2 d\mu(t) = 0.$$

Switching the integral and the sum, and, using Lemma 2 we get

$$\lim_N \frac{1}{N+1}\sum_{k=0}^{N}\left\|\mathbb{E}_A(xu^kx^*)\right\|_\tau^2 = 0.$$

Finally, use the polarization identity for conditional expectations to finish the proof. $\square$

COROLLARY 3. *Let $A \subset M$ be a masa generated by finite projections. A is singular if and only if for $x \in M_\tau$ and $\mathbb{E}_A(x) = 0$ one has*

$$\frac{1}{N+1}\sum_{k=0}^{N}\left\|\mathbb{E}_A(xu^k x^*)\right\|_\tau^2 \to 0 \text{ as } N \to \infty,$$

where $u \in A$ corresponds to the unitary $[0,1] \ni t \mapsto e^{2\pi it}$.

Proof. One way follows directly from Thm. 7. For the reverse direction, suppose to the contrary $A \subsetneq N(A)''$. Then from Thm. 4 there is a nonzero $v \in \mathcal{GN}_M(A) \cap M_\tau$ orthogonal to A_τ. The measure η_v disintegrates as completely atomic measures with atoms (one atom on $t \times [0,1]$ for almost every t on the domain of T_v) located on the partial automorphism graph. By Wiener's theorem,

$$\frac{1}{N+1}\sum_{k=0}^{N}\left|\int_0^1 e^{-2\pi iks} d\eta_v^t(s)\right|^2 \to \sum_{s\in[0,1]} \eta_v^t(\{s\})^2.$$

Thus by using Lemma 2 it follows that $\frac{1}{N+1}\sum_{k=0}^{N}\left\|\mathbb{E}_A(vu^k v^*)\right\|_\tau^2 \to \tau(v^*v) \neq 0$, which is a contradiction. □

Recall that a subset $S \subseteq \mathbb{N} \cup \{0\}$ is said to be of full density or density one if

$$\lim_n \frac{\#(S \cap [0,n])}{n+1} = 1.$$

COROLLARY 4. *Let $A \subset M$ be a masa generated by finite projections. A is singular if and only if A has WAHP.*

Proof. A bounded sequence of positive real numbers $\{a_n\}_{n\in\mathbb{N}}$ converges to zero in the sense of Cesàro if and only if there is a set $P \subseteq \mathbb{N}$ of density 1 such that $a_n \to 0$ along P (see Prop. 6.1.2 [12]). This shows that if A is singular, then A has WAHP (from Cor. 3). The other direction is also easy. If to the contrary $A \subsetneq N(A)''$, then from Thm. 4 there is a nonzero $v \in \mathcal{GN}_M(A) \cap M_\tau$ orthogonal to A_τ. Let $v^*v = p$. Then $p \in A_\tau$. For any unitary $u \in A$, one has $\|\mathbb{E}_A(vuv^*)\|_\tau^2 = \|vuv^*\|_\tau^2 = \tau(vu^*v^*vuv^*) = \tau(vpv^*) = \tau(p)$, which cannot be made arbitrarily small by varying u over the unitaries of A. This completes the argument. □

The next two results characterize the normalizing algebra of a masa. We just state them here, as their proofs are easy exercises and follows from the results in §4.

THEOREM 8. *Let $A \subset M$ be a masa generated by finite projections. Let $u \in A$ be the unitary generator corresponding to the function $[0,1] \ni t \mapsto e^{2\pi it}$. Let $x \in M_\tau$ be such that $\mathbb{E}_A(x) = 0$. Then the following are equivalent.*
(i) $\lim_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\left\|\mathbb{E}_A(xu^k x^*)\right\|_\tau^2 = 0$.
(ii) $\mathbb{E}_{N(A)''}(x) = 0$.

THEOREM 9. *Let $A \subset B \subset M$ be an inclusion of von Neumann algebras, where A is a masa in M generated by finite projections. Suppose there is an unique faithful normal conditional expectation from M onto B preserving τ. Also suppose that for each $x \in M_\tau$ with $\mathbb{E}_B(x) = 0$, one has*

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \left\| \mathbb{E}_A(xu^k x^*) \right\|_\tau^2 = 0,$$

where u is the unitary generator of A corresponding to the function $t \mapsto e^{2\pi it}$. Then $N(A)'' \subseteq B$.

REMARK 5. It is easy to see that the statements in §4 are valid if the special unitary u in the statements is replaced by unitaries that generate the masa.

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