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A NEW UPPER BOUND ON THE LARGEST NORMALIZED LAPLACIAN EIGENVALUE

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Abstract. Let $\mathcal{G}$ be a simple undirected connected graph on n vertices. Suppose that the vertices of $\mathcal{G}$ are labelled $1,2,\ldots,n$. Let d_i be the degree of the vertex i. The Randić matrix of $\mathcal{G}$, denoted by R, is the $n\times n$ matrix whose $(i,j)-$entry is $\frac{1}{\sqrt{d_i d_j}}$ if the vertices i and j are adjacent and 0 otherwise. The normalized Laplacian matrix of $\mathcal{G}$ is $\mathcal{L}=I-R$, where I is the $n\times n$ identity matrix. In this paper, by using an upper bound on the maximum modulus of the subdominant Randić eigenvalues of $\mathcal{G}$, we obtain an upper bound on the largest eigenvalue of $\mathcal{L}$. We also obtain an upper bound on the largest modulus of the negative Randić eigenvalues and, from this bound, we improve the previous upper bound on the largest eigenvalue of $\mathcal{L}$.

1. Introduction

Let $\mathcal{G}=(V,E)$ be a simple undirected graph on n vertices. Some matrices on $\mathcal{G}$ are the adjacency matrix A, the Laplacian matrix $L=D-A$ and the signless Laplacian matrix $Q=D+L$, where D is the diagonal matrix of vertex degrees. It is well known that L and Q are positive semidefinite matrices and that $(0,\mathbf{1})$ is an eigenpair of L where $\mathbf{1}$ is the all ones vector. Fiedler [16] proved that $\mathcal{G}$ is a connected graph if and only if the second smallest eigenvalue of L is positive. This eigenvalue is called the algebraic connectivity of $\mathcal{G}$. The signless Laplacian matrix has recently attracted the attention of several researchers. Recent papers on this matrix are [5, 6, 7, 8, 9] and some of its basic properties [6] are:

1. For a connected graph, the smallest eigenvalue of Q is equal to 0 if and only if the graph is bipartite. In this case, 0 is a simple eigenvalue. Then, for a connected graph, the smallest eigenvalue of Q is positive if and only if the graph is not bipartite.

2. If $\mathcal{G}$ is a bipartite graph then Q and L have the same characteristic polynomial.

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Other matrices on the graph $\mathscr{G}$ are the normalized Laplacian matrix and the Randić matrix of $\mathscr{G}$. Suppose that the vertices of $\mathscr{G}$ are labelled $1,2,\ldots,n$. Let d_i be the degree of the vertex i. Let $D^{-\frac{1}{2}}$ be the diagonal matrix whose diagonal entries are

$$\frac{1}{\sqrt{d_1}},\frac{1}{\sqrt{d_2}},\ldots,\frac{1}{\sqrt{d_n}}$$

whenever $d_i \neq 0$. If $d_i = 0$ for some i then the corresponding diagonal entry of $D^{-\frac{1}{2}}$ is defined to be 0. The normalized Laplacian matrix of $\mathscr{G}$, denoted by $\mathscr{L}$, was introduced by F. Chung [15] as

$$\mathscr{L} = D^{-\frac{1}{2}}LD^{-\frac{1}{2}} = I - D^{-\frac{1}{2}}AD^{-\frac{1}{2}}. \tag{1}$$

The eigenvalues of $\mathscr{L}$ are called the normalized Laplacian eigenvalues of $\mathscr{G}$. From (1), we have

$$D^{\frac{1}{2}}\mathscr{L}D^{\frac{1}{2}} = D - A = L$$

and thus

$$D^{\frac{1}{2}}\mathscr{L}D^{\frac{1}{2}}\mathbf{1} = L\mathbf{1} = 0\mathbf{1}.$$

Hence 0 is an eigenvalue of $\mathscr{L}$ with eigenvector $D^{\frac{1}{2}}\mathbf{1}$.

We recall the following results on $\mathscr{L}$ [15] :

1. The eigenvalues of $\mathscr{L}$ lie in the interval $[0,2]$.
2. 0 is a simple eigenvalue of $\mathscr{L}$ if and only if $\mathscr{G}$ is connected.
3. 2 is an eigenvalue of $\mathscr{L}$ if and only if a connected component of $\mathscr{G}$ is bipartite and nontrivial.

Among papers on $\mathscr{L}$, we mention [10, 11, 13, 14] and [17].

From now on, we assume that $\mathscr{G}$ is connected graph. Then $d_i > 0$ for all i. The notation $i \sim j$ means that the vertices i and j are adjacent. The matrix $R = D^{-\frac{1}{2}}AD^{-\frac{1}{2}}$ in (1) is the Randić matrix of $\mathscr{G}$ in which the (i,j)-entry is $\frac{1}{\sqrt{d_i d_j}}$ if $i \sim j$ and 0 otherwise. Moreover

$$I - \mathscr{L} = R.$$

The eigenvalues of R are called the Randić eigenvalues of $\mathscr{G}$. Clearly $\mathscr{L}$ and R are both real symmetric matrices. The Randić matrix was earlier studied in connection with the Randić index [1, 2, 18] and [19]. Two recent papers on the Randić matrix are [3] and [4].

Throughout this paper

$$0 = \lambda_n \leqslant \lambda_{n-1} \leqslant \ldots \leqslant \lambda_1$$

and

$$\rho_n \leqslant \rho_{n-1} \leqslant \ldots \leqslant \rho_1$$

are the normalized Laplacian eigenvalues and the Randić eigenvalues of $\mathcal{G}$, respectively. It follows that

$$\lambda_i = 1 - \rho_{n-i+1} \ (1 \leqslant i \leqslant n).$$

If M is a nonnegative matrix then, by the Perron-Frobenius Theorem, M has an eigenvalue equal to its spectral radius, called the Perron root of M. In addition, if M is irreducible then the Perron root of M is a simple eigenvalue with a corresponding positive eigenvector, called the Perron vector of M. Since $\mathcal{G}$ is a connected graph, Randić matrix of $\mathcal{G}$ is a irreducible nonnegative matrix. Let $\mathbf{v} = D^{\frac{1}{2}}\mathbf{1}$. Then $\mathbf{v} = \left[\sqrt{d_1}, \sqrt{d_2}, \ldots, \sqrt{d_n}\right]^T$. An easy computation shows that

$$R\mathbf{v} = \mathbf{v}.$$

Hence, 1 and $\mathbf{v}$ are the Perron root and the Perron vector of R, respectively.

Let Δ and δ be the largest and smallest vertex degrees of $\mathcal{G}$, respectively, and let q_n be the smallest eigenvalue of Q.

A recent result involving the largest eigenvalue of $\mathcal{L}$ and the smallest eigenvalue of Q is

THEOREM 1. [17] *Let $\mathcal{G}$ be a connected graph. Then*

$$2 - \frac{q_n}{\delta} \leqslant \lambda_1 \leqslant 2 - \frac{q_n}{\Delta}. \tag{2}$$

We may consider $2 - \frac{q_n}{\Delta}$ as an upper bound on λ_1. Observe that $2 - \frac{q_n}{\Delta} = 2$ if and only if $\mathcal{G}$ is a bipartite graph.

In this paper, we search for a new upper bound on λ_1 not exceeding the trivial upper bound 2.

2. Searching for an upper bound on λ_1

Since $\sum_{i=1}^{n} \rho_i = tr(R) = 0$, it follows that $\rho_n < 0$. We have

$$\lambda_1 = 1 - \rho_n = 1 + |\rho_n|.$$

In order to find an upper bound on λ_1 not exceeding 2, we look for an upper bound on $|\rho_n|$ not exceeding 1.

An eigenvalue of a nonnegative matrix M which is different from the Perron root is called a subdominant eigenvalue of M. Let $\xi(M)$ be the maximum modulus of the subdominant eigenvalues of M. Special attention has been devoted to find upper bounds on $\xi(M)$. In [20], we can find a unified presentation of results concerning upper bounds on $\xi(M)$. These upper bounds are important because $\xi(M)$ plays a major role in convergence properties of powers of M. Since

$$\lambda_1 \leqslant 1 + \xi(R), \tag{3}$$

we focus our attention on upper bounds on $\xi(R)$. We recall the result [12, p. 295] :

THEOREM 2. *If $M = (m_{i,j}) \geqslant 0$ of order $n \times n$ has a positive eigenvector*

$$\mathbf{w} = [w_1, w_2, \dots, w_n]^T$$

corresponding to the spectral radius $\rho(M)$ of M then

$$\xi(M) \leqslant \frac{1}{2} \max_{i<j} \sum_{k=1}^{n} w_k \left| \frac{m_{i,k}}{w_i} - \frac{m_{j,k}}{w_j} \right|.$$

where the maximum is taken over all pairs (i,j), $1 \leqslant i < j \leqslant n$.

In order to apply Theorem 2, it is convenient to observe that the Randić matrix of $\mathscr{G}$ is diagonally similar to the row stochastic matrix

$$S = D^{-\frac{1}{2}} R D^{\frac{1}{2}}. \tag{4}$$

The following lemma gives some immediate properties of S.

LEMMA 1. *1. The $(i,j)-$entry of S is $\frac{1}{d_i}$ if $j \sim i$ and 0 otherwise.*
2. $S\mathbf{1} = \mathbf{1}$ where $\mathbf{1}$ is the all ones vector.
3. $\mathbf{u}$ is an eigenvector for R corresponding to the eigenvalue α if and only if $D^{-\frac{1}{2}}\mathbf{u}$ is an eigenvector for S corresponding to the eigenvalue α.
4. If $\mathscr{G}$ is an $r-$regular graph then $S = R$.

Let N_i be the set of neighbours of the vertex v_i and let $|N_i|$ be the cardinality of N_i.

THEOREM 3. *Let $\mathscr{G}$ be a simple undirected connected graph. If λ_1 is the largest eigenvalue of $\mathscr{L}$ then*

$$|\lambda_1| \leqslant 2 - \min_{i<j} \left\{ \frac{|N_i \cap N_j|}{\max\{d_i, d_j\}} \right\} \tag{5}$$

where the minimum is taken over all pairs (i,j), $1 \leqslant i < j \leqslant n$.

Proof. We know that the Randić matrix of $\mathscr{G}$ is similar to the row stochastic matrix S defined in (4). Then $\xi(R) = \xi(S)$. The eigenvector corresponding to the spectral of S is $\mathbf{w} = \mathbf{1}$. Applying Theorem 2 to $S = (s_{i,j})$, we have

$$\begin{aligned}
\xi(S) &\leqslant \frac{1}{2} \max_{i<j} \sum_{k=1}^{n} |s_{i,k} - s_{j,k}| \\
&= \frac{1}{2} \max_{i<j} \left(\sum_{k \in N_i - N_j} \frac{1}{d_i} + \sum_{k \in N_j - N_i} \frac{1}{d_j} + \sum_{k \in N_i \cap N_j} \left| \frac{1}{d_i} - \frac{1}{d_j} \right| \right) \\
&= \frac{1}{2} \max_{i<j} \left(\frac{|N_i - N_j|}{d_i} + \frac{|N_j - N_i|}{d_j} + \sum_{k \in N_i \cap N_j} \left| \frac{1}{d_i} - \frac{1}{d_j} \right| \right) \\
&= \frac{1}{2} \max_{i<j} \left(2 - \frac{|N_i \cap N_j|}{d_i} - \frac{|N_j \cap N_i|}{d_j} + \sum_{k \in N_i \cap N_j} \left| \frac{1}{d_i} - \frac{1}{d_j} \right| \right).
\end{aligned}$$

Suppose $d_i = \max\left\{d_i, d_j\right\}$. In this case

$$\begin{aligned}
&2-\frac{\left|N_i\cap N_j\right|}{d_i}-\frac{\left|N_j\cap N_i\right|}{d_j}+\sum_{k\in N_i\cap N_j}\left|\frac{1}{d_i}-\frac{1}{d_j}\right| \\
&=2-\frac{\left|N_i\cap N_j\right|}{d_i}-\frac{\left|N_j\cap N_i\right|}{d_j}+\left(\frac{1}{d_j}-\frac{1}{d_i}\right)N_i\cap N_j \\
&=2-\frac{2\left|N_i\cap N_j\right|}{d_i}.
\end{aligned}$$

Similarly, if $d_j = \max\left\{d_i, d_j\right\}$ then

$$\begin{aligned}
&2-\frac{\left|N_i\cap N_j\right|}{d_i}-\frac{\left|N_j\cap N_i\right|}{d_j}+\sum_{k\in N_i\cap N_j}\left|\frac{1}{d_i}-\frac{1}{d_j}\right| \\
&=2-\frac{2\left|N_j\cap N_i\right|}{d_j}.
\end{aligned}$$

Hence

$$\begin{aligned}
\xi(S) &\leqslant \frac{1}{2}\max_{i<j}\sum_{k=1}^{n}\left|s_{i,k}-s_{j,k}\right| \\
&=\frac{1}{2}\max_{i<j}\left\{2-\frac{2\left|N_j\cap N_i\right|}{\max\left\{d_i,d_j\right\}}\right\} \\
&=1-\min_{i<j}\left\{\frac{\left|N_i\cap N_j\right|}{\max\left\{d_i,d_j\right\}}\right\}
\end{aligned}$$

Since $\lambda_1 \leqslant 1+\xi(R) = 1+\xi(S)$, the upper bound in (5) follows. □

REMARK 1. If $\mathscr{G}$ is a bipartite graph then $\left|N_i \cap N_j\right| = 0$, for some $i < j$, and consequently the upper bound in (5) is equal to 2. This is sufficient condition but it is not a necessary condition. In fact, there are other instances in which $N_i \cap N_j = 0$ for some $i < j$. One of them is given by a nonbipartite graph having a bridge. However, if $\min_{i<j}\left|N_i \cap N_j\right| \geqslant 1$ and $q_n < 1$ then

$$2-\min_{i<j}\left\{\frac{\left|N_i\cap N_j\right|}{\max\left\{d_i,d_j\right\}}\right\}<2-\frac{q_n}{\Delta}. \tag{6}$$

In fact

$$q_n < 1 \leqslant \left|N_i\cap N_j\right| \text{ for } i<j$$

and

$$\frac{q_n}{\Delta} \leqslant \frac{1}{\max\left\{d_i,d_j\right\}} \text{ for } i<j.$$

Then

$$\frac{q_n}{\Delta} < \frac{\left|N_i \cap N_j\right|}{\max\left\{d_i, d_j\right\}} \text{ for } i < j.$$

It follows

$$2 - \frac{q_n}{\Delta} > 2 - \min_{i<j} \frac{\left|N_i \cap N_j\right|}{\max\left\{d_i, d_j\right\}}.$$

Hence, if $\min_{i<j}\left|N_i \cap N_j\right| \geqslant 1$ and $q_n < 1$ then (5) gives a better upper bound for λ_1 than the second inequality in (2) does.

3. Improving the upper bound on λ_1

We have

$$\lambda_1 = u1 + |q_n| \leqslant 1 + \xi(R) = 1 + \xi(S).$$

The upper bound on λ_1 in (5) was obtained by using an upper bound on $\xi(R)$. In this section, in order to get an improved upper bound on λ_1, we search for an upper bound on $|q_n|$, that is, on the largest modulus of the negative Randić eigenvalues.

THEOREM 4. *Let $\mathcal{G}$ be a simple undirected connected graph. If ρ_n is eigenvalue with the largest modulus among the negative Randić eigenvalues of $\mathcal{G}$ then*

$$|\rho_n| \leqslant 1 - \min_{i\sim j}\left\{\frac{\left|N_i \cap N_j\right|}{\max\left\{d_i, d_j\right\}}\right\}$$

where the minimum is taken over all pairs (i, j), $1 \leqslant i < j \leqslant n$, such that the vertices i and j are adjacent.

Proof. Let ρ_n be the largest modulus of the negative Randić eigenvalues of $\mathcal{G}$. Let

$$\mathbf{x} = [x_1, x_2, \ldots, x_n]^T$$

be such that

$$S\mathbf{x} = \rho_n \mathbf{x}. \tag{7}$$

From Lemma 1, we have $\mathbf{x} = D^{-\frac{1}{2}}\mathbf{u}$ where $R\mathbf{u} = \rho_n\mathbf{u}$. Since $\mathbf{u}$ is orthogonal to the Perron vector $\mathbf{v} = \left[\sqrt{d_1}, \sqrt{d_2}, \ldots, \sqrt{d_n}\right]^T$, the vector $\mathbf{u}$ has at least one positive component and at least one negative component. Since $\mathbf{x} = D^{-\frac{1}{2}}\mathbf{u}$, this is also true for the vector $\mathbf{x}$. Let

$$\max\{x_1, x_2, \ldots, x_n\} = x_i$$

and let

$$x_j = \min\{x_k : k \sim i\}.$$

Since $\mathbf{x}$ has at least one positive component, $x_i > 0$. Let $S = (s_{i,j})$. From (7)

$$\rho_n x_j = \sum_{k=1}^{n} s_{j,k} x_k = \frac{1}{d_j} \sum_{k \in N_j} x_k \tag{8}$$

and

$$\rho_n x_i = \sum_{k=1}^{n} s_{i,k} x_k = \frac{1}{d_i} \sum_{k \in N_i} x_k. \tag{9}$$

Subtracting (9) from (8), we get

$$\rho_n (x_j - x_i) = \frac{1}{d_j} \sum_{k \in N_j} x_k - \frac{1}{d_i} \sum_{k \sim i} x_k.$$

Then

$$q_n (x_j - x_i) \tag{10}$$
$$= \frac{1}{d_j} \sum_{k \in N_j - N_i} x_k + \frac{1}{d_j} \sum_{k \in N_j \cap N_i} x_k - \frac{1}{d_i} \sum_{k \in N_i - N_j} x_k - \frac{1}{d_i} \sum_{k \in N_j \cap N_i} x_k.$$

By definition, $x_j \leqslant x_k$ for all $k \sim i$ and $x_k \leqslant x_i$ for all k. Hence

$$\sum_{k \in N_j - N_i} x_k \leqslant \left| N_j - N_i \right| x_i \tag{11}$$

and

$$- \sum_{k \in N_i - N_j} x_k \leqslant - \left| N_i - N_j \right| x_j. \tag{12}$$

Replacing the inequalities (11) and (12) in (10), we obtain

$$q_n (x_j - x_i)$$
$$\leqslant \frac{1}{d_j} \left| N_j - N_i \right| x_i - \frac{1}{d_i} \left| N_i - N_i \right| x_j + \sum_{k \in N_j \cap N_i} \left(\frac{1}{d_j} - \frac{1}{d_i} \right) x_k.$$

Thus

$$\begin{aligned} q_n (x_j - x_i) \leqslant & \frac{1}{2} \frac{1}{d_j} \left| N_j - N_i \right| (x_i - x_j) + \frac{1}{2} \frac{1}{d_i} \left| N_i - N_j \right| (x_i - x_j) \\ & + \frac{1}{2} \left(\frac{1}{d_j} \left| N_j - N_i \right| - \frac{1}{d_i} \left| N_i - N_j \right| \right) (x_i + x_j) \\ & + \sum_{k \in N_i \cap N_j} \left(\frac{1}{d_j} - \frac{1}{d_i} \right) x_k. \end{aligned}$$

Clearly

$$\frac{1}{d_j} \left| N_j - N_i \right| - \frac{1}{d_i} \left| N_i - N_j \right| = \left(\frac{1}{d_i} - \frac{1}{d_j} \right) \left| N_i \cap N_j \right|.$$

Hence

$$\begin{aligned}\rho_n(x_j - x_i) &\leqslant \frac{1}{2}\frac{1}{d_j}\left|N_j - N_i\right|(x_i - x_j) + \frac{1}{2}\frac{1}{d_i}\left|N_i - N_j\right|(x_i - x_j) \\ &\quad + \frac{1}{2}\left(\frac{1}{d_i} - \frac{1}{d_j}\right)\left|N_i \cap N_j\right|(x_i + x_j) + \frac{1}{2}\sum_{k \in N_i \cap N_j}\left(\frac{1}{d_j} - \frac{1}{d_i}\right)(x_k + x_k) \\ &= \frac{1}{2}\frac{1}{d_j}\left|N_j - N_i\right|(x_i - x_j) + \frac{1}{2}\frac{1}{d_i}\left|N_i - N_j\right|(x_i - x_j) \\ &\quad + \frac{1}{2}\sum_{k \in N_i \cap N_j}\left(\frac{1}{d_i} - \frac{1}{d_j}\right)(x_i - x_k + x_j - x_k).\end{aligned}$$

Moreover

$$\begin{aligned}&\sum_{k \in N_i \cap N_j}\left(\frac{1}{d_i} - \frac{1}{d_j}\right)(x_i - x_k + x_j - x_k) \\ &\leqslant \sum_{k \in N_i \cap N_j}\left|\frac{1}{d_i} - \frac{1}{d_j}\right|(x_i - x_k) + \sum_{k \in N_i \cap N_j}\left|\frac{1}{d_i} - \frac{1}{d_j}\right|(x_k - x_j) \\ &= \sum_{k \in N_i \cap N_j}\left|\frac{1}{d_i} - \frac{1}{d_j}\right|(x_i - x_j).\end{aligned}$$

Therefore

$$\begin{aligned}\rho_n(x_j - x_i) \leqslant\ &\frac{1}{2}\frac{1}{d_j}\left|N_j - N_i\right|(x_i - x_j) + \frac{1}{2}\frac{1}{d_i}\left|N_i - N_j\right|(x_i - x_j) \\ &+ \frac{1}{2}\sum_{k \in N_i \cap N_j}\left|\frac{1}{d_i} - \frac{1}{d_j}\right|(x_i - x_j).\end{aligned} \tag{13}$$

If $x_j = x_i$ then $x_k = x_i$ for all $k \sim i$. Consequently, from $S\mathbf{x} = \rho_n \mathbf{x}$, we have

$$q_n x_i = \sum_{k \in N_i}\frac{1}{d_i}x_k = \frac{1}{d_i}\sum_{k \in N_i}x_i = \frac{x_i}{d_i}d_i = x_i.$$

Thus $\rho_n = 1$, which is a contradiction. Hence $x_i - x_j > 0$. Dividing both sides of (13) by $(x_i - x_j)$, we obtain

$$-\rho_n \leqslant \frac{1}{2}\frac{1}{d_j}\left|N_j - N_i\right| + \frac{1}{2}\frac{1}{d_i}\left|N_i - N_j\right| + \frac{1}{2}\sum_{k \in N_i \cap N_j}\left|\frac{1}{d_i} - \frac{1}{d_j}\right|. \tag{14}$$

As in the proof of Theorem 3, we get

$$\begin{aligned}&\frac{1}{2}\frac{1}{d_j}\left|N_j - N_i\right| + \frac{1}{2}\frac{1}{d_i}\left|N_i - N_j\right| + \frac{1}{2}\sum_{k \in N_i \cap N_j}\left|\frac{1}{d_i} - \frac{1}{d_j}\right| \\ &= 1 - \frac{\left|N_i \cap N_j\right|}{\max\left\{d_i, d_j\right\}}.\end{aligned}$$

Consequently

$$|\rho_n| \leqslant 1 - \frac{|N_i \cap N_j|}{\max\{d_i, d_j\}}.$$

Observe that the vertices v_i and v_j are adjacent. Hence

$$|\rho_n| \leqslant \max_{i\sim j}\left\{1 - \frac{|N_i \cap N_j|}{\max\{d_i, d_j\}}\right\} = 1 - \min_{i\sim j}\left\{\frac{|N_i \cap N_j|}{\max\{d_i, d_j\}}\right\}.$$

The proof is complete. □

Finally, we have

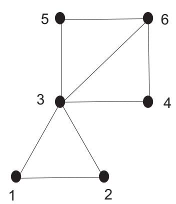
THEOREM 5. *Let $\mathscr{G}$ be a simple undirected connected graph. If λ_1 is the largest normalized Laplacian eigenvalue of $\mathscr{G}$ then*

$$\lambda_1 \leqslant 2 - \min_{i\sim j}\left\{\frac{|N_i \cap N_j|}{\max\{d_i, d_j\}}\right\}$$

where the minimum is taken over all pairs (i, j), $1 \leqslant i < j \leqslant n$, such that the vertices i and j are adjacent.

Proof. Since $\lambda_1 = 1 - \rho_n = 1 + |\rho_n|$, the proof is immediate using the upper bound on $|\rho_n|$ given by Theorem 4. □

EXAMPLE 1. $\mathscr{G}$:



Let

$$b(i, j) = \frac{|N_i \cap N_j|}{\max\{d_i, d_j\}}$$

For this graph

$$b(1,2) = \frac{1}{2}$$

$$b(1,3) = b(2,3) = b(3,4) = b(3,5) = \frac{1}{5},\ b(3,6) = \frac{2}{5}$$

$$b(4,6) = b(5,6) = \frac{1}{3}.$$

Then $\min_{i\sim j} b(i, j) = \frac{1}{5}$. Hence the largest modulus of the negative Randić eigenvalues is bounded above by $\frac{4}{5}$ and the largest normalized Laplacian eigenvalue is bounded above by $\frac{9}{5} = 1.8$. To four decimal places the smallest signless Laplacian eigenvalue of $\mathscr{G}$ is 0.7411. Since $\Delta = 5$, the upper bound in (2) becomes $2 - \frac{0.7411}{5} = 1.8518$.

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