Operators and Matrices
Volume 7, Number 2 (2013), 333–355
doi:10.7153/oam-07-20

# UNITAL FULL AMALGAMATED FREE PRODUCTS OF MF C*–ALGEBRAS

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(*Communicated by D. Hadwin*)

*Abstract.* In the paper, we consider the question whether a unital full amalgamated free product of MF algebras is MF. First, we show that, under a natural condition, a unital full free product of two MF algebras with amalgamation over a finite-dimensional C*-algebra is again an MF algebra. As an application, we show that a unital full free product of two AF algebras with amalgamation over an AF algebra is an MF algebra if there are faithful tracial states on each of these two AF algebras such that the restrictions on the common subalgebra agree.

## 1. Introduction

The concept of MF algebras was first introduced by Blackadar and Kirchberg in [3]. If a separable C*- algebra $\mathscr{A}$ can be embedded into $\prod_k \mathscr{M}_{n_k}(\mathbb{C})/\sum_k \mathscr{M}_{n_k}(\mathbb{C})$ for a sequence of positive integers $\{n_k\}_{k=1}^{\infty}$, then $\mathscr{A}$ is called an MF algebra. Many properties of MF algebras were discussed in [3]. For example, it was shown there that an inductive limit of MF algebras is an MF algebra and every subalgebra of an MF algebra is an MF algebra. This class of C*-algebras is of interest for many reasons. For example, it plays an important role in the classification of C$^*$-algebras and it is connected to the question whether the Ext semigroup, in the sense of Brown, Douglas and Fillmore [5], of a unital C$^*$-algebra is a group (see the striking result of Haagerup and Thorbjørnsen on $Ext(C_r^*(F_2)$ in [15]). This notion is also closely connected to Voiculescu's topological free entropy dimension for a family of self-adjoint elements $x_1,\cdots,x_n$ in a unital C*-algebra $\mathscr{A}$ [29].

Recall that a C*-algebra is said to be residually finite-dimensional (RFD) if it has a separating family of finite-dimensional representations. In [24], a necessary and sufficient condition was given for a unital full free product of RFD C*-algebras with amalgamation over a finite-dimensional C*-algebra to be RFD again. For MF algebras, Hadwin, Li and Shen have shown that a unital full free product of MF algebras is MF. Based on this result and the results on unital full amalgmated free products of RFD C*-algebras in [24], it is natural to ask whether the similar results hold for MF algebras. For the case when the common part of two C*-algebras in a unital full amalgamated

*Mathematics subject classification* (2010): 46L09, 46L35.
*Keywords and phrases*: MF C*-algebras, unital full amalgamated free products of C*-algebras.
The research of second author is partially sponsored by an NSF grant.

free product is *-isomorphic to a full matrix algebra (Proposition 1, [24]), we know that this unital full amalgamated free product is MF (respectively, RFD, quasidiagonal) if and only if these two C*-algebras are both MF (respectively, RFD, quasidiagonal).

In this paper, we consider the case when the common part in a unital full amalgamated free product is a finite-dimensional C*-algebra or an AF algebra. First, we show that, under a natural condition, a unital full free product of two MF algebras with amalgamation over a finite-dimensional C*-algebra is MF. As an application, we show that a unital full free product of two AF algebras with amalgamation over an AF algebra is an MF algebra if there are faithful tracial states on each of these two AF algebras such that the restrictions on the common subalgebra agree.

Recall the definition of full amalgamated free product of unital C*-algebras as follows:

*Given C*-algebras $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{D}$ with unital embeddings (injective $*$-homomorphisms) $\psi_{\mathscr{A}} : \mathscr{D} \to \mathscr{A}$ and $\psi_{\mathscr{B}} : \mathscr{D} \to \mathscr{B}$, the corresponding full amalgamated free product C*-algebra is the C*-algebra $\mathscr{C}$, equipped with unital embeddings $\sigma_{\mathscr{A}} : \mathscr{A} \to \mathscr{C}$ and $\sigma_{\mathscr{B}} : \mathscr{B} \to \mathscr{C}$ such that $\sigma_{\mathscr{A}} \circ \psi_{\mathscr{A}} = \sigma_{\mathscr{B}} \circ \psi_{\mathscr{B}}$, such that $\mathscr{C}$ is generated by $\sigma_{\mathscr{A}}(\mathscr{A}) \cup \sigma_{\mathscr{B}}(\mathscr{B})$ and satisfying the universal property that whenever $\mathscr{E}$ is a C*-algebra and $\pi_{\mathscr{A}} : \mathscr{A} \to \mathscr{E}$ and $\pi_{\mathscr{B}} : \mathscr{B} \to \mathscr{E}$ are $*$-homomorphisms satisfying $\pi_{\mathscr{A}} \circ \psi_{\mathscr{A}} = \pi_{\mathscr{B}} \circ \psi_{\mathscr{B}}$, there is a $*$-homomorphism $\pi : \mathscr{C} \to \mathscr{E}$ such that $\pi \circ \sigma_{\mathscr{A}} = \pi_{\mathscr{A}}$ and $\pi \circ \sigma_{\mathscr{B}} = \pi_{\mathscr{B}}$. The full amalgamated free product C*-algebra $\mathscr{C}$ is commonly denoted by $\mathscr{A} \underset{\mathscr{D}}{*} \mathscr{B}$.*

When $D = \mathbb{C}I$, the above definition is the unital full free product $\mathscr{A} *_{\mathbb{C}} \mathscr{B}$ of $\mathscr{A}$ and $\mathscr{B}$. Our main results are as follows:

THEOREM 2. *Let $\mathscr{A}$ and $\mathscr{B}$ be unital MF-algebras and $\mathscr{D}$ be a finite-dimensional C*-algebra. Let $\psi_1 : \mathscr{D} \to \mathscr{A}$ and $\psi_2 : \mathscr{D} \to \mathscr{B}$ be unital embeddings. Then $\mathscr{A} \underset{\mathscr{D}}{*} \mathscr{B}$ is an MF algebra if and only if there is a sequence $\{k_n\}_{n=1}^{\infty}$ of integers with unital embeddings $q_1 : \mathscr{A} \to \prod_{n=1}^{\infty} \mathscr{M}_{k_n}(\mathbb{C}) / \sum_{n=1}^{\infty} \mathscr{M}_{k_n}(\mathbb{C})$ and $q_2 : \mathscr{B} \to \prod_{n=1}^{\infty} \mathscr{M}_{k_n}(\mathbb{C}) / \sum_{n=1}^{\infty} \mathscr{M}_{k_n}(\mathbb{C})$ such that the following diagram commutes*

$$\begin{array}{ccc} \mathscr{D} & \overset{\psi_{\mathscr{A}}}{\hookrightarrow} & \mathscr{A} \\ \psi_{\mathscr{B}} \downarrow & & \downarrow q_1 \\ \mathscr{B} & \overset{q_2}{\hookrightarrow} & \prod_{m=1}^{\infty} \mathscr{M}_{k_m}(\mathbb{C}) / \sum \mathscr{M}_{k_m}(\mathbb{C}) \end{array}$$

When the common C*-subalgebra $\mathscr{D}$ is an AF algebra, we obtain following results.

THEOREM 4. *Suppose that $\mathscr{A} \supset \mathscr{D} \subset \mathscr{B}$ are unital inclusions of unital MF algebras where $\mathscr{D}$ is an AF algebra. Then the unital full free product $\mathscr{A} *_{\mathscr{D}} \mathscr{B}$ of $\mathscr{A}$ and $\mathscr{B}$ with amalgamation over $\mathscr{D}$ is an MF algebra if and only if there is an MF algebra $\mathscr{E}$ such that*

$$\mathscr{E} \supseteq \mathscr{A} \supset \mathscr{D} \subset \mathscr{B} \subseteq \mathscr{E}.$$

THEOREM 5. *Suppose that* $\mathscr{A} \supset \mathscr{D} \subset \mathscr{B}$ *are unital inclusions of AF* $C^*$ *-algebras. If there are faithful tracial states* $\tau_{\mathscr{A}}$ *and* $\tau_{\mathscr{B}}$ *on* $\mathscr{A}$ *and* $\mathscr{B}$ *respectively, such that*

$$\tau_{\mathscr{A}}(x) = \tau_{\mathscr{B}}(x), \qquad \forall\, x \in \mathscr{D},$$

*then* $\mathscr{A} *_{\mathscr{D}} \mathscr{B}$ *is an MF algebra.*

COROLLARY 2. *Suppose that* $\mathscr{A} \supseteq \mathscr{D} \subseteq \mathscr{B}$ *are unital inclusions of C*-algebras where* $\mathscr{A}$, $\mathscr{B}$ *are UHF algebras and* $\mathscr{D}$ *is an AF algebra. Then* $\mathscr{A} \underset{\mathscr{D}}{*} \mathscr{B}$ *is an MF algebra if and only if*

$$\tau_{\mathscr{A}}(z) = \tau_{\mathscr{B}}(x) \text{ for each } z \in \mathscr{D},$$

*where* $\tau_{\mathscr{A}}$ *and* $\tau_{\mathscr{B}}$ *are the unique faithful tracial states on* $\mathscr{A}$ *and* $\mathscr{B}$ *respectively.*

A brief overview of this paper is as follows. For the sake of completeness, in Section 2, we fix our notation and recall the basic properties of MF C*-algebras and quasidiagonal C*-algebras. Section 3 is devoted to results on the full amalgamated free products of unital MF C*-algebras. We first consider unital full free products of unital MF algebras with amalgamation over finite-dimensional C*-subalgebras. Then we consider the case when the common unital C*-subalgebra in a unital full amalgamated free product is an infinite-dimensional C*-algebra. More precisely, we consider the case when the common part is an AF algebra.

## 2. Definitions and preliminaries

### 2.1. Noncommutative polynomials and MF algebras

In this article, we always assume that all C*-algebras are unital separable $C^*$-algebras. We use notation $C^*(x_1, x_2, \cdots)$ to denote the unital C*-algebra generated by $\{x_1, x_2, \cdots\}$. Let $\mathbb{C}\langle \mathbf{X}_1, \ldots, \mathbf{X}_n\rangle$ be the set of all noncommutative polynomials in the indeterminants $\mathbf{X}_1, \ldots, \mathbf{X}_n$. Let $\mathbb{C}_{\mathbb{Q}} = \mathbb{Q} + i\mathbb{Q}$ denote the complex-rational numbers, i.e., the numbers whose real and imaginary parts are rational. Then the set $\mathbb{C}_{\mathbb{Q}}\langle \mathbf{X}_1, \ldots, \mathbf{X}_n\rangle$ of noncommutative polynomials with complex-rational coefficients is countable. Throughout this paper we write

$$\mathbb{C}\langle \mathbf{X}_1, \mathbf{X}_2, \cdots\rangle = \cup_{m=1}^{\infty} \mathbb{C}\langle \mathbf{X}_1, \mathbf{X}_2, \cdots \mathbf{X}_m\rangle$$

and

$$\mathbb{C}_{\mathbb{Q}}\langle \mathbf{X}_1, \mathbf{X}_2, \cdots\rangle = \cup_{m=1}^{\infty} \mathbb{C}_{\mathbb{Q}}\langle \mathbf{X}_1, \mathbf{X}_2, \cdots \mathbf{X}_m\rangle.$$

Let $\{P_r\}_{r=1}^{\infty}$ be the collection of all noncommutative polynomials in $\mathbb{C}_{\mathbb{Q}}\langle \mathbf{X}_1, \mathbf{X}_2, \cdots\rangle$ with rational complex coefficients.

We need more notation and concepts for recalling a theorem which gives an equivalent condition for the MF property. We assume that $\mathscr{H}$ is a separable complex Hilbert space and $\mathscr{B}(\mathscr{H})$ is the set of all bounded operators on $\mathscr{H}$. Let $SOT$ denote the strong operator topology on $\mathscr{B}(\mathscr{H})$. A sequence $\{T_k\}$ of operators converges to an operator $T$ is the $*$-strong operator topology ($*$-$SOT$) if and only if $T_k \longrightarrow T \;\; (SOT)$

and $T_k^* \longrightarrow T^*$ $(SOT)$. We say that a sequence $\left\{\left\langle T_1^{(k)}, \cdots, T_n^{(k)} \right\rangle\right\}$ of $n$-tuples converges to $\langle T_1, \cdots, T_n\rangle$ if and only if

$$T_i^{(k)} \to T_i \quad *\text{-}SOT$$

as $k \to \infty$.

Suppose $\mathscr{A}$ is a separable unital C*-algebra on a Hilbert space $\mathscr{H}$. Let $\mathscr{H}^\infty = \oplus_{\mathbb{N}} \mathscr{H}$ and, for any $x \in \mathscr{A}$, let $x^\infty$ be the element $\oplus_{\mathbb{N}} x = (x, x, x, \ldots)$ in $\prod_{k\in\mathbb{N}} \mathscr{A}^{(k)} \subset B(\mathscr{H}^\infty)$, where $\mathscr{A}^{(k)}$ is the $k$-th copy of $\mathscr{A}$.

The following theorem is one of the key ingredients for showing our main results in this paper.

THEOREM 1. (Theorem 5.1.2, [16]) *Suppose that $\mathscr{A}$ is a unital $C^*$-algebra generated by a sequence of self-adjoint elements $x_1, x_2, \cdots$ in $\mathscr{A}$. Then the following are equivalent:*

1. *$\mathscr{A}$ is an MF algebra*
2. *For each $n \in \mathbb{N}$, there are a sequence of positive integers $\{m_k\}_{k=1}^\infty$ and self-adjoint matrices $A_1^{(k)}, \ldots, A_n^{(k)}$ in $\mathscr{M}_{m_k}^{s.a.}(\mathbb{C})$ for $k = 1, 2, \ldots$, such that, $\forall\, P \in \mathbb{C}\langle X_1, \ldots, X_n\rangle$,*

$$\lim_{k\to\infty} \|P(A_1^{(k)}, \ldots, A_n^{(k)})\| = \|P(x_1, \ldots, x_n)\|.$$

The examples of MF algebras contain all finite dimensional C*-algebras, AF (approximately finite dimensional) algebras and quasidiagonal C*-algebras. In [15], Haagerup and Thorbjørnsen showed that $C_r^*(F_n)$ is an MF algebra for $n \geqslant 2$. For more examples of MF algebras, we refer the reader to [3] and [20].

### 2.2. Basic properties of quasidiagonal algebras

Quasidiagonal operators on separable Hilbert spaces were defined by P. R. Halmos [22] as compact perturbations of block-diagonal operators. A generalized notion of quasidiagonal operators to sets of operators is the concept of quasidiagonal sets of operators. A C*-algebra $\mathscr{A}$ is quasidiagonal (QD) if there is a faithful representation $\rho$ such that $\rho(\mathscr{A})$ is a quasidiagonal set of operators. This class of C*-algebras has been studied for more than 30 years. In [4], it has been shown that a full free product of two unital QD C*-algebras amalgamated over units is QD.

The next lemma is a fundamental result about representations of quasidiagonal C*-algebras. Recall that a faithful representation $\pi : \mathscr{A} \to B(\mathscr{H})$ is called essential if $\pi(\mathscr{A})$ contains no non-zero finite rank operators.

LEMMA 1. (Theorem 1.7, [28]) *Let $\pi : \mathscr{A} \to \mathscr{B}(\mathscr{H})$ be a faithful essential representation. Then $\mathscr{A}$ is quasidiagonal if and only if $\pi(\mathscr{A})$ is a quasidiagonal set of operators.*

The following lemma is an important ingredient in the proof of Proposition 1.

LEMMA 2. (Lemma 2.1, [20]) *Suppose that $\mathscr{A} \subset \mathscr{B}(\mathscr{H})$ is a separable unital quasidiagonal C*-algebra and $x_1, \cdots, x_n$ are self-adjoint elements in $\mathscr{A}$. For any $\varepsilon > 0$, any finite subset $\{f_1, \cdots, f_r\}$ of $\mathbb{C}\langle \mathbf{X}_1, \cdots, \mathbf{X}_n\rangle$ and any finite subset $\{\xi_1, \cdots, \xi_r\}$ of $\mathscr{H}$, there is a finite rank projection $p$ in $\mathscr{B}(\mathscr{H})$ such that:*

*(i)* $\|\xi_k - p\xi_k\| < \varepsilon$, $\|(px_ip - x_i)\,\xi_k\| < \varepsilon$, *for all* $1 \leqslant i \leqslant n$ *and* $1 \leqslant k \leqslant r$;

*(ii)* $\left| \|f_j\,(px_1p, \cdots, px_np)\|_{\mathscr{B}(p\mathscr{H})} - \|f_j\,(x_1, \cdots, x_n)\| \right| < \varepsilon$, *for all* $1 \leqslant j \leqslant r$.

Using Lemma 2, it is easy to see that all quasidiagonal C*-algebras are MF algebras.

LEMMA 3. (Proposition 7.4, [7]) *If $\{A_n\}$ is a sequence of C*-algebras then $\prod_{n \in \mathbb{N}} A_n$ is QD if and only if each $A_n$ is QD.*

The examples of quasidiagonal C*-algebras include all abelian C*-algebras and finite-dimensional C*-algebras as well as residually finite-dimensional C*-algebras.

## 3. Full amalgamated free product of unital MF-algebras

### 3.1. $\mathscr{D}$ is a finite-dimensional C*-algebra

In this subsection, we will consider unital full free products of unital MF algebras with amalgamation over finite-dimensional C*-algebras. To state and prove our main result, we need following lemmas.

LEMMA 4. *Suppose $\mathscr{A} = C^*(x_1, x_2, \cdots)$ and $\mathscr{B} = C^*(y_1, y_2, \cdots)$ are unital C*-algebras. Then there is a unital *-homomorphism from $\mathscr{A}$ to $\mathscr{B}$ sending each $x_k$ to $y_k$, if and only if, for each $*$-polynomial $P \in \mathbb{C}_{\mathbb{Q}}\langle \mathbf{X}_1, \mathbf{X}_2, \cdots\rangle$, we have*

$$\|P\,(x_1, x_2, \cdots)\| \geqslant \|P\,(y_1, y_2, \cdots)\|.$$

LEMMA 5. (Theorem 5.1., [17]) *Suppose $\mathscr{A}$ is a separable unital C*-algebra, $\mathscr{H}_1$, $\mathscr{H}_2$ are separable infinite-dimensional Hilbert spaces and $\pi_i : \mathscr{A} \to \mathscr{B}(\mathscr{H}_i)$ are unital *-representations for $i = 1,\ 2$. If, for each $x \in \mathscr{A}$,*

$$rank(\pi_1(x)) \leqslant rank\,(\pi_2\,(x)),$$

*then there is a sequence $\{U_n\}$ of unitary operators, $U_n : \mathscr{H}_1 \to \mathscr{H}_2$, such that, for each $x \in \mathscr{A}$,*

$$U_n^*\pi_2\,(x)\,U_n \to \pi_1\,(x) \quad *\text{-}SOT$$

*as $n \to \infty$.*

The following lemma leads us to ask whether a similar result holds for MF algebras.

LEMMA 6. (Corollary 1, [24]) *Let $\mathscr{A} \supseteq \mathscr{D} \subseteq \mathscr{B}$ be unital C*-inclusions of C*-algebras in $\prod_{n=1}^{\infty} \mathscr{M}_{k_n}(\mathbb{C})$ and $\mathscr{D}$ is a unital finite-dimensional C*-subalgebra. Then $\mathscr{A} \underset{\mathscr{D}}{*} \mathscr{B}$ is RFD.*

The following lemma can be found in [14], which concerns some elementary and useful facts about elements in ultraproducts of C*-algebras and their representatives.

LEMMA 7. (Proposition 2.1, [14]) *Let $\mathscr{A}_i$, $i \in \mathbb{Z}$, be unital C*-algebras and $\alpha$ an ultrafilter on $\mathbb{Z}$. Then*

1. *If $P$ is a projection in $\prod^{\alpha} \mathscr{A}_l$, then there are projections $P_l$ in $\mathscr{A}_l$ such that $P = [(P_l)]$;*

2. *If $P = [(P_l)]$, $Q = [(Q_l)]$ are in $\prod^{\alpha} \mathscr{A}_l$ and all $P_l$, $Q_l$ are projections and if $V \in \prod^{\alpha} \mathscr{A}_l$ is a partial isometry with $V^*V = P$ and $VV^* = Q$, then there are $V_l$ in $\mathscr{A}_l$ such that, eventually along $\alpha, V_l^* V_l = P_l$ and $V_l V_l^* = Q_l$;*

3. *If $P = [(P_l)] \in \prod^{\alpha} \mathscr{A}_l$ and each $P_l$ is a projection, and if $Q$ is a projection in $\prod^{\alpha} \mathscr{A}_l$ such that $Q \leqslant P$, then there are projections $Q_l \in \mathscr{A}_l$ with $Q_l \leqslant P_l$, such that $Q = [(Q_l)]$.*

The next lemma is a technical result.

LEMMA 8. *Let $\mathscr{A} \supset \mathscr{D} \subset \mathscr{B}$ be unital inclusions of MF-algebras. Suppose that $\mathscr{D}$ is a finite-dimensional abelian C*-algebra generated by a family $\{z_1, z_2, \cdots z_l\}$ of self-adjoint elements, and $\mathscr{A}$ is generated by a family $\{z_1, z_2, \cdots z_l, x_1, x_2 \cdots\}$ of self-adjoint elements, $\mathscr{B}$ is generated by a family $\{z_1, z_2, \cdots z_l, y_1, y_2, \cdots\}$ of self-adjoint elements. Let $\varphi : \mathscr{A} \underset{\mathscr{D}}{*} \mathscr{B} \to \mathscr{B}(\mathscr{H})$ be a faithful representation of the full amalgamated free product $\mathscr{A} \underset{\mathscr{D}}{*} \mathscr{B}$ on a separable Hilbert space $\mathscr{H}$. Assume that there is a sequence $\{k_n\}_{n=1}^{\infty}$ of integers with unital embeddings*

$$q_1 : \mathscr{A} \to \prod_{n=1}^{\infty} \mathscr{M}_{k_n}(\mathbb{C}) / \sum_{n=1}^{\infty} \mathscr{M}_{k_n}(\mathbb{C}),$$

*and*

$$q_2 : \mathscr{B} \to \prod_{n=1}^{\infty} \mathscr{M}_{k_n}(\mathbb{C}) / \sum_{n=1}^{\infty} \mathscr{M}_{k_n}(\mathbb{C})$$

*such that $q_1(z_i) = q_2(z_i)$ for each $1 \leqslant i \leqslant l$. Also assume that, for a large enough $r \in \mathbb{N}$,*

$$\{P_1, \cdots, P_{2r}\} \subset \mathbb{C}_{\mathbb{Q}} \langle \mathbf{Z}_1, \cdots, \mathbf{Z}_l, \mathbf{X}_1, \cdots, \mathbf{X}_r \rangle$$

*with* $\{\mathbf{Z}_1,\cdots,\mathbf{Z}_l,\mathbf{X}_1,\cdots,\mathbf{X}_r\}\subset\{P_1,\cdots,P_{2r}\}$ *, and*

$$\{Q_1,\cdots,Q_{2r}\}\subset\mathbb{C}_{\mathbb{Q}}\langle\mathbf{Z}_1,\cdots,\mathbf{Z}_l,\mathbf{Y}_1,\cdots,\mathbf{Y}_r\rangle$$

*with* $\{\mathbf{Z}_1,\cdots,\mathbf{Z}_l,\mathbf{Y}_1,\cdots,\mathbf{Y}_r\}\subset\{Q_1,\cdots,Q_{2r}\}$. *Then there are sequences* $\left\{E^i_{m,r}\right\}_{m=1}^{\infty}$, $\left\{F^i_{m,r}\right\}_{m=1}^{\infty}$ *of operators in* $\mathscr{B}(\mathscr{H})$ *for each* $i\in\mathbb{N}$, *and a sequence* $\left\{G^j_{m,r}\right\}_{m=1}^{\infty}$ *of operators in* $\mathscr{B}(\mathscr{H})$ *for each* $j\in\{1,\cdots,l\}$ *such that*

$$\left|\left\|P_s\left(G^1_{m,r},\cdots,G^l_{m,r},E^1_{m,r},\cdots,E^r_{m,r}\right)\right\|-\left\|P_s\left(z_1,z_2,\cdots z_l,x_1,\cdots,x_r\right)\right\|\right|<\frac{1}{2r},\tag{5.1}$$

$$\left|\left\|Q_s\left(G^1_{m,r},\cdots,G^l_{m,r},F^1_{m,r},\cdots F^r_{m,r}\right)\right\|-\left\|Q_s\left(z_1,\cdots,z_l,y_1,\cdots,y_r\right)\right\|\right|<\frac{1}{2r}\tag{5.2}$$

*for each* $1\leqslant s\leqslant 2r,\ m\in\mathbb{N}$. *We also have that, for a fixed* $r$,

$$C^*\left(G^1_{m,r},\cdots,G^l_{m,r},E^1_{m,r},\cdots,E^r_{m,r}\right)$$

*is an RFD C*-algebra for each* $m\in\mathbb{N}$, *and*

$$E^i_{m,r}\to\varphi(x_i)\ \text{ as }\ m\to\infty \text{ in *-SOT for } i\in\mathbb{N};\tag{5.3}$$

$$F^i_{m,r}\to\varphi(y_i)\ \text{ as }\ m\to\infty \text{ in *-SOT for } i\in\mathbb{N};\tag{5.4}$$

$$G^j_{m,r}\to\varphi(z_j)\ \text{ as }\ m\to\infty \text{ in *-SOT for } j\in\{1,\cdots,l\}.\tag{5.5}$$

*Proof.* Without loss of generality, we suppose that $z_1,\cdots z_l$ are orthogonal projections with $\sum_{i=1}^{l} z_i=I$ and

$$\|x_i\|=\|y_i\|=1\quad\text{for each } i\in\mathbb{N}.$$

From Theorem 1, we may assume that for each $i\in\mathbb{N},\ j\in\{1,\cdots,l\}$, there are families

$$\{A^m_1,A^m_2,\cdots\},\{D^m_1,D^m_2,\cdots,D^m_l\}\text{ and }\{B^m_1,B^m_2,\cdots\}\subset\mathscr{M}^{s.a}_{k_m}(\mathbb{C})$$

for each $k_m\in\{k_n\}_{n=1}^{\infty}$ satisfying

$$\lim_{m\to\infty}\|P(D^m_1,\cdots,D^m_l,A^m_1,A^m_2\ldots)\|=\|P(z_1,z_2,\cdots z_l,x_1,x_2\cdots)\|\tag{5.6}$$

for any $P\in\mathbb{C}_{\mathbb{Q}}\langle\mathbf{Z}_1,\cdots\mathbf{Z}_l,\mathbf{X}_1,\mathbf{X}_2\cdots\rangle$, and

$$\lim_{m\to\infty}\|Q(D^m_1,\cdots,D^m_l,B^m_1,B^m_2\ldots)\|=\|Q(z_1,\cdots,z_l,y_1,y_2\ldots)\|\tag{5.7}$$

for any $Q\in\mathbb{C}_{\mathbb{Q}}\langle\mathbf{Z}_1,\cdots\mathbf{Z}_l,\mathbf{Y}_1,\mathbf{Y}_2,\cdots\rangle$.

If $r$ is large enough, we can assume that $D^m_1,D^m_2,\cdots,D^m_l$ are orthogonal projections with $\sum_{j=1}^{l}D^m_j=I\in\mathscr{M}_{k_m}(\mathbb{C})$ for $m\geqslant r$ by Lemma 7. If $\{P_1,\cdots,P_{2r}\}\subset\mathbb{C}_{\mathbb{Q}}\langle\mathbf{Z}_1,\cdots,\mathbf{Z}_l,\mathbf{X}_1,\cdots,\mathbf{X}_r\rangle$ satisfying

$$\{\mathbf{X}_1,\cdots,\mathbf{X}_r,\mathbf{Z}_1,\cdots,\mathbf{Z}_l\}\subset\{P_1,\cdots,P_{2r}\}\tag{5.8}$$

and $\{Q_1,\cdots,Q_{2r}\}\subset\mathbb{C}_{\mathbb{Q}}\langle\mathbf{Z}_1,\cdots,\mathbf{Z}_l,\mathbf{Y}_1,\cdots,\mathbf{Y}_r\rangle$ satisfying

$$\{\mathbf{Y}_1,\cdots,\mathbf{Y}_r,\mathbf{Z}_1,\cdots,\mathbf{Z}_l\}\subset\{Q_1,\cdots,Q_{2r}\} \tag{5.9}$$

then there is an integer $N_r$ such that, for $A_i(N_r)=\prod\limits_{r\geqslant N_r}A_i^r, B_i(N_r)=\prod\limits_{r\geqslant N_r}B_i^r$ for each $i\in\mathbb{N}$ and $D_j(N_r)=\prod\limits_{r\geqslant N_r}D_i^r$ for $j\in\{1,\cdots,l\}$, we have

$$|\|P_s(D_1(N_r),\cdots,D_l(N_r),A_1(N_r),\cdots,A_r(N_r))\|-\|P_s(z_1,z_2,\cdots z_l,x_1,\cdots,x_r)\||<\frac{1}{2r} \tag{5.10}$$

for each $1\leqslant s\leqslant 2r$ by (5.6), and

$$|\|Q_t(D_1(N_r),\cdots,D_l(N_r),B_1(N_r),\cdots B_r(N_r))\|-\|Q_t(z_1,\cdots,z_l,y_1,\cdots,y_r)\||<\frac{1}{2r} \tag{5.11}$$

for each $1\leqslant t\leqslant 2r$ by (5.7). Combining with (5.8) and (5.9), we have

$$\|A_i(N_r)\|\leqslant 1+\frac{1}{2r}\ \text{ for }\ 1\leqslant i\leqslant r \tag{5.12}$$

$$\|B_i(N_r)\|\leqslant 1+\frac{1}{2r}\ \text{ for }\ 1\leqslant i\leqslant r \tag{5.13}$$

and

$$\|D_i(N_r)\|\leqslant 1+\frac{1}{2r}\ \text{ for }\ 1\leqslant i\leqslant l. \tag{5.14}$$

Let

$$\mathscr{A}_{N_r}=C^*(A_1(N_r),A_2(N_r)\cdots,D_1(N_r),\cdots,D_l(N_r)),$$
$$\mathscr{B}_{N_r}=C^*(B_1(N_r),B_2(N_r)\cdots,D_1(N_r),\cdots,D_l(N_r))$$

be C*-subalgebras in $\prod\limits_{r\geqslant N_r}\mathscr{M}_{k_r}(\mathbb{C})$ and

$$\mathscr{D}_{N_r}=C^*(D_1(N_r),\cdots,D_l(N_r))$$

be a common unital finite-dimensional C*-subalgebra of $\mathscr{A}_{N_r}$ and $\mathscr{B}_{N_r}$.

Since, by 5.6 and 5.7, we have

$$\begin{aligned}&\|P(D_1(N_r),\cdots,D_l(N_r),A_1(N_r),A_2(N_r)\cdots)\|_{\prod\limits_{r\geqslant N}\mathscr{M}_{k_r}(\mathbb{C})}\\&=\sup_{k\geqslant N_r}\left\|P(D_1^k,\cdots,D_l^k,A_1^k,A_2^k\cdots)\right\|_{\mathscr{M}_{k_r}(\mathbb{C})}\geqslant\|P(z_1,z_2,\cdots z_l,x_1,x_2\cdots)\|_{\mathscr{A}\underset{\mathscr{D}}{*}\mathscr{B}}\end{aligned}$$

for any $P\in\mathbb{C}_{\mathbb{Q}}\langle\mathbf{Z}_1,\cdots,\mathbf{Z}_l,\mathbf{X}_1,\mathbf{X}_2\cdots\rangle$, and

$$\begin{aligned}&\|Q(D_1(N_r),\cdots,D_l(N_r),B_1(N_r),B_2(N_r)\cdots)\|_{\prod\limits_{r\geqslant N}\mathscr{M}_{k_r}(\mathbb{C})}\\&=\sup_{k\geqslant N_r}\left\|Q(D_1^k,\cdots,D_l^k,B_1^k,B_2^k\cdots)\right\|_{\mathscr{M}_{k_r}(\mathbb{C})}\geqslant\|Q(z_1,\cdots,z_l,y_1,y_2\cdots)\|_{\mathscr{A}\underset{\mathscr{D}}{*}\mathscr{B}}\end{aligned}$$

for any $Q \in \mathbb{C}_{\mathbb{Q}} \langle \mathbf{Z}_1, \cdots, \mathbf{Z}_l, \mathbf{Y}_1, \mathbf{Y}_2 \cdots \rangle$, there are *-homomorphisms

$$\rho_{N_r}^{\mathscr{A}} : \mathscr{A}_{N_r} \to \varphi\left(\mathscr{A} \underset{\mathscr{D}}{*} \mathscr{B}\right);$$

$$\rho_{N_r}^{\mathscr{B}} : \mathscr{B}_{Nr} \to \varphi\left(\mathscr{A} \underset{\mathscr{D}}{*} \mathscr{B}\right).$$

such that $\rho_{N_r}^{\mathscr{A}}(A_i(N_r)) = \varphi(x_i)$, $\rho_{N_r}^{\mathscr{B}}(B_i(N_r)) = \varphi(y_i)$ and $\rho_{N_r}^{\mathscr{A}}(D_j(N_r)) = \rho_{N_r}^{\mathscr{B}}(D_j(N_r)) = \varphi(z_j)$ for $i \in \mathbb{N}$ and $j \in \{1, \cdots, l\}$ by Lemma 4. It follows that there is a *-homomorphism

$$\rho_{N_r} : \mathscr{A}_{N_r} \underset{\mathscr{D}_{N_r}}{*} \mathscr{B}_{N_r} \to \varphi\left(\mathscr{A} \underset{\mathscr{D}}{*} \mathscr{B}\right)$$

satisfying $\rho_{N_r}(A_i(N_r)) = \varphi(x_i)$ and $\rho_{N_r}(B_i(N_r)) = \varphi(y_i)$ as well as $\rho_{N_r}(D_j(N_r)) = \varphi(z_j)$ for each $i \in \mathbb{N}$ and $j \in \{1, \cdots, l\}$. We also know that $\mathscr{A}_{N_r} \underset{\mathscr{D}_{N_r}}{*} \mathscr{B}_{N_r}$ is an RFD C*-algebra by Lemma 6.

Let $\pi_{N_r} : \mathscr{A}_{N_r} \underset{\mathscr{D}_{N_r}}{*} \mathscr{B}_{N_r} \to \mathscr{B}(\mathscr{H}_{N_r})$ be a faithful essential representation of $\mathscr{A}_{N_r} \underset{\mathscr{D}_{N_r}}{*} \mathscr{B}_{N_r}$. Then $\pi_{N_r}\left(\mathscr{A}_{N_r} \underset{\mathscr{D}_{N_r}}{*} \mathscr{B}_{N_r}\right)$ is an RFD C*-algebra and

$$\pi_{N_r}\left(\mathscr{A}_{N_r} \underset{\mathscr{D}_{N_r}}{*} \mathscr{B}_{N_r}\right) = C^*\left(\pi_{N_r}(A_1(N_r)), \cdots, \pi_{N_r}(B_1(N_r)), \cdots, \pi_{N_r}(D_l(N_r))\right). \quad (5.15)$$

Since

$$rank\left(\pi_{N_r}(x)\right) \geqslant rank\left(\rho_{N_r}(x)\right)$$

for every $x \in \mathscr{A}_{N_r} \underset{\mathscr{D}_{N_r}}{*} \mathscr{B}_{N_r}$, Lemma 5 implies that there is a sequence of unitary operators $\{U_m^{N_r}\}_{m=1}^{\infty} \subset \mathscr{B}(\mathscr{H}, \mathscr{H}_{N_r})$ such that

$$\rho_{N_r}(x) = *\text{-}SOT\text{-}\lim_{m \to \infty} U_m^{N_r *} \pi_{N_r}(x) U_m^{N_r} \quad (5.16)$$

for each $x \in \mathscr{A}_{N_r} \underset{\mathscr{D}_{N_r}}{*} \mathscr{B}_{N_r}$. So, for $i \in \mathbb{N}$ and $j \in \{1, \cdots, l\}$, if we put

$$E_{m,r}^{i} = U_m^{N_r *} \pi_{N_r}(A_i(N_r)) U_m^{N_r}, \quad (5.17)$$

$$F_{m,r}^{i} = U_m^{N_r *} \pi_{N_r}(B_i(N_r)) U_m^{N_r} \quad (5.18)$$

and

$$G_{m,r}^{j} = U_m^{N_r *} \pi_{N_r}(D_j(N_r)) U_m^{N_r}, \quad (5.19)$$

then, for every $m \in \mathbb{N}, i \in \mathbb{N}, j \in \{1, \cdots, l\}$, we have

$$\left\| P\left(G_{m,r}^{1}, \cdots, G_{m,r}^{l}, E_{m,r}^{1}, \cdots, E_{m,r}^{r}\right) \right\| = \left\| P\left(D_1(N_r), \cdots, D_l(N_r), A_1(N_r), \cdots, A_r(N_r)\right) \right\|, \quad (5.20)$$

for every $P \in \mathbb{C}\langle Z_1,\cdots,Z_l,X_1,X_2,\cdots\rangle$, and

$$\left\|Q\left(G_{m,r}^1,\cdots,G_{m,r}^l,F_{m,r}^1,\cdots F_{m,r}^r\right)\right\| = \|Q\left(D_1\left(N_r\right),\cdots,D_l\left(N_r\right),B_1\left(N_r\right),\cdots B_r\left(N_r\right),\right)\| \tag{5.21}$$

for every $Q \in \mathbb{C}\langle Z_1,\cdots,Z_l,Y_1,Y_2,\cdots\rangle$. It follows that, for each $P_s \in \{P_1,\cdots,P_{2r}\} \subseteq \mathbb{C}_{\mathbb{Q}}\langle Z_1,\cdots,Z_l,X_1,\cdots,X_r\rangle$ and each $Q_t \in \{Q_1,\cdots,Q_{2r}\} \subseteq \mathbb{C}_{\mathbb{Q}}\langle Z_1,\cdots,Z_l,Y_1,\cdots,Y_r\rangle$

$$\left|\left\|P_s\left(G_{m,r}^1,\cdots,G_{m,r}^l,E_{m,r}^1,\cdots,E_{m,r}^r\right)\right\| - \|P_s\left(z_1,\cdots,z_l,x_1,\cdots,x_r\right)\|\right| < \frac{1}{2r},$$

$$\left|\left\|Q_t\left(G_{m,r}^1,\cdots,G_{m,r}^l,F_{m,r}^1,\cdots F_{m,r}^r\right)\right\| - \|Q_t\left(z_1,\cdots,z_l,y_1,\cdots,y_r\right)\|\right| < \frac{1}{2r}$$

for each $m \in \mathbb{N}$ by (5.10), (5.11), and (5.20), (5.21). Since

$$C^*\left(G_{m,r}^1,\cdots,G_{m,r}^l,E_{m,r}^1,\cdots,F_{m,r}^1,\cdots\right)$$

is *-isomorphic to the C*-algebra

$$C^*\left(\pi_{N_r}\left(\pi_{N_r}\left(D_1\left(N_r\right)\right),\cdots,\pi_{N_r}\left(D_l\left(N_r\right)\right),A_1\left(N_r\right)\right),\cdots,\pi_{N_r}\left(B_1\left(N_r\right)\right),\cdots\right),$$

we have

$$C^*\left(G_{m,r}^1,\cdots,G_{m,r}^l,E_{m,r}^1,\cdots,F_{m,r}^1,\cdots\right)$$

is an RFD C*-algebra for each $m \in \mathbb{N}$. We also get

$$\begin{aligned}
&E_{m,r}^i \to \varphi(x_i) \ \text{ as } \ m \to \infty \ \text{ in *-SOT } \ \text{for } \ i \in \mathbb{N};\\
&F_{m,r}^i \to \varphi(y_i) \ \text{ as } \ m \to \infty \ \text{ in *-SOT } \ \text{for } \ i \in \mathbb{N};\\
&G_{m,r}^j \to \varphi(z_j) \ \text{ as } \ m \to \infty \ \text{ in *-SOT } \ \text{for } \ j \in \{1,\cdots,l\}.
\end{aligned}$$

from the definition of representation $\rho_{N_r}$ and equations (5.16), (5.17), (5.18), (5.19). □

The next proposition is a key ingredient for proving our main theorem in this subsection.

PROPOSITION 1. *Let $\mathscr{A} \supset \mathscr{D} \subset \mathscr{B}$ be unital inclusions of MF-algebras, where $\mathscr{D}$ is a finite-dimensional abelian C*-algebra. Let $\varphi : \mathscr{A} \underset{\mathscr{D}}{*} \mathscr{B} \to \mathscr{B}(\mathscr{H})$ be a faithful representation of the full amalgamated free product $\mathscr{A} \underset{\mathscr{D}}{*} \mathscr{B}$. Suppose that $\mathscr{D}$ is generated by a family $\{z_1,z_2,\cdots z_l\}$ of self-adjoint elements, $\mathscr{A}$ is generated by a family $\{z_1,z_2,\cdots z_l,x_1,x_2\cdots\}$ of self-adjoint elements and $\mathscr{B}$ is generated by a family $\{z_1,z_2,\cdots z_l,y_1,y_2,\cdots\}$ of self-adjoint elements. Suppose that there is a sequence $\{k_n\}_{n=1}^{\infty}$ of integers with unital embeddings $q_1 : \mathscr{A} \to \prod_{n=1}^{\infty}\mathscr{M}_{k_n}(\mathbb{C})/\sum_{n=1}^{\infty}\mathscr{M}_{k_n}(\mathbb{C})$, and $q_2 : \mathscr{B} \to \prod_{n=1}^{\infty}\mathscr{M}_{k_n}(\mathbb{C})/\sum_{n=1}^{\infty}\mathscr{M}_{k_n}(\mathbb{C})$ such that $q_1(z_i) = q_2(z_i)$ for each $1 \leqslant i \leqslant l$. Then* *there is a sequence* $\{t_m\}_{m=1}^{\infty}$ *of integers such that, for each* $t_r \in \{t_m\}_{m=1}^{\infty}$, *there exist sequences*

$$\{X_1^r, X_2^r, \cdots\}, \{Y_1^r, Y_2^r, \cdots\} \text{ and } \{Z_1^r, \cdots Z_l^r\}$$

*in* $\mathscr{M}_{t_r}(\mathbb{C})$ *and a unitary operator* $W_r : \mathscr{H} \to (\mathbb{C}^{t_r})^{\infty}$ *satisfying*

$$W_r^* (X_i^r)^{(\infty)} W_r \to \varphi(x_i) \text{ in SOT as } r \to \infty \text{ for } i \in \mathbb{N}$$

$$W_r^* (Y_i^r)^{(\infty)} W_r \to \varphi(y_i) \text{ in SOT as } r \to \infty \text{ for } i \in \mathbb{N}$$

*and*

$$W_r^* (Z_i^r)^{(\infty)} W_r \to \varphi(z_i) \text{ in SOT as } r \to \infty \text{ for } i \in \{1, \cdots, l\}$$

*as well as*

$$\|P(z_1, z_2, \cdots z_l, x_1, x_2 \cdots)\| = \lim_{r \to \infty} \|P(Z_1^r, \cdots, Z_l^r, X_1^r, X_2^r, \cdots)\|$$
$$\|Q(z_1, \cdots z_l, y_1, y_2, \cdots)\| = \lim_{r \to \infty} \|Q(Z_1^r, \cdots, Z_l^r, Y_1^r, Y_2^r, \cdots)\|$$

*for any* $P \in \mathbb{C}_{\mathbb{Q}}\langle \mathbf{Z}_1, \cdots \mathbf{Z}_l, \mathbf{X}_1, \mathbf{X}_2 \cdots \rangle$ *and* $Q \in \mathbb{C}_{\mathbb{Q}}\langle \mathbf{Z}_1, \cdots \mathbf{Z}_l, \mathbf{Y}_1, \mathbf{Y}_2 \cdots \rangle$.

*Proof.* Suppose $z_1, \cdots, z_l$ are orthogonal projections with $\sum_{i=1}^{l} z_i = I$ and

$$\|x_i\| = \|y_i\| = 1 \quad \text{for each } i \in \mathbb{N}.$$

Assume $\{e_1, e_2, \cdots\}$ is a family of orthonormal basis of $\mathscr{H}$. With notation as in Lemma 8, for a large enough integer $r$ and a subset $\{e_1, \cdots, e_r\} \subseteq \{e_1, e_2 \cdots\}$, there is an integer $M$ such that, for $1 \leqslant i \leqslant r, j \in \{1, \cdots, l\}$ and $1 \leqslant k \leqslant r$

$$\left\|E_{M,r}^i e_k - \varphi(x_i) e_k\right\| < \frac{1}{2r} \tag{5.22}$$

$$\left\|F_{M,r}^i e_k - \varphi(y_j) e_k\right\| < \frac{1}{2r} \tag{5.23}$$

and

$$\left\|G_{M,r}^j e_k - \varphi(z_j) e_k\right\| < \frac{1}{2r}. \tag{5.24}$$

Note that $\left\{G_{M,r}^1, \cdots G_{M,r}^l, E_{M,r}^1, E_{M,r}^2, \cdots, F_{M,r}^1, \cdots\right\}$ is a family of self-adjoint elements in $\mathscr{B}(\mathscr{H})$ from the proof of Lemma 8. So, by Lemma 2 and the fact that $C^*\left(G_{M,r}^1, \cdots G_{M,r}^l, E_{M,r}^1, E_{M,r}^2, \cdots, F_{M,r}^1, \cdots\right)$ is a quasidiagonal C*-algebra (actually it is RFD), there is a projection $\mathscr{P}_r \in \mathscr{B}(\mathscr{H})$ such that, for $1 \leqslant i \leqslant r$, $j \in \{1, \cdots, l\}$, $1 \leqslant k \leqslant r$,

$$\|e_k - \mathscr{P}_r e_k\| < \frac{1}{6r} \tag{5.25}$$

and

$$\left\| \mathscr{P}_r E^i_{M,r} \mathscr{P}_r e_k - E^i_{M,r} e_k \right\| < \frac{1}{6r} \tag{5.26}$$

$$\left\| \mathscr{P}_r F^i_{M,r} \mathscr{P}_r e_k - F^i_{M,r} e_k \right\| < \frac{1}{6r} \tag{5.27}$$

$$\left\| \mathscr{P}_r G^j_{M,r} \mathscr{P}_r e_k - G^j_{M,r} e_k \right\| < \frac{1}{6r} \tag{5.28}$$

as well as

$$\left| \left\| P_s \left( \mathscr{P}_r G^l_{M,r} \mathscr{P}_r, \cdots, \mathscr{P}_r E^1_{M,r} \mathscr{P}_r \right) \right\| - \left\| P_s \left( G^1_{M,r}, \cdots G^l_{M,r}, E^1_{M,r}, \cdots E^r_{M,r} \right) \right\| \right| < \frac{1}{2r} \quad \text{for } 1 \leqslant s \leqslant 2r, \tag{5.29}$$

$$\left| \left\| Q_t \left( \mathscr{P}_r G^l_{M,r} \mathscr{P}_r, \cdots, \mathscr{P}_r F^1_{M,r} \mathscr{P}_r \right) \right\| - \left\| Q_t \left( G^1_{M,r}, \cdots G^l_{M,r}, F^1_{M,r}, \cdots F^r_{M,r} \right) \right\| \right| < \frac{1}{2r} \quad \text{for } 1 \leqslant t \leqslant 2r. \tag{5.30}$$

By (5.20), (5.21) and (5.29), (5.30), we have that

$$\begin{aligned} &\left| \left\| P_s \left( \mathscr{P}_r G^1_{M,r} \mathscr{P}_r, \cdots \mathscr{P}_r G^l_{M,r} \mathscr{P}_r, \mathscr{P}_r E^1_{M,r} \mathscr{P}_r, \cdots, \mathscr{P}_r E^r_{M,r} \mathscr{P}_r \right) \right\| \right. \\ &\quad \left. - \left\| P_s \left( C_1 (N_r), \cdots C_l (N_r), A_1 (N_r), \cdots, A_r (N_r) \right) \right\| \right| < \frac{1}{2r} \quad \text{for } 1 \leqslant s \leqslant 2r \end{aligned} \tag{5.31}$$

$$\begin{aligned} &\left| \left\| Q_t \left( \mathscr{P}_r G^1_{M,r} \mathscr{P}_r, \cdots \mathscr{P}_r G^l_{M,r} \mathscr{P}_r, \mathscr{P}_r F^1_{M,r} \mathscr{P}_r, \cdots, \mathscr{P}_r F^r_{M,r} \mathscr{P}_r, \right) \right\| \right. \\ &\quad \left. - \left\| Q_t \left( C_1 (N_r), \cdots C_l (N_r), B_1 (N_r), \cdots B_r (N_r) \right) \right\| \right| < \frac{1}{2r} \quad \text{for } 1 \leqslant t \leqslant 2r. \end{aligned} \tag{5.32}$$

Let $t_r = \dim \mathscr{P}_r \mathscr{H}$ and $\widetilde{W}_r : \mathscr{P}_r \mathscr{H} \to \mathbb{C}^{t_r}$ be a unitary operator. Putting $X^r_i = \widetilde{W}_r \mathscr{P}_r E^i_{M,r} \mathscr{P}_r \widetilde{W}^*_r$, $Y^r_i = \widetilde{W}_r \mathscr{P}_r F^i_{M,r} \mathscr{P}_r \widetilde{W}^*_r$ for $i \in \mathbb{N}$ and $Z^r_j = \widetilde{W}_r \mathscr{P}_r G^j_{M,r} \mathscr{P}_r \widetilde{W}^*_r$ for $j \in \{1, \cdots, l\}$, and combining (5.31), (5.32) and (5.10), (5.11), we have

$$|\|P_s (Z^r_1, \cdots, Z^r_l, X^r_1, \cdots, X^r_r)\| - \|P_s(z_1, \cdots, z_l, x_1, \cdots, x_r\|| < \frac{1}{r} \tag{5.33}$$

$$|\|Q_t (Z^r_1, \cdots, Z^r_l, Y^r_1, \cdots, Y^r_r)\| - \|Q_t(z_1, \cdots, z_l, y_1, \cdots, y_r\|| < \frac{1}{r} \tag{5.34}$$

for $1 \leqslant s,t \leqslant 2r$. Hence we can find a unitary $W_r : \mathscr{H} \to (\mathbb{C}^{t_r})^\infty$ such that $W_r$ is unitary equivalent to $\left(\widetilde{W}_r\right)^\infty$ and $W_r \mathscr{P}_r = \widetilde{W}_r$. It follows that, for $1 \leqslant i \leqslant r$, $j \in \{1, \cdots, l\}$ and

$1 \leqslant k \leqslant r$, we have

$$\begin{aligned}
&\left\|W_r^*\left(X_i^r\right)^{\infty} W_r e_k - E_{m_r}^i e_k\right\| \\
\leqslant &\left\|W_r^*\left(X_i^r\right)^{\infty} W_r\right\| \left\|e_k - \mathscr{P}_r e_k\right\| + \left\|W_r^*\left(X_i^r\right)^{\infty} W_r \mathscr{P}_r e_k - E_{m_r}^i e_k\right\| \\
\leqslant &\left(1+\frac{1}{2r}\right)\frac{1}{6r} + \left\|W_r^*\left(\widetilde{W}_r \mathscr{P}_r E_{M,r}^i \mathscr{P}_r \widetilde{W}_r^*\right)^{\infty} \widetilde{W}_r e_k - E_{m_r}^i e_k\right\| \\
= &\left(1+\frac{1}{2r}\right)\frac{1}{6r} + \left\|\mathscr{P}_r E_{M,r}^i \mathscr{P}_r e_k - E_{M,r}^i e_k\right\| \\
\leqslant &\left(1+\frac{1}{2r}\right)\frac{1}{6r} + \frac{1}{6r} < \frac{1}{2r},
\end{aligned} \tag{5.35}$$

and

$$\left\|W_r^*\left(Y_i^r\right)^{\infty} W_r e_k - F_{M,r}^i e_k\right\| < \frac{1}{2r}, \tag{5.36}$$

$$\left\|W_r^*\left(Z_j^r\right)^{\infty} W_r e_k - G_{M,r}^j e_k\right\| < \frac{1}{2r} \tag{5.37}$$

by the definition of $X_i^r$, $Y_i^r$ and $Z_j^r$ and (5.25), (5.26), (5.27) and (5.28). Combining the inequalities from above with (5.22), (5.23) and (5.24), we have, for $1 \leqslant i \leqslant r$, $j \in \{1, \cdots, l\}$ and $1 \leqslant k \leqslant r$,

$$\begin{aligned}
\left\|W_r^*\left(X_i^r\right)^{\infty} W_r e_k - \varphi\left(x_i\right) e_k\right\| &< \frac{1}{r}; \\
\left\|W_r^*\left(Y_i^r\right)^{\infty} W_r e_k - \varphi\left(y_i\right) e_k\right\| &< \frac{1}{r}; \\
\left\|W_r^*\left(Z_j^r\right)^{\infty} W_r e_k - \varphi\left(z_j\right) e_k\right\| &< \frac{1}{r}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&W_r^*\left(X_i^r\right)^{(\infty)} W_r \to \varphi\left(x_i\right) &&\text{in SOT as } r \to \infty \\
&W_r^*\left(Y_i^r\right)^{(\infty)} W_r \to \varphi\left(y_i\right) &&\text{in SOT as } r \to \infty \\
&W_r^*\left(Z_j^r\right)^{(\infty)} W_r \to \varphi\left(z_j\right) &&\text{in SOT as } r \to \infty
\end{aligned}$$

for $i \in \mathbb{N}$, $j \in \{1, \cdots, l\}$ and

$$\begin{aligned}
\left\|P(z_1, z_2, \cdots z_l, x_1, x_2 \cdots)\right\| &= \lim_{r \to \infty} \left\|P\left(Z_1^r, \cdots, Z_l^r, X_1^r, X_2^r, \cdots\right)\right\| \\
\left\|Q(z_1, z_2 \cdots, z_l, y_1, y_2, \cdots)\right\| &= \lim_{r \to \infty} \left\|Q\left(Z_1^r, \cdots, Z_l^r, Y_1^r, Y_2^r, \cdots\right)\right\|
\end{aligned}$$

for any $P \in \mathbb{C}_{\mathbb{Q}}\langle \mathbf{Z}_1, \cdots, \mathbf{Z}_l, \mathbf{X}_1, \mathbf{X}_2 \cdots\rangle$ and $Q \in \mathbb{C}_{\mathbb{Q}}\langle \mathbf{Z}_1, \cdots, \mathbf{Z}_l, \mathbf{Y}_1, \mathbf{Y}_2 \cdots\rangle$ as desired. □

Before giving our main result in this subsection, we need one more lemma.

LEMMA 9. (Lemma 2.2., [6]) *Let $\mathscr{A}$ and $\mathscr{B}$ be unital C*-algebras having $\mathscr{D}$ embedded as a unital C*-subalgebra of each of them. Let*

$$\mathscr{C} = \mathscr{A} \underset{\mathscr{D}}{*} \mathscr{B}$$

*be the full amalgamated free product of $\mathscr{A}$ and $\mathscr{B}$ over $\mathscr{D}$. If there is a projection $p \in \mathscr{D}$ and there are partial isometries $v_1, \cdots, v_n \in \mathscr{D}$ such that $v_i^* v_i \leqslant p$ and $\sum_{i=1}^{n} v_i v_i^* = 1 - p$, then*

$$p\mathscr{C}p \cong (p\mathscr{A}p) \underset{p\mathscr{D}p}{*} (p\mathscr{B}p).$$

REMARK 1. Suppose $\mathscr{A}$ is a unital C*-algebra and suppose there is a projection $p \in \mathscr{A}$ and there are partial isometries $v_1, \cdots, v_n \in \mathscr{A}$ such that $v_i^* v_i \leqslant p$ and $\sum_{i=1}^{n} v_i v_i^* = 1 - p$. By emulating the argument in the proof of Lemma 2.1 in [6], we know that $\mathscr{A}$ is MF if and only if $p\mathscr{A}p$ is MF.

THEOREM 2. *Let $\mathscr{A}$ and $\mathscr{B}$ be unital MF-algebras and $\mathscr{D}$ be a finite-dimensional C*-algebra. Suppose $\psi_1 : \mathscr{D} \to \mathscr{A}$ and $\psi_2 : \mathscr{D} \to \mathscr{B}$ are unital embeddings. Then $\mathscr{A} \underset{\mathscr{D}}{*} \mathscr{B}$ is an MF algebra if and only if there are unital embeddings*

$$q_1 : \mathscr{A} \to \prod_{n=1}^{\infty} \mathscr{M}_{k_n}(\mathbb{C}) / \sum_{n=1}^{\infty} \mathscr{M}_{k_n}(\mathbb{C})$$

*and*

$$q_2 : \mathscr{B} \to \prod_{n=1}^{\infty} \mathscr{M}_{k_n}(\mathbb{C}) / \sum_{n=1}^{\infty} \mathscr{M}_{k_n}(\mathbb{C})$$

*for a sequence $\{k_n\}_{n=1}^{\infty}$ of integers such that the following diagram commutes*

$$\begin{array}{ccc} \mathscr{D} & \overset{\psi_{\mathscr{A}}}{\hookrightarrow} & \mathscr{A} \\ \psi_{\mathscr{B}} \downarrow & & \downarrow q_1 \\ \mathscr{B} & \overset{q_2}{\hookrightarrow} & \prod_{m=1}^{\infty} \mathscr{M}_{k_m}(\mathbb{C}) / \sum \mathscr{M}_{k_m}(\mathbb{C}) \end{array}$$

*Proof.* If $\mathscr{A} \underset{\mathscr{D}}{*} \mathscr{B}$ is an MF algebra, then there is a unital embedding

$$\Phi : \mathscr{A} \underset{\mathscr{D}}{*} \mathscr{B} \to \prod_{n=1}^{\infty} \mathscr{M}_{k_n}(\mathbb{C}) / \sum_{n=1}^{\infty} \mathscr{M}_{k_n}(\mathbb{C})$$

for a sequence $\{k_n\}_{n=1}^{\infty}$ of integers. Let $q_1$ and $q_2$ be the restrictions of $\Phi$ on $\mathscr{A}$ and $\mathscr{B}$ respectively. Then the above diagram is commutative.

Conversely, suppose $\mathscr{D}$, $\mathscr{A}$ and $\mathscr{B}$ are generated by families

$$\{z_1,\cdots,z_l\},$$

$$\{\psi_1(z_1),\cdots,\psi_1(z_l),x_1,x_2\cdots\}$$

and

$$\{\psi_2(z_1),\cdots,\psi_2(z_l),y_1,y_2,\cdots\}$$

respectively with $\|x_i\|=\|y_i\|=\|z_j\|=1$ for each $i\in\mathbb{N}$ and $j\in\{1,\cdots,l\}$. By Remark 1 and Lemma 9, we may assume that $\mathscr{D}$ is a finite-dimensional abelian C*-algebra and $z_1,\cdots z_l$ are orthogonal projections with $\sum_{i=1}^{l} z_i = I$. Without loss of generality, we may assume that $\mathscr{A} \underset{\mathscr{D}}{*} \mathscr{B}$ is generated by a sequence $\{z_1,\cdots,z_l,x_1,x_2\cdots,y_1,y_2\cdots\}$. Let $\varphi$ : $\mathscr{A} \underset{\mathscr{D}}{*} \mathscr{B} \to \mathscr{B}(\mathscr{H})$ be a faithful representation of full amalgamated free product $\mathscr{A} \underset{\mathscr{D}}{*} \mathscr{B}$. Applying Proposition 1, there is a sequence $\{t_m\}_{m=1}^{\infty}$ of integers such that, for each $t_r \in \{t_m\}_{m=1}^{\infty}$, there exist sequences $\{X_1^r,X_2^r,\cdots\},\{Y_1^r,Y_2^r,\cdots\}$ and $\{Z_1^r,\cdots,Z_2^r\}$ in $\mathscr{M}_{t_r}(\mathbb{C})$ and a unitary operator $W_r:\mathscr{H}\to(\mathbb{C}^{t_r})^{\infty}$ such that

$$W_r^*(X_i^r)^{(\infty)}W_r \to \varphi(x_i) \qquad \text{in SOT as } r\to\infty \text{ for } i\in\mathbb{N}$$

$$W_r^*(Y_i^r)^{(\infty)}W_r \to \varphi(y_i) \qquad \text{in SOT as } r\to\infty \text{ for } i\in\mathbb{N}$$

and

$$W_r^*\left(Z_j^r\right)^{(\infty)}W_r \to \varphi(z_j) \qquad \text{in SOT as } r\to\infty \text{ for } j\in\{1,\cdots,l\}$$

as well as

$$\|P(z_1,z_2,\cdots z_l,x_1,x_2\cdots)\| = \lim_{r\to\infty}\|P(Z_1^r,\cdots,Z_l^r,X_1^r,X_2^r,\cdots)\|$$
$$\|Q(z_1,z_2,\cdots,z_l,y_1,y_2,\cdots)\| = \lim_{r\to\infty}\|Q(Z_1^r,\cdots,Z_l^r,Y_1^r,Y_2^r,\cdots)\|$$

for any $P\in\mathbb{C}_{\mathbb{Q}}\langle\mathbf{Z}_1,\cdots,\mathbf{Z}_l,\mathbf{X}_1,\mathbf{X}_2\cdots\rangle$ and $Q\in\mathbb{C}_{\mathbb{Q}}\langle\mathbf{Z}_1,\cdots,\mathbf{Z}_l,\mathbf{Y}_1,\mathbf{Y}_2\cdots\rangle$. Therefore, we can define unital embeddings

$$q_1:\mathscr{A}\to\prod M_{t_r}(\mathbb{C})/\sum M_{t_r}(\mathbb{C})$$

and

$$q_2:\mathscr{B}\to\prod M_{t_r}(\mathbb{C})/\sum M_{t_r}(\mathbb{C})$$

so that $q_1(x_i)=[(X_i^r)]$, $q_1(\psi_1(z_j))=\left[\left(Z_j^r\right)\right]$, and $q_2(y_i)=[(Y_i^r)]$, $q_2(\psi_2(z_j))=\left[\left(Z_j^r\right)\right]$ for $i\in\mathbb{N}, j\in\{1,\cdots,l\}$. From the definition of full amalgamated free product, there is a $*$-homomorphism

$$\Phi:\mathscr{A}\underset{\mathscr{D}}{*}\mathscr{B}\to\prod M_{t_r}(\mathbb{C})/\sum M_{t_r}(\mathbb{C})$$

such that $\Phi(x_i) = [(X_i^r)]$, $\Phi(y_i) = [(Y_i^r)]$, $\Phi(z_j) = \left[\left(Z_j^r\right)\right]$ where $i \in \mathbb{N}$ and $j \in \{1, \cdots, l\}$. Furthermore, for any $\Psi_j \in \mathbb{C}_{\mathbb{Q}} \langle \mathbf{X}_1, \mathbf{X}_2 \cdots, \mathbf{Y}_1, \mathbf{Y}_2 \cdots, \mathbf{Z}_1, \cdots, \mathbf{Z}_l \rangle$, we have

$$\begin{aligned}
&\left\|\Psi_j\left([(Z_1^r)], \cdots, [(Z_l^r)], [(X_1^r)], [(X_2^r)] \cdots, [(Y_1^r)], [(Y_2^r)] \cdots\right)\right\| \\
&= \limsup_{r\to\infty} \left\|\Psi_j\left(Z_1^r, \cdots, Z_l^r, X_1^r, X_2^r \cdots, Y_1^r, Y_2^r \cdots\right)\right\|_{M_{k_r}(\mathbb{C})} \\
&\leqslant \left\|\Psi_j\left(z_1, \cdots, z_l, x_1, x_2 \cdots, y_1, y_2 \cdots\right)\right\|_{\mathscr{A} \underset{\mathscr{D}}{*} \mathscr{B}}. \qquad (5.38)
\end{aligned}$$

Meanwhile,

$$\begin{aligned}
&\Psi_j\left(W_r^*\left(Z_1^r\right)^\infty W_r, \cdots, W_r^*\left(Z_l^r\right)^\infty W_r, W_r^*\left(X_1^r\right)^\infty W_r, \cdots, W_r^*\left(Y_1^r\right)^\infty W_r, \cdots,\right) \\
&\to \Psi_j\left(z_1, \cdots, z_l, x_1, x_2 \cdots, y_1, y_2 \cdots\right) \text{ in SOT as } r \to \infty,
\end{aligned}$$

and therefore

$$\begin{aligned}
&\liminf_{r\to\infty} \left\|\Psi_j\left(Z_1^r, \cdots, Z_l^r, X_1^r, X_2^r \cdots, Y_1^r, Y_2^r \cdots\right)\right\|_{\mathscr{M}_{k_r}(\mathbb{C})} \\
&= \liminf_{r\to\infty} \left\|\Psi_j\left(W_r^*\left(Z_1^r\right)^\infty W_r, \cdots, W_r^*\left(Z_l^r\right)^\infty W_r, W_r^*\left(X_1^r\right)^\infty W_r, \cdots, W_r^*\left(Y_1^r\right)^\infty W_r, \cdots\right)\right\| \\
&\geqslant \left\|\Psi_j\left(z_1, \cdots, z_l, x_1, x_2 \cdots, y_1, y_2 \cdots\right)\right\|_{\mathscr{A} \underset{\mathscr{D}}{*} \mathscr{B}}. \qquad (5.39)
\end{aligned}$$

Combining (5.38) and (5.39), it follows that

$$\left\|\Psi\left(z_1, \cdots, z_l, x_1, x_2 \cdots, y_1, y_2 \cdots\right)\right\| = \lim_{r\to\infty} \left\|\Psi_j\left(Z_1^r, \cdots, Z_l^r, X_1^r, X_2^r \cdots, Y_1^r, Y_2^r \cdots\right)\right\|$$

for any $\Psi \in \mathbb{C}_{\mathbb{Q}} \langle \mathbf{Z}_1, \cdots, \mathbf{Z}_l, \mathbf{X}_1, \cdots, \mathbf{Y}_1, \cdots \rangle$. Then $\Phi$ is a unital injective $*$- homomorphism. It follows that $\mathscr{A} \underset{\mathscr{D}}{*} \mathscr{B}$ is an MF algebra. $\square$

The following corollary is an easy consequence of Theorem 2.

COROLLARY 1. *Let $\mathscr{A}$ be an MF algebra and $\mathscr{D}$ be a finite-dimensional C*-algebra. If there is a unital embedding $q: \mathscr{D} \to \mathscr{A}$, then $\mathscr{A} \underset{\mathscr{D}}{*} \mathscr{A}$ is an MF algebra with respect to the embedding $q$.*

Applying Theorems 2, we can obtain the following result.

PROPOSITION 2. *Let $\mathscr{A}$ and $\mathscr{B}$ be unital MF algebras and let $\mathscr{D}$ be the direct sum of n copies of the set of all complex numbers, that is,*

$$\mathscr{D} = \mathbb{C} \oplus \mathbb{C} \oplus \cdots \oplus \mathbb{C}.$$

*Suppose $\psi_{\mathscr{A}} : \mathscr{D} \to \mathscr{A}$ and $\psi_{\mathscr{B}} : \mathscr{D} \to \mathscr{B}$ are unital embeddings. If $\psi_{\mathscr{A}}$ and $\psi_{\mathscr{B}}$ can be extended to unital embeddings $\widetilde{\psi}_{\mathscr{A}} : \mathscr{M}_n(\mathbb{C}) \to \mathscr{A}$ and $\widetilde{\psi}_{\mathscr{B}} : \mathscr{M}_n(\mathbb{C}) \to \mathscr{B}$ respectively, then $\mathscr{A} \underset{\mathscr{D}}{*} \mathscr{B}$ is an MF algebra.*

*Proof.* Let

$$E_1 = 1 \oplus 0 \oplus \cdots \oplus 0, E_2 = 0 \oplus 1 \oplus 0 \oplus \cdots \oplus 0, \cdots, E_n = 0 \oplus 0 \oplus \cdots \oplus 0 \oplus 1$$

in $\mathscr{D}$. Then $\mathscr{D} = \mathbb{C}E_1 + \cdots + \mathbb{C}E_n$. Suppose C*-algebras $\mathscr{A}$ and $\mathscr{B}$ are generated by families

$$\{\psi_{\mathscr{A}}(E_1), \cdots, \psi_{\mathscr{A}}(E_n), x_1, x_2 \cdots\}$$

of self-adjoint elements and

$$\{\psi_{\mathscr{B}}(E_1), \cdots, \psi_{\mathscr{B}}(E_n), y_1, y_2 \cdots\}$$

of self-adjoint elements respectively. Without loss of generality, we may assume that $\mathscr{A}$ and $\mathscr{B}$ can be embedded as unital C*-subalgebras of $\prod_{m=1}^{\infty} \mathscr{M}_{k_m}(\mathbb{C}) / \sum \mathscr{M}_{k_m}(\mathbb{C})$, respectively, for a sequence $\{k_m\}_{m=1}^{m}$ of integers with sequences

$$\{A_1^m, A_2^m, \cdots\}, \{C_1^m, \cdots, C_n^m\} \text{ and } \{B_1^m, B_2^m, \cdots\}, \{D_1^m, \cdots, D_n^m\} \subset \mathscr{M}_{k_m}^{s.a.}(\mathbb{C})$$

for each $k_m \in \{k_n\}_{n=1}^{\infty}$ satisfying

$$\lim_{m\to\infty} \|P(C_1^m, \cdots, C_n^m, A_1^m, A_2^m \ldots, )\| = \|P(\psi_{\mathscr{A}}(E_1), \cdots, \psi_{\mathscr{A}}(E_n), x_1, x_2 \cdots)\|$$

for any $P \in \mathbb{C}_{\mathbb{Q}} \langle \mathbf{Z}_1, \cdots, \mathbf{Z}_n, \mathbf{X}_1, \mathbf{X}_2 \cdots \rangle$, and

$$\lim_{m\to\infty} \|Q(D_1^m, \cdots, D_n^m, B_1^m, B_2^m \ldots)\| = \|Q(\psi_{\mathscr{B}}(E_1), \cdots, \psi_{\mathscr{B}}(E_n), y_1, y_2 \cdots)\|$$

for any $Q \in \mathbb{C}_{\mathbb{Q}} \langle \mathbf{Z}_1, \cdots, \mathbf{Z}_n, \mathbf{Y}_1, \mathbf{Y}_2, \cdots \rangle$. Since the images of $\psi_{\mathscr{A}}(E_1), \cdots, \psi_{\mathscr{A}}(E_n)$ under the embedding from $\mathscr{A}$ to $\prod_{m=1}^{\infty} \mathscr{M}_{k_m}(\mathbb{C}) / \sum \mathscr{M}_{k_m}(\mathbb{C})$ are $[(C_1^m)], \cdots, [(C_n^m)]$ and $\psi_{\mathscr{A}}$ can be extended to a unital embedding $\widetilde{\psi}_{\mathscr{A}} : \mathscr{M}_n(\mathbb{C}) \to \mathscr{A}$, it follows that there are partial isometries $[(V_1^m)], \cdots, [(V_n^m)]$ in $\prod_{m=1}^{\infty} \mathscr{M}_{k_m}(\mathbb{C}) / \sum \mathscr{M}_{k_m}(\mathbb{C})$ such that $[(V_s^m)]^* [(V_s^m)] = [(C_1^m)]$ and $[(V_s^m)] [(V_s^m)]^* = [(C_s^m)]$ for each $s \in \{1, \cdots, n\}$. By Lemma 7, we may assume that, $C_s^m \in \mathscr{M}_{k_m}(\mathbb{C})$ is a projection for each $m \in \mathbb{N}$ and $s \in \{1, \cdots, n\}$. We may conclude further that, when $k_m$ is large enough, $V_s^m$ is a partial isometry such that $V_s^{m*} V_s^m = C_1^m$ and $V_s^m V_s^{m*} = C_s^m$ for $1 \leqslant s \leqslant n$ in $\mathscr{M}_{k_m}(\mathbb{C})$ by Lemma 7. So it follows that $C_1^m$ is equivalent to $C_s^m$ in $\mathscr{M}_{k_m}(\mathbb{C})$ for $1 \leqslant s \leqslant n$ and $\sum_{s=1}^{n} C_s^m = I$, $C_i^m C_j^m = 0$ for $1 \leqslant i \neq j \leqslant n$. Similarly, we can assume that $D_s^m$ is a projection in $\mathscr{M}_{k_m}(\mathbb{C})$ for $1 \leqslant s \leqslant n$. When $k_m$ is large enough, we conclude that $D_1^m$ is equivalent to $D_s^m$ for each $1 \leqslant s \leqslant n$, $\sum_{s=1}^{n} D_s^m = I$ and $D_i^m D_j^m = 0$ as $1 \leqslant i \neq j \leqslant n$ in $\mathscr{M}_{k_m}(\mathbb{C})$. Hence, there exists an integer $K$ such that, for each $k_m > K$, there exists a unitary $U^m \in \mathscr{M}_{k_m}(\mathbb{C})$ satisfying $U^m C_s^m U^{m*} = D_s^m$ for each $s \in \{1, 2, \cdots, n\}$ in $\mathscr{M}_{k_m}(\mathbb{C})$. It follows that there is a unitary $[(U^m)] \in \prod_{m=1}^{\infty} \mathscr{M}_{k_m}(\mathbb{C}) / \sum \mathscr{M}_{k_m}(\mathbb{C})$ satisfying $[(U^m)] [(C_i^m)] [(U^m)]^* = [D_i^m]$ for $1 \leqslant i \leqslant n$. Now we define embeddings

$$q_1 : \mathscr{A} \to \prod_{m=1}^{\infty} \mathscr{M}_{k_m}(\mathbb{C}) / \sum \mathscr{M}_{k_m}(\mathbb{C})$$

so that $q_1(x_i) = [(U^m)][(A_i^m)][(U^m)]^*$ for $i \in \mathbb{N}$, $q_1(\psi_{\mathscr{A}}(z_j)) = \left[\left(D_j^m\right)\right]$ for $1 \leqslant j \leqslant n$ and

$$q_2 : \mathscr{B} \to \prod_{m=1}^{\infty} \mathscr{M}_{k_m}(\mathbb{C}) / \sum \mathscr{M}_{k_m}(\mathbb{C})$$

so that $q_2(y_i) = [(B_i^m)]$ for $i \in \mathbb{N}$, $q_2(\psi_{\mathscr{B}}(z_i)) = \left[\left(D_j^m\right)\right]$ for $1 \leqslant j \leqslant n$. It is clear that the following diagram is commutative

$$\begin{array}{ccc} \mathscr{D} & \overset{\psi_{\mathscr{A}}}{\hookrightarrow} & \mathscr{A} \\ \psi_{\mathscr{B}} \downarrow & & \downarrow q_1 \\ \mathscr{B} & \overset{q_2}{\hookrightarrow} & \prod_{m=1}^{\infty} \mathscr{M}_{k_m}(\mathbb{C}) / \sum \mathscr{M}_{k_m}(\mathbb{C}) \end{array}$$

So $\mathscr{A} \underset{\mathscr{D}}{*} \mathscr{B}$ is MF by Theorem 2. $\square$

### 3.2. $\mathscr{D}$ is an infinite-dimensional C*-algebra

In this subsection, we will consider the case when $\mathscr{D}$ is an infinite-dimensional C*-algebra. More precisely, we will consider the case when $\mathscr{D}$ can be written as a norm closure of the union of an increasing sequence of C*-algebras.

THEOREM 3. *Suppose that $\mathscr{A} \supseteq \mathscr{D} \subseteq \mathscr{B}$ are unital inclusions of unital separable C*-algebras and $\{\mathscr{D}_k\}_{k=1}^{\infty}$ is an increasing sequence of unital $C^*$-subalgebras of $\mathscr{D}$ such that $\cup_{k \geqslant 1} \mathscr{D}_k$ is norm dense in $\mathscr{D}$. Let $\mathscr{A} *_{\mathscr{D}} \mathscr{B}$ and $\mathscr{A} *_{\mathscr{D}_k} \mathscr{B}$ for $k \geqslant 1$ be the unital full free products of $\mathscr{A}$ and $\mathscr{B}$ with amalgamation over $\mathscr{D}$ and $\mathscr{D}_k$ for $k \geqslant 1$ respectively. If $\mathscr{A} *_{\mathscr{D}_k} \mathscr{B}$ is an MF algebra for each $k \geqslant 1$, then $\mathscr{A} *_{\mathscr{D}} \mathscr{B}$ is an MF algebra.*

*Proof.* Note that $\mathscr{A}, \mathscr{B}$ and $\mathscr{D}$ are unital separable $\mathrm{C}^*$-algebras. We might assume that $\{x_n\}_{n \in \mathbb{N}} \subseteq \mathscr{A}$, and $\{y_n\}_{n \in \mathbb{N}} \subseteq \mathscr{B}$ are families of generators of $\mathscr{A}$ and $\mathscr{B}$ respectively.

Assume that $\sigma : \mathscr{A} \to \mathscr{A} *_{\mathscr{D}} \mathscr{B}$ and $\sigma_k : \mathscr{A} \to \mathscr{A} *_{\mathscr{D}_k} \mathscr{B}$ are natural unital embeddings from $\mathscr{A}$ into $\mathscr{A} *_{\mathscr{D}} \mathscr{B}$ and $\mathscr{A} *_{\mathscr{D}_k} \mathscr{B}$, respectively, for each $k \geqslant 1$. Assume that $\rho : \mathscr{B} \to \mathscr{A} *_{\mathscr{D}} \mathscr{B}$ and $\rho_k : \mathscr{B} \to \mathscr{A} *_{\mathscr{D}_k} \mathscr{B}$ are natural unital embeddings from $\mathscr{B}$ into $\mathscr{A} *_{\mathscr{D}} \mathscr{B}$ and into $\mathscr{A} *_{\mathscr{D}_k} \mathscr{B}$, respectively, for each $k \geqslant 1$.

Consider the unital $\mathrm{C}^*$-algebra

$$\prod_{k \geqslant 1} \mathscr{A} *_{\mathscr{D}_k} \mathscr{B} / \sum_{k \geqslant 1} \mathscr{A} *_{\mathscr{D}_k} \mathscr{B}.$$

From Corollary 3.4.3 in [3] and the fact that, for each $k \geqslant 1$, $\mathscr{A} *_{\mathscr{D}_k} \mathscr{B}$ is an MF algebra, we know that every separable $\mathrm{C}^*$-subalgebra of

$$\prod_{k \geqslant 1} \mathscr{A} *_{\mathscr{D}_k} \mathscr{B} / \sum_{k \geqslant 1} \mathscr{A} *_{\mathscr{D}_k} \mathscr{B}$$

is an MF algebra. Let

$$a_n = [(\sigma_k(x_n))_k] \in \prod_{k\geqslant 1} \mathscr{A} *_{\mathscr{D}_k} \mathscr{B} / \sum_{k\geqslant 1} \mathscr{A} *_{\mathscr{D}_k} \mathscr{B}, \qquad \forall\, n \in \mathbb{N}$$

and

$$b_n = [(\rho_k(y_n))_k] \in \prod_{k\geqslant 1} \mathscr{A} *_{\mathscr{D}_k} \mathscr{B} / \sum_{k\geqslant 1} \mathscr{A} *_{\mathscr{D}_k} \mathscr{B}, \qquad \forall\, n \in \mathbb{N}.$$

Let $\mathscr{Q}$ be the unital $\mathrm{C}^*$-subalgebra generated by $\{a_n, b_n\}_{n\in\mathbb{N}}$ in

$$\prod_{k\geqslant 1} \mathscr{A} *_{\mathscr{D}_k} \mathscr{B} / \sum_{k\geqslant 1} \mathscr{A} *_{\mathscr{D}_k} \mathscr{B}.$$

Thus $\mathscr{Q}$ is an MF algebra.

Next we shall show that there is a $*$-isomorphic from $\mathscr{Q}$ onto $\mathscr{A} *_{\mathscr{D}} \mathscr{B}$ by sending each $a_n$ to $\sigma(x_n)$ and $b_n$ to $\rho(y_n)$. This will induce that $\mathscr{A} *_{\mathscr{D}} \mathscr{B}$ is also an MF algebra. In order to obtain such $*$-isomorphism from $\mathscr{Q}$ onto $\mathscr{A} *_{\mathscr{D}} \mathscr{B}$, it suffices to show that $\forall\, N \in \mathbb{N}$ *and* $\forall\, P \in \mathbb{C}\langle X_1, \ldots X_N, Y_1, \ldots Y_N\rangle$, *we have*

$$\begin{aligned} &\|P(\sigma(x_1),\cdots,\sigma(x_N),\rho(y_1),\cdots,\rho(y_N))\|_{\mathscr{A} *_{\mathscr{D}} \mathscr{B}} \\ &= \|P(a_1,\cdots,a_N,b_1,\cdots,b_N)\|_{\prod_{k\geqslant 1} \mathscr{A} *_{\mathscr{D}_k} \mathscr{B} / \sum \mathscr{A} *_{\mathscr{D}_k} \mathscr{B}} \end{aligned} \tag{5.40}$$

By the definition of full amalgamated free product, we know, for each $k \geqslant 1$, there is a $*$-homomorphism from $\mathscr{A} *_{\mathscr{D}_k} \mathscr{B}$ to $\mathscr{A} *_{\mathscr{D}} \mathscr{B}$, which send $\sigma_k(x_n)$ to $\sigma(x_n)$ and $\rho_k(y_n)$ to $\rho(y_n)$ respectively, for every $n \in \mathbb{N}$. Hence

$$\begin{aligned} &\|P(\sigma(x_1),\cdots,\sigma(x_N),\rho(y_1),\cdots,\rho(y_N))\|_{\mathscr{A} *_{\mathscr{D}} \mathscr{B}} \\ &\leqslant \|P(\sigma_k(x_1),\cdots,\sigma_k(x_N),\rho_k(y_1),\cdots,\rho_k(y_N))\|_{\mathscr{A} *_{\mathscr{D}_k} \mathscr{B}} \quad \text{for all } k \geqslant 1 \end{aligned}$$

and, consequently,

$$\begin{aligned} &\|P(\sigma(x_1),\cdots,\sigma(x_N),\rho(y_1),\cdots,\rho(y_N))\|_{\mathscr{A} *_{\mathscr{D}} \mathscr{B}} \\ &\leqslant \|P(a_1,\cdots,a_N,b_1,\cdots,b_N)\|_{\prod_{k\geqslant 1} \mathscr{A} *_{\mathscr{D}_k} \mathscr{B} / \sum_{k\geqslant 1} \mathscr{A} *_{\mathscr{D}_k} \mathscr{B}} \end{aligned} \tag{5.41}$$

We will show that $[(\sigma_k(z))_k] = [(\rho_k(z))_k]$ for every $z \in \mathscr{D}$. Suppose $z \in \mathscr{D}$ and $\varepsilon > 0$. Then there exist a positive integer $p$ and an element $z_p$ in $\mathscr{D}_p$ such that $\|z - z_p\| < \varepsilon$. Since $\{\mathscr{D}_k\}_k$ is an increasing sequence of $\mathrm{C}^*$-algebras, we know that $z_p \in \mathscr{D}_k$ for $k \geqslant p$. It follows that $[(\sigma_k(z_p))_k] = [(\rho_k(z_p))_k]$. So,

$$\begin{aligned} &\|[(\sigma_k(z))_k] - [(\rho_k(z))_k]\|_{\prod_{k\geqslant 1} \mathscr{A} *_{\mathscr{D}_k} \mathscr{B} / \sum_{k\geqslant 1} \mathscr{A} *_{\mathscr{D}_k} \mathscr{B}} \\ &\quad = \limsup_k \|\sigma_k(z) - \rho_k(z)\|_{\mathscr{A} *_{\mathscr{D}_k} \mathscr{B}} \end{aligned}$$

$$\begin{aligned}&\leqslant \limsup_k(\|\sigma_k(z)-\sigma_k(z_p)\|_{\mathscr{A}*_{\mathscr{D}_k}\mathscr{B}}+\|\sigma_k(z_p)-\rho_k(z_p)\|_{\mathscr{A}*_{\mathscr{D}_k}\mathscr{B}}\\&\qquad\qquad+\|\rho_k(z_p)-\rho_k(z)\|_{\mathscr{A}*_{\mathscr{D}_k}\mathscr{B}})\\&\leqslant 2\varepsilon \qquad \text{for all } \varepsilon>0.\end{aligned}$$

Thus we obtain that $[(\sigma_k(z))_k]=[(\rho_k(z))_k]$.

Now it follows from the definitions of full amalgamated free product and of the $C^*$-algebra $\mathscr{Q}$, together with the fact that $[(\sigma_k(z))_k]=[(\rho_k(z))_k]$ for every $z\in\mathscr{D}$, we know there is a $*$-homomorphism from $\mathscr{A}*_{\mathscr{D}}\mathscr{B}$ onto $\mathscr{Q}$, which maps each $\sigma(x_n),\rho(y_n)$ to $a_n,b_n$ respectively for $n\in\mathbb{N}$. Therefore,

$$\begin{aligned}&\|P(\sigma(x_1),\cdots,\sigma(x_N),\rho(y_1),\cdots,\rho(y_N))\|_{\mathscr{A}*_{\mathscr{D}}\mathscr{B}}\\&\geqslant \|P(a_1,\cdots,a_N,b_1,\cdots,b_N)\|_{\prod_{k\geqslant 1}\mathscr{A}*_{\mathscr{D}_k}\mathscr{B}/\sum_{k\geqslant 1}\mathscr{A}*_{\mathscr{D}_k}\mathscr{B}}\end{aligned}\tag{5.42}$$

Now equation (5.40) follows easily from inequalities (5.41) and (5.42). This ends our proof. □

Once we get the preceding theorem, we are ready to consider the case when $\mathscr{D}$ is an AF algebra. The following theorem is an analogous result to Theorem 2

THEOREM 4. *Suppose that $\mathscr{A}\supset\mathscr{D}\subset\mathscr{B}$ are unital inclusions of unital MF algebras where $\mathscr{D}$ is an AF algebra. Then the unital full free product $\mathscr{A}*_{\mathscr{D}}\mathscr{B}$ of $\mathscr{A}$ and $\mathscr{B}$ with amalgamation over $\mathscr{D}$ is an MF algebra if and only if there is an MF algebra $\mathscr{E}$ such that*

$$\mathscr{E}\supseteq\mathscr{A}\supset\mathscr{D}\subset\mathscr{B}\subseteq\mathscr{E}.$$

*Proof.* If $\mathscr{A}*_{\mathscr{D}}\mathscr{B}$ is MF, then let $\mathscr{E}=\mathscr{A}*_{\mathscr{D}}\mathscr{B}$. It is clear that

$$\mathscr{E}\supseteq\mathscr{A}\supset\mathscr{D}\subset\mathscr{B}\subseteq\mathscr{E}.$$

For another direction, suppose there is an MF algebra $\mathscr{E}$ such that $\mathscr{E}\supseteq\mathscr{A}\supset\mathscr{D}\subset\mathscr{B}\subseteq\mathscr{E}$. Note that $\mathscr{D}$ is an AF algebra, therefore there is an increasing sequence of unital finite-dimensional $C^*$-subalgebra $\{\mathscr{D}_p\}_{p\geqslant 1}$ of $\mathscr{D}$ such that $\cup_p\mathscr{D}_p$ is norm dense in $\mathscr{D}$. Hence we can find a sequnece of positive integers $\{n_k\}_{k=1}^{\infty}$ and a unital embedding $q:\mathscr{E}\to\prod_k\mathscr{M}_{n_k}(\mathbb{C})/\sum_k\mathscr{M}_{n_k}(\mathbb{C})$ such that the following diagram

$$\begin{array}{ccc}\mathscr{D}_p & \subseteq & \mathscr{A}\\ \cap & & \downarrow q|_{\mathscr{A}}\\ \mathscr{B} & \xrightarrow{q|_{\mathscr{B}}} & \prod_k\mathscr{M}_{n_k}(\mathbb{C})/\sum_k\mathscr{M}_{n_k}(\mathbb{C})\end{array}$$

commutes for each $p\geqslant 1$ where $q|_{\mathscr{A}}$ and $q|_{\mathscr{B}}$ are restrictions of $q$ onto $\mathscr{A}$ and $\mathscr{B}$ respectively. By Theorem 2, we obtain that $\mathscr{A}*_{\mathscr{D}_p}\mathscr{B}$ is an MF algebra for each $p\geqslant 1$. Now it follows from Theorem 3 that $\mathscr{A}*_{\mathscr{D}}\mathscr{B}$ is an MF algebra. □

Since every AF algebra has a faithful tracial state, we are able to consider the case when $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{D}$ are all AF algebras and give a sufficient condition in terms of faithful tracial states.

THEOREM 5. *Suppose that* $\mathscr{A} \supset \mathscr{D} \subset \mathscr{B}$ *are unital inclusions of AF* $C^*$*-algebras. If there are faithful tracial states* $\tau_{\mathscr{A}}$ *and* $\tau_{\mathscr{B}}$ *on* $\mathscr{A}$ *and* $\mathscr{B}$ *respectively, such that*

$$\tau_{\mathscr{A}}(x) = \tau_{\mathscr{B}}(x), \qquad \forall\, x \in \mathscr{D},$$

*then* $\mathscr{A} *_{\mathscr{D}} \mathscr{B}$ *is an MF algebra.*

*Proof.* Assume that $\{x_n\}_{n=1}^{\infty} \subseteq \mathscr{A}$, $\{y_n\}_{n=1}^{\infty} \subseteq \mathscr{B}$ and $\{z_n\}_{n=1}^{\infty} \subseteq \mathscr{D}$ are families of generators in $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{D}$ respectively. Note that $\mathscr{A}, \mathscr{B}$ and $\mathscr{D}$ are AF algebras. For each $N \in \mathbb{N}$, there are finite dimensional $\mathrm{C}^*$-subalgebras $\mathscr{D}_N \subseteq \mathscr{D}$, $\mathscr{A}_N \subseteq \mathscr{A}$ and $\mathscr{B}_N \subseteq \mathscr{B}$ such that

$$\max_{1 \leqslant n \leqslant N} \{dist(x_n, \mathscr{A}_N), dist(y_n, \mathscr{B}_N), dist(z_n, \mathscr{D}_N)\} \leqslant \frac{1}{N} \tag{5.43}$$

and

$$\mathscr{A}_N \supset \mathscr{D}_N \subset \mathscr{B}_N \tag{5.44}$$

Note that $\tau_{\mathscr{A}}(x) = \tau_{\mathscr{B}}(x)$, $\forall\, x \in \mathscr{D}$. From the argument in the proof of Theorem 4.2 [1], there are rational faithful tracial states on $\mathscr{A}_N$ and $\mathscr{B}_N$ such that their restrictions on $\mathscr{D}_N$ agree. This implies that there is a positive integer $k_N$ such that

$$\mathscr{M}_{k_N}(\mathbb{C}) \supseteq \mathscr{A}_N \supseteq \mathscr{D}_N \subseteq \mathscr{B}_N \subseteq \mathscr{M}_{k_N}(\mathbb{C}). \tag{5.45}$$

Combining (5.43), (5.44) and (5.45), we know that there is a sequence of positive integers $\{k_N\}_{N=1}^{\infty}$ such that

$$\prod_N \mathscr{M}_{k_N}(\mathbb{C}) / \sum_N \mathscr{M}_{k_N}(\mathbb{C}) \supseteq \mathscr{A} \supset \mathscr{D} \subset \mathscr{B} \subseteq \prod_N \mathscr{M}_{k_N}(\mathbb{C}) / \sum_N \mathscr{M}_{k_N}(\mathbb{C}).$$

By Theorem 4, we obtain that $\mathscr{A} *_{\mathscr{D}} \mathscr{B}$ is an MF algebra. $\square$

It is well-known that the tracial state on each UHF algebra is unique. Therefore we can restate Theorem 5 when $\mathscr{A}$, $\mathscr{B}$ are both UHF algebras and $\mathscr{D}$ is an AF algebra. A necessary and sufficient condition can be given in this case.

COROLLARY 2. *Suppose that* $\mathscr{A} \supseteq \mathscr{D} \subseteq \mathscr{B}$ *are unital inclusions of C*-algebras where* $\mathscr{A}$, $\mathscr{B}$ *are UHF algebras and* $\mathscr{D}$ *is an AF algebra. Then* $\mathscr{A} \underset{\mathscr{D}}{*} \mathscr{B}$ *is an MF algebra if and only if*

$$\tau_{\mathscr{A}}(z) = \tau_{\mathscr{B}}(x) \text{ for each } z \in \mathscr{D},$$

*where* $\tau_{\mathscr{A}}$ *and* $\tau_{\mathscr{B}}$ *are faithful tracial states on* $\mathscr{A}$ *and* $\mathscr{B}$ *respectively.*

*Proof.* From the fact that every MF algebra has a tracial state and the tracial state on UHF algebra is unique and faithful, one direction of the proof is obvious. Another direction is followed by applying Theorem 5. $\square$

**Acknowledgement**

It is a pleasure to thank Don Hadwin, who is the first author's advisor, for many helpful discussions and suggestions during the preparation of this paper. The research of second author is partially sponsored by an NSF grant.

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(Received November 23, 2011)


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