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AN ORTHOGONALITY PROPERTY FOR REAL SYMMETRIC MATRIX POLYNOMIALS WITH APPLICATION TO THE INVERSE PROBLEM

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Abstract. An orthogonality property common to a broad class of real symmetric matrix polynomials is developed generalizing earlier results concerning polynomials of second degree. This property is obtained with the help of canonical forms expressed in terms of a triple of *real* matrices (even though there may be complex spectrum) and it is used in the solution of an inverse spectral problem. The distribution of eigenvalues on the real line is discussed and earlier results for quadratic polynomials are generalized, in which the inertias of coefficient matrices are expressed in terms of the canonical forms.

1. Introduction

We consider real selfadjoint matrix polynomials of any degree. With the exception of Appendix A, it is assumed for simplicity that *all eigenvalues are semisimple*. The main objective is a generalization of an orthogonality property originating in [15], and developed in [10], for quadratic polynomials having positive definite leading coefficient and no real eigenvalues. See also [9] for the (hyperbolic) quadratic case of *all* real eigenvalues. This orthogonality property has also been studied recently by Al-Ammari and Tisseur in [1] for general quadratic matrix polynomials.

Here, a general nonsingular leading coefficient is admitted as well as mixed real/non-real spectrum. We take advantage of the corresponding spectral theory of *selfadjoint* matrix polynomials initiated by Gohberg, Lancaster, and Rodman in 1978 in [2], continued in [4], [5], [7], and revisited more recently in [12]. We study $n \times n$ polynomials of the form

$$L(\lambda) = L_\ell \lambda^\ell + L_{\ell-1}\lambda^{\ell-1} + \cdots + L_1\lambda + L_0, \quad \ell \geqslant 1 \tag{1}$$

with *real and symmetric* coefficients and $\det L_\ell \neq 0$. The spectrum generally consists of both real and non-real eigenvalues - the latter in conjugate pairs.

The notions of *real selfadjoint Jordan triple* and *sign characteristic* will play an important role in our development. These are reviewed in Section 2, and in Section 3 a specific real selfadjoint Jordan triple is constructed (see Theorem 1) that is well-suited

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to derivation of the orthogonality property. Theorem 10 of Appendix A also has a role in this theory.

Section 4 contains the basic orthogonality results. It is shown in Theorem 2 that, for $n \times n$ real symmetric matrix polynomials of even degree $\ell = 2m$, "half" of the spectral data (eigenvalues, eigenvectors and sign characteristic) is determined by the other "half" together with a real orthogonal matrix of size $mn \times mn$. There is an analogous statement for the case of polynomials of odd degree.

A discussion of our basic hypotheses in the context of more general complex systems appears in Section 5.

Section 6 provides an investigation of the orthogonal matrix mentioned above, and characterizes admissible eigenvalue and sign characteristic distributions for symmetric matrix polynomials of even degree, $\ell = 2m$. The main result is Theorem 5 in which a connection is made between admissible canonical structures and certain $mn \times mn$ real orthogonal matrices. It is known that, for Hermitian matrix polynomials with positive definite leading coefficients, some restrictions apply to the spectral data (see Example 1.5 of [5]) and this idea is developed further in Theorem 7.

In Section 8 we focus on the important special case of the inverse *quadratic* eigenvalue problem, $\ell = 2$, and show how the general theory developed in Section 6 applies in this important case. Proposition 8 shows how the inertias of the matrix coefficients can be expressed in terms of a canonical triple. Corollary 9 demonstrates the role played by $n \times n$ real orthogonal matrices in this construction.

In the case of *real symmetric* polynomials with $L_\ell > 0$, a result of Gohberg, Lancaster, and Rodman [6] shows how to take advantage of conjugate complex symmetry in the formulation of Jordan triples. A generalization to admit general nonsingular L_ℓ is the subject of Appendix A

2. Canonical forms

An early comprehensive study of canonical structures for selfadjoint matrix polynomials can be found in [6], but is confined to the case of positive definite leading coefficient, L_ℓ. However, the theory in [7] is more general in that L_ℓ is to be invertible, but may be *indefinite*. Furthermore, this degree of generality is maintained in the recent survey [12]. This section provides a survey of necessary canonical forms for real symmetric matrix polynomials of the form (1).

Briefly, this structure includes a complete summary of the eigenvalue/eigenvector data for $L(\lambda)$ and the so-called "sign characteristic" of the real eigenvalues. Basic references are [6, 7, 12].

Let $\mathbb{F}$ denote the field of either the real numbers or the complex numbers. Two matrices $X \in \mathbb{F}^{n \times \ell n}$ and $T \in \mathbb{F}^{\ell n \times \ell n}$ form a "standard pair" if

$$C(X,T) = \begin{bmatrix} X \\ XT \\ \vdots \\ XT^{\ell-1} \end{bmatrix} \tag{2}$$

is nonsingular and it is a standard pair for $L(\lambda)$ if, in addition,

$$L(X,T) := L_\ell X T^\ell + \cdots + L_1 X T + L_0 X = 0.$$

Then three matrices $X \in \mathbb{F}^{n\times \ell n}$, $T \in \mathbb{F}^{\ell n \times \ell n}$ and $Y \in \mathbb{F}^{\ell n \times n}$ are said to form a *standard triple* if (X,T) is a standard pair and

$$Y = \begin{bmatrix} X \\ XT \\ \vdots \\ XT^{\ell-1} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ Q \end{bmatrix}. \tag{3}$$

for some nonsingular matrix $Q \in \mathbb{F}^{n\times n}$. If (X,T) is a standard pair for $L(\lambda)$ and (3) holds with $Q = L_\ell^{-1}$ then (X,T,Y) is a *standard triple for* $L(\lambda)$.

All standard triples for $L(\lambda)$ are similar in the sense that (X_1,T_1,Y_1) and (X_2,T_2,Y_2) are standard triples for $L(\lambda)$ if and only if

$$(X_2,T_2,Y_2) = (X_1 S, S^{-1}T_1 S, S^{-1}Y_1)$$

for some invertible matrix S. In particular, if (X,T,Y) is a standard triple for $L(\lambda)$ then, by applying a similarity transformation, we can obtain a triple $(XS, S^{-1}TS, S^{-1}Y)$ with $S^{-1}TS$ in Jordan canonical form. Such a triple is called a *Jordan triple* for $L(\lambda)$, the columns of XS are right Jordan chains of $L(\lambda)$ and the rows of $S^{-1}Y$ are left Jordan chains. Thus, Jordan triples convey complete information about the eigenvalue/eigenvector data of $L(\lambda)$.

Now, if $L(\lambda)$ is real symmetric ($L_i^T = L_i$, $i = 0,1,\ldots,\ell$) or complex Hermitian ($L_i^* = L_i$, $i = 0,1,\ldots,\ell$) then the non-real eigenvalues appear in conjugate pairs and the relationship between the right and left Jordan chains is stronger than (3). It is related to the *sign characteristic* as follows. Since $L(\lambda) = L^\star(\lambda)$ ($\star = {}^*$ or ^{T}), both (X,T,Y) and $(Y^\star, T^\star, X^\star)$ are standard triples of $L(\lambda)$. So there is an invertible matrix S such that

$$Y^\star = XS, \quad T^\star = S^{-1}TS, \quad X^\star = S^{-1}Y. \tag{4}$$

It turns out (see [6]) that such a matrix S is *unique* and that (see [12]) S is symmetric or Hermitian according as $L(\lambda)$ is real symmetric or complex Hermitian, respectively. Thus, (4) reduces to

$$Y^\star = XH, \quad T^\star = H^{-1}TH, \quad (H^\star = H). \tag{5}$$

Triples (X,T,Y) satisfying (5) are called *selfadjoint standard triples* when X, T and Y are matrices with complex entries and $\star = {}^*$ (conjugate transposition). And if X, T and Y are real matrices and $\star = {}^T$ (transposition) then they are called *real selfadjoint standard triples*. When T is in (real) Jordan form then they are called *(real) selfadjoint Jordan triples*.

The general theory asserts (see [5, 12]), that if $L(\lambda)$ is *complex Hermitian* then there is a selfadjoint Jordan triple $(X, J, P_{\varepsilon,J}X^*)$ for $L(\lambda)$ for which:

$$J = \bigoplus_{j=1}^{r} J_{l_j}(\alpha_j) \bigoplus \bigoplus_{k=1}^{s} \begin{bmatrix} J_{m_k}(\bar{\beta}_k) & 0 \\ 0 & J_{m_k}(\beta_k) \end{bmatrix}, \tag{6}$$

and

$$P_{\varepsilon,J} = \bigoplus_{j=1}^{q} \varepsilon_j F_{l_j} \bigoplus \bigoplus_{k=1}^{s} F_{2m_k}, \tag{7}$$

where

- $\alpha_1, \ldots, \alpha_r$ are the (not necessarily distinct) real eigenvalues of $L(\lambda)$ with partial multiplicities $l_1, \ldots, l_r$,
- $(\beta_1, \bar{\beta}_1), \ldots, (\beta_s, \bar{\beta}_s)$ are the (not necessarily distinct) pairs of non-real conjugate eigenvalues of $L(\lambda)$ with partial multiplicities $m_1, \ldots, m_s$,
- $J_k(\lambda_0) = \begin{bmatrix} \lambda_0 & & & \\ 1 & \lambda_0 & & \\ & \ddots & \ddots & \\ & & 1 & \lambda_0 \end{bmatrix} \in \mathbb{F}^{k \times k}$, $F_k = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{bmatrix} \in \mathbb{F}^{k \times k}$, and
- $\varepsilon_1, \ldots, \varepsilon_q$ are each equal to either +1 or -1 and, together, they are known as the *sign characteristic* of $L(\lambda)$.

Furthermore, if $L(\lambda)$ is *real and symmetric* then there is a *real* selfadjoint Jordan triple for $L(\lambda)$ of the form $(X, J, P_{\varepsilon,J} X^T)$ where $P_{\varepsilon,J}$ is as in (7) and

$$J = \bigoplus_{j=1}^{r} J_{l_j}(\alpha_j) \bigoplus \bigoplus_{k=1}^{s} K_{2m_k}(\beta_k) \tag{8}$$

is a matrix in *real* Jordan form with

$$K_j(\beta_j) = \begin{bmatrix} U_j & & & \\ I_2 & U_j & & \\ & \ddots & \ddots & \\ & & I_2 & U_j \end{bmatrix} \in \mathbb{R}^{j \times j}, \tag{9}$$

and

$$U_j = \begin{bmatrix} \mu_j & -\nu_j \\ \nu_j & \mu_j \end{bmatrix}, \; \beta_j = \mu_j + i\nu_j, \; \nu_j > 0.$$

In both cases the columns of X and the rows of $P_{\varepsilon,J} X^\star$ (recall $^\star = {}^*$ or ^{T}) are real or complex right and left Jordan chains, respectively, of $L(\lambda)$ according as this matrix is real symmetric or complex Hermitian (see [12] for the notion of real Jordan chain of non-real eigenvalues). In particular, if $L(\lambda)$ is a real symmetric matrix polynomial with non-real eigenvalues, then it admits both real and complex Jordan forms and chains. The relationship between them will be used in the next section and is explored further in Appendix A.

3. Convenient real canonical structures

As mentioned in the introduction, we will focus on semisimple real symmetric matrix polynomials with nonsingular leading coefficents. If $L(\lambda)$ is such a matrix function, the partial multiplicities of its eigenvalues are all equal to one. If $2s$ is the total number of non-real eigenvalues (counting multiplicities) then the remaining $n\ell-2s$ eigenvalues are real. Denote the *signature* of L_ℓ by δ, i.e. δ is the difference between the number of positive and negative real eigenvalues of L_ℓ, so that $-n \leqslant \delta \leqslant n$.

By Proposition 4.2 of [4], if the sign characteristic of $L(\lambda)$ associated with the real eigenvalue λ_i $(1 \leqslant i \leqslant n\ell-2s)$ is $\varepsilon_i = \pm 1$, it follows that

$$\sum_{i=1}^{n\ell-2s} \varepsilon_i = \begin{cases} 0 & \text{if } \ell \text{ is even,} \\ \delta & \text{if } \ell \text{ is odd,} \end{cases}$$

Now define χ to be 0 or 1 according as ℓ is even or odd, respectively, and if q is the number of real eigenvalues of *negative type* (negative sign characteristic), then $q+\chi\delta$ is the number of real eigenvalues of *positive type* (positive sign characteristic). Thus, $q \geqslant 0$ and $q+\chi\delta \geqslant 0$ but it may happen that $q+\chi\delta = 0$ or $q=0$.

Let $r_1,\ldots,r_{q+\chi\delta}$ be the real eigenvalues of positive type, $r_{q+\chi\delta+1},\ldots,r_{2q+\chi\delta}$ be those of negative type and construct diagonal matrices of size $q+\chi\delta$ and q:

$$R_+ = \operatorname{Diag}(r_1,\ldots,r_{q+\chi\delta}), \qquad R_- = \operatorname{Diag}(r_{q+\chi\delta+1},\ldots,r_{2q+\chi\delta}). \tag{10}$$

Write the $2s$ conjugate pairs of eigenvalues as follows:

$$\beta_j = \mu_j + i\nu_j, \qquad \beta_{j+1} = \bar{\beta}_j = \mu_j - i\nu_j \qquad (\nu_j > 0), \qquad j = 1,3,\ldots,2s-1.$$

The semisimple case of Theorem 10 of Appendix A implies that there is a (generally) complex Jordan triple for $L(\lambda)$ of the form $(X_c,\, J_c,\, P_cX_c^*)$ where $X_c \in \mathbb{C}^{n\times n\ell}, J_c \in \mathbb{C}^{n\ell\times n\ell}, P_c \in \mathbb{R}^{n\ell\times n\ell}$ and

$$J_c = \operatorname{Diag}(R_+,R_-,\bar{\beta}_1,\beta_1,\ldots,\bar{\beta}_s,\beta_s),$$

$$P_c = \operatorname{Diag}\left(I_{q+\chi\delta}, -I_q, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right), \tag{11}$$

$$X_c = \left[X_+ \; X_- \; u_1 - iv_1 \; u_1 + iv_1 \; \cdots \; u_s - iv_s \; u_s + iv_s\right], \tag{12}$$

and $X_+ \in \mathbb{R}^{n\times(q+\chi\delta)}, X_- \in \mathbb{R}^{n\times q}$, $u_j, v_j \in \mathbb{R}^{n\times 1}$, $j = 1,\ldots,s$.

We are to transform such a triple to a *real* selfadjoint Jordan triple by applying a suitable transformation as follows:

$$(X_c,\, J_c,\, P_cX_c^*) \to (X_c\hat{U},\, \hat{U}^*J_c\hat{U},\, \hat{U}^*P_cX_c^*) = \left(X_c\hat{U},\, \hat{U}^*J_c\hat{U},\, (\hat{U}^*P_c\hat{U})(X_c\hat{U})^*\right),$$

where $\hat{U}$ is unitary.

First consider the primitive unitary matrix $W = \frac{1}{\sqrt{2}}\begin{bmatrix} i & 1 \\ -i & 1 \end{bmatrix}$ and observe that, if $x = u+iv$, then

$$\left[u - iv \;\; u + iv\right] W = \sqrt{2}\left[v \;\; u\right].$$

Also, if $\beta = \mu + i\nu$,

$$W^* \begin{bmatrix} \bar{\beta} & 0 \\ 0 & \beta \end{bmatrix} W = \begin{bmatrix} \mu & -\nu \\ \nu & \mu \end{bmatrix}, \quad \text{and} \quad W^* \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} W = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

By applying corresponding unitary transformations to the complex Jordan triple, $(X_c,\, J_c,\, P_c X_c^*)$, one for each conjugate pair of non-real eigenvalues, we arrive at a *real* Jordan triple defined by:

$$J_R = \mathrm{Diag}\left(R_+,\, R_-,\, \begin{bmatrix} \mu_1 & -\nu_1 \\ \nu_1 & \mu_1 \end{bmatrix}, \ldots, \begin{bmatrix} \mu_s & -\nu_s \\ \nu_s & \mu_s \end{bmatrix}\right),$$

$$P_R = \mathrm{Diag}(I_{q+\chi\delta}, -I_q, -1, +1, \ldots, -1, +1),$$

$$X_R = \begin{bmatrix} X_+ & X_- & \sqrt{2}v_1 & \sqrt{2}u_1 & \cdots & \sqrt{2}v_s & \sqrt{2}u_s \end{bmatrix}.$$

Now apply the (unitary) permutation $\Pi = \mathrm{Diag}\left(I_{2q+\chi\delta},\, P_0\right)$ where

$$P_0 = \begin{bmatrix} e_1 & e_3 & e_2 & e_4 & \cdots & e_{2s} \end{bmatrix},$$

and, defining

$$M = \mathrm{Diag}(\mu_1, \mu_3, \ldots, \mu_{2s-1}), \quad N = \mathrm{Diag}(\nu_1, \nu_3, \ldots, \nu_{2s-1}) > 0, \tag{13}$$

we obtain the *final real canonical forms*:

$$J = \Pi^T J_R \Pi = \mathrm{Diag}\left(R_+, R_-, \begin{bmatrix} M & -N \\ N & M \end{bmatrix}\right) \in \mathbb{R}^{n\ell \times n\ell}, \tag{14}$$

$$P = \Pi^T P_R \Pi = \mathrm{Diag}\left(I_{q+\chi\delta}, -I_q, -I_s, I_s\right) \in \mathbb{R}^{n\ell \times n\ell}, \tag{15}$$

$$X = X_R \Pi = \begin{bmatrix} X_+ & X_- & V & U \end{bmatrix} \in \mathbb{R}^{n \times n\ell}, \tag{16}$$

where $X_+ \in \mathbb{R}^{n\times(q+\chi\delta)}, X_- \in \mathbb{R}^{n\times q}$ and

$$V = \sqrt{2}\begin{bmatrix} v_1 & \cdots & v_s \end{bmatrix} \in \mathbb{R}^{n\times s}, \qquad U = \sqrt{2}\begin{bmatrix} u_1 & \cdots & u_s \end{bmatrix} \in \mathbb{R}^{n\times s}. \tag{17}$$

Note the fundamental symmetry property, $(JP)^T = JP$. This particular canonical form has the advantage of making the inertia of P explicit.

Bearing in mind the results of Section 2 and, in particular, the definition (5) of a real selfadjoint Jordan triple we have:

THEOREM 1. *A semisimple real matrix polynomial of the form* (1) *with L_ℓ nonsingular has a real selfadjoint canonical triple (X, J, PX^T) where X, J, P have the real canonical forms* (16), (14), (15)*, respectively. Conversely, such a real canonical triple uniquely defines a semisimple real selfadjoint matrix polynomial $L(\lambda)$ with L_ℓ nonsingular.*

The last part of the theorem is a straightforward consequence of Theorems 2.4 and 3.5 of [12].

REMARK 1. We note that, for the real selfadjoint Jordan triple (X,J,PX^T) of Theorem 1, we have the following properties:

(i) The "moment conditions" hold (see (3)):

$$XJ^kPX^T = 0, \quad k = 0,1,\ldots,\ell-2 \tag{18}$$

and, furthermore, the leading coefficient is given by

$$XJ^{\ell-1}PX^T = L_\ell^{-1}. \tag{19}$$

Indeed, it is generally possible to express all the coefficients of $L(\lambda)$ in terms of the moments.

(ii) More generally, when λ is not an eigenvalue of $L(\lambda)$:

$$\lambda^r L(\lambda)^{-1} = \begin{cases} XJ^r(I\lambda - J)^{-1}PX^T, & r = 0,1,\ldots,\ell-1, \\ XJ^\ell(I\lambda I - J)^{-1}PX^T + L_\ell^{-1}, & r = \ell, \end{cases}$$

for all $\lambda \notin \sigma(L)$, the spectrum of $L(\lambda)$. The case $r = 0$ is known as the "resolvent form" (see [3], Corol. to Thm. 1).

(iii) When ℓ (the degree of $L(\lambda)$) is odd ($\chi = 1$) there are at least $|\delta|$ real eigenvalues r_j (see [4], Thm. 3.1).

REMARK 2. If J and P are matrices as in (14) and (15) and X satisfies (18) and (19) then (X,J,PX^T) is a real selfadjoint Jordan triple. In fact, $C(X,J)$ of (2) is invertible because

$$C(X,J)C(XP,J^T)^T = \begin{bmatrix} 0 & \cdots & 0 & L_\ell^{-1} \\ 0 & \cdots & L_\ell^{-1} & \star \\ \vdots & \iddots & \vdots & \vdots \\ L_\ell^{-1} & \cdots & \star & \star \end{bmatrix}$$

is invertible and (X,J,PX^T) clearly satisfies (3) and (5).

EXAMPLE 1. To illustrate, consider the real matrix polynomial

$$L(\lambda) := \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}\lambda^2 + \begin{bmatrix} 1 & 3 \\ 3 & -5 \end{bmatrix}\lambda + \begin{bmatrix} 1 & 2 \\ 2 & -6 \end{bmatrix}$$

of [9], Example 4. The four eigenvalues are: a real eigenvalue -1 of positive type, a real eigenvalue -2 of negative type, and a conjugate pair $-2 \pm i$.

There is a complex Jordan triple:

$$J_c = \mathrm{Diag}(-1,-2,-2-i,-2+i),$$

$$P_c = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad X_c = \begin{bmatrix} 1 & 0 & \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ 0 & 1 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

It is found that, as required, $X_c P_c X_c^* = 0$ and

$$X_c J_c P_c X_c^* = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}^{-1}.$$

In terms of the *real* canonical structures of (14), (15) and (16) we have:

$$J = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & 1 & -2 \end{bmatrix}, \qquad P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$X = \begin{bmatrix} X_+ & X_- & V & U \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

The moment condition (18) can be verified: $XPX^T = 0$ and (as in (19))

$$XJPX^T = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}^{-1}.$$

4. An orthogonality property

Using the canonical forms (14) - (17), and for $k = 0, 1, \ldots, \ell - 1$, form $n \times 2s$ matrices

$$\begin{bmatrix} V_k & U_k \end{bmatrix} := \begin{bmatrix} V & U \end{bmatrix} \begin{bmatrix} M & -N \\ N & M \end{bmatrix}^k \tag{20}$$

and $n \times (q + \chi\delta + s)$, $n \times (q + s)$ matrices:

$$A_k := \begin{bmatrix} X_+ R_+^k & U_k \end{bmatrix}, \quad B_k := \begin{bmatrix} X_- R_-^k & V_k \end{bmatrix}, \tag{21}$$

respectively. Then, with the definitions (14), (16) and (20),

$$XJ^k = \begin{bmatrix} X_+ R_+^k & X_- R_-^k & V_k & U_k \end{bmatrix} \in \mathbb{R}^{n \times n\ell}, \tag{22}$$

and if $\ell = 2m + \chi$ then the moment conditions of (18) can be written collectively as a matrix product:

$$\begin{bmatrix} X \\ XJ \\ \vdots \\ XJ^{m+\chi-1} \end{bmatrix} \begin{bmatrix} PX^T & JPX^T & \cdots & J^{m-1}PX^T \end{bmatrix} = 0, \tag{23}$$

the zero matrix of size $n(m+\chi)\times nm$. (Actually, this is the submatrix of B^{-1} on p. 34 of [5] formed by its first $m+\chi$ block-rows and m block-columns.)

Notice now that, by definition of a selfadjoint standard triple (see (5)), $PJP=J^T$. Therefore $J^kP=P(J^T)^k$ and equation (23) takes the symmetric form

$$\begin{bmatrix} X \\ XJ \\ \vdots \\ XJ^{m+\chi-1}\end{bmatrix} P\begin{bmatrix} X^T & J^TX^T & \cdots & (J^T)^{m-1}X^T\end{bmatrix}=0. \tag{24}$$

However, using (22) and the definition of (21) it follows that, for $i=0,1,\dots,m+\chi-1$ and $k=0,1,\dots,m-1$, the (i,k) block-entry of this matrix is:

$$\begin{aligned} XJ^iP(J^T)^kX^T &= \begin{bmatrix} X_+R_+^i & -X_-R_-^i & -V_i & U_i\end{bmatrix}\begin{bmatrix}(R_+^T)^kX_+^k \\ (R_-^T)^kX_-^k \\ V_k^T \\ U_k^T\end{bmatrix}, \\ &= X_+R_+^i(R_+^T)^kX_+^T+U_iU_k^T-X_-R_-^i(R_-^T)^kX_-^K-V_iV_k^T, \\ &= A_iA_k^T-B_iB_k^T. \end{aligned}$$

Hence (24) is equivalent to

$$A_iA_k^T=B_iB_k^T, \qquad \text{for} \quad i=0,1,2,\dots,m+\chi-1 \quad \text{and} \quad k=0,1,2,\dots,m-1. \tag{25}$$

Define

$$A=\begin{bmatrix} A_0 \\ A_1 \\ \vdots \\ A_{m-1}\end{bmatrix}=\begin{bmatrix} X_+ & U_0 \\ X_+R_+ & U_1 \\ \vdots & \vdots \\ X_+R_+^{m-1} & U_{m-1}\end{bmatrix}, \quad B=\begin{bmatrix} B_0 \\ B_1 \\ \vdots \\ B_{m-1}\end{bmatrix}=\begin{bmatrix} X_- & V_0 \\ X_-R_- & V_1 \\ \vdots & \vdots \\ X_-R_-^{m-1} & V_{m-1}\end{bmatrix}, \tag{26}$$

and let us compute the sizes of A and B. It follows from $\ell=2m+\chi$ that $n\ell$ is both $2mn+n\chi$ and (counting the eigenvalues) $2s+2q+\delta\chi$. Thus

$$q+s+\delta\chi=nm+\frac{(n+\delta)\chi}{2}, \quad \text{and hence} \quad q+s=nm+\frac{(n-\delta)\chi}{2}.$$

Then A and B are $nm\times\left(nm+\frac{(n+\delta)\chi}{2}\right)$ and $nm\times\left(nm+\frac{(n-\delta)\chi}{2}\right)$ real matrices respectively.

It is convenient, at this point, to separate the cases of even and odd degree polynomials, i.e. $\chi=0$ or $\chi=1$.

(i) $\underline{\chi=0}$: $L(\lambda)$ has degree, $\ell=2m$, A and B are $nm\times nm$ real matrices and (25) is equivalent to $AA^T=BB^T$. In turn, this condition is equivalent (see Appendix B) to the existence of a real orthogonal $nm\times nm$ matrix Θ such that

$$B=A\Theta. \tag{27}$$

(ii) $\underline{\chi = 1}$: $L(\lambda)$ has degree $\ell = 2m+1$ and (25) is equivalent, simultaneously, to $AA^T = BB^T$ and $A_m A^T = B_m B^T$. If $p := \frac{(n+\delta)}{2}$ then the sizes of A and B are $nm \times (nm+p)$ and $nm \times (nm+p-\delta)$, respectively.

(a) If $\delta \geqslant 0$ then (see Appendix B), $AA^T = BB^T$ is equivalent to the existence of a real $(nm+p) \times (nm+p-\delta)$ matrix Θ with orthonormal rows such that

$$B = A\Theta \tag{28}$$

and, under this condition, $A_m A^T = B_m B^T$ is equivalent to $(A_m - B_m \Theta^T) A^T = 0$; i.e. the rows of $A_m - B_m \Theta^T$ and A are mutually orthogonal.

(b) If $\delta \leqslant 0$ then (see Appendix B), $AA^T = BB^T$ is equivalent to the existence of a real $(nm+p-\delta) \times (nm+p)$ matrix Θ with orthonormal rows such that

$$A = B\Theta \tag{29}$$

and, under this condition, $A_m A^T = B_m B^T$ is equivalent to $(B_m - A_m \Theta^T) B^T = 0$; i.e. the rows of $B_m - A_m \Theta^T$ and B are mutually orthogonal.

(c) When $\delta = 0$ and n is even Θ is a real orthogonal matrix.

Observe that the definitions of (26) imply that

$$\begin{bmatrix} A & B \end{bmatrix} = \begin{bmatrix} X \\ XJ \\ \vdots \\ XJ^{m-1} \end{bmatrix} \begin{bmatrix} I_{q+\delta\chi} & 0 & 0 & 0 \\ 0 & 0 & I_q & 0 \\ 0 & 0 & 0 & I_s \\ 0 & I_s & 0 & 0 \end{bmatrix},$$

so that $\begin{bmatrix} A & B \end{bmatrix}$ is simply a column permutation of the matrix

$$Q = \begin{bmatrix} X \\ XJ \\ \vdots \\ XJ^{m-1} \end{bmatrix} \in \mathbb{R}^{mn \times 2mn}.$$

Given that (X, J) is a right Jordan pair, $\operatorname{rank} Q = nm$ and, since either $A = B\Theta$ or $B = A\Theta$ for some full-rank matrix Θ, $\operatorname{rank} A = \operatorname{rank} B = nm$, and

$$\operatorname{rank} \begin{bmatrix} X_+ & U_k \end{bmatrix} = \operatorname{rank} \begin{bmatrix} X_- & V_k \end{bmatrix} = n, \qquad k = 0, 1 \ldots, m-1. \tag{30}$$

We have established the following general orthogonality property for real-symmetric matrix polynomials:

THEOREM 2. *Let $L(\lambda)$ be as in (1) and let (X, J, PX^T) be a real selfadjoint triple as defined in (14) - (17) where $J, P \in \mathbb{R}^{n\ell \times n\ell}$, $\ell = 2m + \chi$ ($\chi = 0$ or 1) and the sizes of the submatrices R_+ and R_- of J are $q + \delta\chi$ and q, respectively. Let $p = \frac{n+\delta}{2}$ and let full-rank real matrices A, B of sizes $nm \times (nm + p\chi)$ and $nm \times (nm + (p - \delta)\chi)$ be formed from this triple as in (21) and (26). Then:*

(i) If $\ell = 2m$ *there is a real orthogonal matrix* $\Theta \in \mathbb{R}^{mn\times mn}$ *such that* $B = A\Theta$.

(ii) If $\ell = 2m+1$ *and* $\delta > 0$ *there is a real* $(nm+p-\delta)\times(nm+p)$ *matrix* Θ *with orthonormal rows such that* $B = A\Theta$ *and* $(A_m - B_m\Theta^T)A^T = 0$.
If $\ell = 2m+1$ *and* $\delta < 0$ *there is a real* $(nm+p)\times(nm+p-\delta)$ *matrix* Θ *with orthonormal rows such that* $A = B\Theta$ *and* $(B_m - A_m\Theta^T)B^T = 0$.

Conversely, let J *and* P *be canonical matrices of the form* (14) *and* (15)*, respectively, and let* $X \in \mathbb{R}^{n\times \ell n}$ *be such that:*

(a) $XJ^{\ell-1}PX^T$ *is nonsingular, and*

(b) There is a real orthogonal $\Theta \in \mathbb{R}^{mn\times mn}$ *such that matrices* A *and* B*, formed as in* (21) *and* (26)*, satisfy condition (i) or (ii), as appropriate,*

then (X, J, PX^T) *is a real selfadjoint triple.*

The proof of the last part of this Theorem is straightforward: If A and B satisfy either condition (i) or (ii) then $XJ^kPX^T = 0$ for $k = 1,\dots,\ell n-2$ and, by Remark 2, if $XJ^{\ell-1}PX^T$ is nonsingular then (X, J, PX^T) is a real selfadjoint Jordan triple.

Theorem 2 provides a generalization of results in [10] (Section 3) and [9] (Section 9) in two important respects: polynomials of any degree are admitted and, also, polynomials with mixed real/non-real spectra. It also implies that real symmetric matrix polynomials with any prescribed degree and semi-simple spectrum can be constructed by appropriate choices of real matrices A and Θ, or B and Θ, as in (27), (28), or (29), as appropriate. We develop this idea for matrix polynomials of even degree in Section 6.

Notice also that if $L(\lambda)$ has even degree and $L_{2m} > 0$ then $L(\lambda)$ can be factorized into a product of matrix polynomials of degree m and the real matrices A and B of (27) are necessarily nonsingular (see Theorems 11.2 of [6] and 12.3.2 of [7]). Here, it has been shown that the matrices A and B are nonsingular under the weaker condition $\det L_{2m} \neq 0$.

5. Remarks on more general polynomial systems

The existence of the real transformation Θ with orthogonal rows is, of course, a direct consequence of equation (23). This, in turn, is part of the lower block-triangular Hankel structure of the matrix H^{-1} where

$$H = \begin{bmatrix} L_1 & L_2 & \cdots & & L_\ell \\ L_2 & \cdots & & L_\ell & 0 \\ \vdots & & & & \vdots \\ L_\ell & 0 & \cdots & & 0 \end{bmatrix}$$

(as in [5]). Now these triangular structures are shared with more general systems (with odd or even degree, complex hermitian coefficients, and with no hypotheses on the degrees of elementary divisors). Furthermore, this triangular structure can always be expressed in the form (24) but with (generally) *complex* selfadjoint standard triples (X_c, J_c, PX_c^*) where, perhaps, J_c is not a Jordan matrix but easily constructed from it (as in (14)). Matrix P, however, is a diagonal matrix of 1 and -1 exhibiting the inertia of the original matrix $P_{\varepsilon,J}$. This, in turn, leads to generalizations of (27) - but a (generally) *complex* matrix with orthonormal rows (*unitary* if ℓ is even) will play the role of Θ.

This is the approach taken in [1] for the general Hermitian quadratic case, but with a substantial difference: In [1] selfadjoint standard triples (X,T,PX^*) are obtained for Hermitian quadratic matrix polynomials with $P = \mathrm{Diag}(I,-I)$ but T does not reflect, in general, the Jordan structure of the matrix polynomial and so, the columns of X are not, in general, eigenvectors of that matrix polynomial. However, with this matrix P, $XPX^* = 0$ does reveal orthogonality properties of the columns of X.

We anticipate that properties of a *real orthogonal* Θ will be more readily visualized (as in [10] and [9]) and this, together with the frequent occurrence of real symmetric systems, is the rationale for our focus on analysis of the *real* selfadjoint Jordan triples of Theorem 1.

6. The inverse problem

Our goal in this section is investigation of the role of the real orthogonal matrix Θ of Theorem 2 in representations of the coefficients of $L(\lambda)$ – keeping in mind the constraint of equation (19) defining the leading coefficient. For simplicity, we will focus on symmetric matrix polynomials of *even* degree, so that $\ell = 2m$. (A similar study (though probably more difficult) could be carried out for odd degree matrix polynomials.)

Systems with invertible leading coefficient (as in Theorem 1) are of great importance and suggest the following definitions of "admissible" structures:

DEFINITION 3.

(a) A *real selfadjoint Jordan structure* is a pair of matrices $(J,P) \in \mathbb{R}^{2mn\times 2mn} \times \mathbb{R}^{2mn\times 2mn}$ with the form (14), (15) for some real diagonal matrices $R_+, R_- \in \mathbb{R}^{q\times q}$ and $M, N \in \mathbb{R}^{s\times s}$ $(q+s = nm)$ and $N > 0$.

(b) A real selfadjoint Jordan structure (J,P) is said to be *admissible* if there is an $X \in \mathbb{R}^{n\times \ell n}$ for which equations (18) and (19) hold (in particular, $XJ^{\ell-1}PX^T$ is nonsingular).

(c) An admissible real selfadjoint Jordan structure, (J,P), will be said to be *admissible positive* if $XJ^{\ell-1}PX^T > 0$.

Recall Remark 2: If J, P and X satisfy condition (18) and $XJ^{\ell-1}PX^T$ is nonsingular then (X,J,PX^T) is a real selfadjoint Jordan triple and (Theorem 1) it defines a unique real symmetric matrix polynomial. The coefficients of such a matrix polynomial can then be expressed in terms of the moments. In Section 8 we will be more specific about this construction in the quadratic case.

As mentioned in the introduction, there are real selfadjoint Jordan structures which are not admissible positive. For example, there is no 2×2 quadratic matrix polynomial with positive definite leading coefficient and Jordan structure

$$J = \mathrm{Diag}(1,2,3,4), \quad P = \mathrm{Diag}(1,1,-1-1).$$

This is because, by Example 1.5 of [5], for all matrix polynomials of even degree with positive definite leading coefficient, the sign characteristic of the largest real eigenvalue must be positive and that of the smallest real eigenvalue must be negative (see also Theorem 7 below).

With the help of Theorem 2 and the notions of subspaces which are nondegenerate, positive, or neutral with respect to a (possibly singular) symmetric matrix, the next theorem provides a characterization of the admissible real selfadjoint Jordan structures for matrix polynomials of *even* degree, $\ell = 2m$.

More notation will be needed: Given a Jordan matrix as in (14) with diagonal matrices M and N as in (13), define sequences of $2s\times 2s$ diagonal matrices $\{M_r\}_{r=0}^{\infty}$ and $\{N_r\}_{r=0}^{\infty}$ recursively by setting $M_0 = I_s$, $N_0 = 0_{s\times s}$ and

$$\begin{bmatrix} M_{r+1} & N_{r+1} \\ N_{r+1} & -M_{r+1} \end{bmatrix} = \begin{bmatrix} M & -N \\ N & M \end{bmatrix} \begin{bmatrix} M_r & N_r \\ N_r & -M_r \end{bmatrix}, \qquad r\in\mathbb{Z}, \tag{31}$$

and observe that $M_1 = M$, $N_1 = N$. Indeed,

$$\begin{bmatrix} M_r & N_r \\ N_r & -M_r \end{bmatrix} = \begin{bmatrix} M & -N \\ N & M \end{bmatrix}^r \begin{bmatrix} I_s & 0 \\ 0 & -I_s \end{bmatrix}, \qquad r\in\mathbb{Z}.$$

Let $\Theta\in\mathbb{R}^{nm\times nm}$ be an orthogonal matrix and define symmetric matrices in $\mathbb{R}^{nm\times nm}$,

$$H_k(\Theta) := \begin{bmatrix} I_{nm} & \Theta \end{bmatrix} \begin{bmatrix} R_+^k & 0 & 0 & 0 \\ 0 & M_k & 0 & -N_k \\ 0 & 0 & -R_-^k & 0 \\ 0 & -N_k & 0 & -M_k \end{bmatrix} \begin{bmatrix} I_{nm} \\ \Theta^T \end{bmatrix}, \quad k\in\mathbb{Z}. \tag{32}$$

and note that $H_0(\Theta) = 0$.

We recall next the definition of nondegenerate, positive and neutral subspaces with respect to a symmetric matrix (see [13] and, for the case when H is nonsingular, [7, Ch. 2]):

Given a symmetric matrix $H\in\mathbb{R}^{p\times p}$, a *subspace* $\mathscr{S}\subset\mathbb{R}^p$ is said to be:
(a) H-*nondegenerate* if $x\in\mathscr{S}$ and $x^THy = 0$ for all $y\in\mathscr{S}$ implies that $x = 0$,
(b) H-*positive* if $x^THx > 0$ for all $x\in\mathscr{S}$,
(c) H-*neutral* if $x^THx = 0$ for all $x\in\mathscr{S}$.

The following lemma will be useful:

LEMMA 4. *Let $H\in\mathbb{R}^{p\times p}$ be a symmetric matrix and S be a subspace of $\mathbb{R}^p$ of dimension d. Then:*
*(i) $\mathscr{S}$ is H-*nondegenerate *if and only if X^THX is invertible for any matrix $X\in\mathbb{R}^{p\times d}$ such that* $\mathrm{Im}X = \mathscr{S}$.

(ii) $\mathscr{S}$ *is* H-positive *if and only if* $X^T HX$ *is positive definite for any matrix* $X \in \mathbb{R}^{p\times d}$ *such that* $\mathrm{Im}X = \mathscr{S}$.
(iii) $\mathscr{S}$ *is* H-neutral *if and only if* $X^T HX = 0$ *for any matrix* $X \in \mathbb{R}^{p\times d}$ *such that* $\mathrm{Im}X = \mathscr{S}$.

Proof. (i) Assume that $\mathrm{Im}X = \mathscr{S}$ and $X^T HX$ is invertible. We are to show that $x \in \mathscr{S}$ and $x^T Hy = 0$ for all $y \in \mathscr{S}$ implies $x = 0$. We have $x = X\alpha$ for some $\alpha \in \mathbb{R}^d$ so, for any $y = Xu \in \mathscr{S}$ ($u \in \mathbb{R}^d$),

$$0 = x^T Hy = \alpha^T X^T HXu.$$

But this holds for all $u \in \mathbb{R}^d$, so $\alpha^T X^T HX = 0$ and, because $X^T HX$ is invertible, it follows that $\alpha = 0$ and, finally, $x = X\alpha = 0$, as required.

Conversely, if $\mathscr{S} = \mathrm{Im}X$ and $X^T HX$ is not invertible then there is an $\alpha \neq 0$ such that $\alpha^T X^T HX = 0$. Thus $\alpha^T X^T HX\beta = 0$ for all $\beta \in \mathbb{R}^d$ and so $\mathscr{S}$ is degenerate.

(ii) If $X^T HX > 0$, $\mathrm{Im}X = \mathscr{S}$ and $0 \neq y \in \mathscr{S}$ then $y = Xz$ for some $0 \neq z \in \mathbb{R}^d$ and $y^T Hy = z^T X^T HXz > 0$.

Conversely, assume that $\mathscr{S}$ is H-positive and let $X \in \mathbb{R}^{p\times d}$ be a matrix whose columns span $\mathscr{S}$. Take any nonzero $y \in \mathbb{R}^d$. Then $0 \neq z = Xy \in \mathscr{S}$ and $y^T X^T HXy = z^T Hz > 0$. This means that $X^T HX > 0$, as desired.

(iii) This follows at once from the fact that $x^T Hx = 0$ for all $x \in \mathscr{S}$ implies $x^T Hy = 0$ for all $x, y \in \mathscr{S}$ (see [7, p. 13]). □

THEOREM 5. *Let* $\ell = 2m$ *and* J, P *be canonical matrices of the form* (14) *and* (15). *Then* (J,P) *is an admissible (admissible positive) real selfadjoint Jordan structure if and only if there is a real orthogonal matrix* $\Theta \in \mathbb{R}^{nm\times nm}$ *such that, for* $k = 0,1,\ldots,2m-2$, *the real symmetric matrix* $H_k(\Theta)$ *of* (32) *is either the zero matrix or has a neutral subspace of dimension* n *which is non-degenerate (resp. positive) with respect to* $H_{2m-1}(\Theta)$.

Proof. Let (X,J,PX^T) be a real selfadjoint Jordan triple of some matrix polynomial of even degree $\ell = 2m$ with positive definite leading coefficient, L_{2m}. Then (18) and (19) are satisfied and

$$\begin{aligned} J^k &= \mathrm{Diag}\left[R_+^k \; R_-^k \; \begin{bmatrix} M & -N \\ N & M \end{bmatrix}^k \right], \\ &= \mathrm{Diag}\left[R_+^k \; R_-^k \; \begin{bmatrix} M_k & N_k \\ N_k & -M_k \end{bmatrix} \right] \mathrm{Diag}\left[I_{2q}, \begin{bmatrix} I_s & 0 \\ 0 & -I_s \end{bmatrix} \right]. \end{aligned}$$

Hence, for $k = 0,1,\ldots,2m-2$,

$$XJ^kPX^T = X\,\mathrm{Diag}\left[R_+^k \; -R_-^k \; \begin{bmatrix} -M_k & -N_k \\ -N_k & M_k \end{bmatrix} \right] X^T = 0, \tag{33}$$

and

$$L_{2m}^{-1} = X\,\mathrm{Diag}\left[R_+^{2m-1} \; -R_-^{2m-1} \; \begin{bmatrix} -M_{2m-1} & -N_{2m-1} \\ -N_{2m-1} & M_{2m-1} \end{bmatrix} \right] X^T. \tag{34}$$

Now,

$$X = \begin{bmatrix} X_+ \; X_- \; V \; U \end{bmatrix} = \begin{bmatrix} X_+ \; U \; X_- \; V \end{bmatrix} \begin{bmatrix} I_q & 0 & 0 & 0 \\ 0 & 0 & 0 & I_s \\ 0 & I_q & 0 & 0 \\ 0 & 0 & I_s & 0 \end{bmatrix}$$

and, using (27), Theorem 2 implies that $\begin{bmatrix} X_- \; V \end{bmatrix} = \begin{bmatrix} X_+ \; U \end{bmatrix} \Theta$, for some orthogonal matrix $\Theta \in \mathbb{R}^{nm \times nm}$. Hence

$$X = \begin{bmatrix} X_+ \; U \end{bmatrix} \begin{bmatrix} I_{nm} \; \Theta \end{bmatrix} \begin{bmatrix} I_q & 0 & 0 & 0 \\ 0 & 0 & 0 & I_s \\ 0 & I_q & 0 & 0 \\ 0 & 0 & I_s & 0 \end{bmatrix}.$$

Bearing in mind the definition (32) of $H_k(\Theta)$, (33) and (34) give

$$XJ^kPX^T = \begin{bmatrix} X_+ \; U \end{bmatrix} H_k(\Theta) \begin{bmatrix} X_+^T \\ U^T \end{bmatrix} = 0, \qquad k = 0, 1, \ldots, 2m-2, \tag{35}$$

and

$$L_{2m}^{-1} = \begin{bmatrix} X_+ \; U \end{bmatrix} H_{2m-1}(\Theta) \begin{bmatrix} X_+^T \\ U^T \end{bmatrix}. \tag{36}$$

Let $\mathscr{S}$ be the subspace spanned by the columns of $\begin{bmatrix} X_+^T \\ U^T \end{bmatrix}$. By (30), $\mathscr{S}$ has dimension n and by (35), (36) and Lemma 4, $\mathscr{S}$ is $H_k(\Theta)$-neutral for $k = 1, \ldots, 2m-2$, and $H_{2m-1}(\Theta)$-nondegenerate. Furthermore, if L_{2m} is positive definite, then $\mathscr{S}$ must be $H_{2m-1}(\Theta)$-positive.

Conversely, assume that for some real orthogonal matrix $\Theta \in \mathbb{R}^{nm \times nm}$ there is a subspace $\mathscr{S} \subset \mathbb{R}^{nm \times nm}$ of dimension n which is $H_k(\Theta)$-neutral for $k = 1, \ldots, 2m-2$ and $H_{2m-1}(\Theta)$-nondegenerate. Let $Y = \begin{bmatrix} X_+^T \\ U^T \end{bmatrix}$ be any full rank matrix for which $\operatorname{Im} Y = \mathscr{S}$. Define

$$\begin{bmatrix} X_- \; V \end{bmatrix} = \begin{bmatrix} X_+ \; U \end{bmatrix} \Theta,$$

and we have

$$X = \begin{bmatrix} X_+ \; X_- \; V \; U \end{bmatrix}.$$

Then

$$X = \begin{bmatrix} X_+ \; U \; X_- \; V \end{bmatrix} \begin{bmatrix} I_q & 0 & 0 & 0 \\ 0 & 0 & 0 & I_s \\ 0 & I_q & 0 & 0 \\ 0 & 0 & I_s & 0 \end{bmatrix} = \begin{bmatrix} X_+ \; U \end{bmatrix} \begin{bmatrix} I_{nm} \; \Theta \end{bmatrix} \begin{bmatrix} I_q & 0 & 0 & 0 \\ 0 & 0 & 0 & I_s \\ 0 & I_q & 0 & 0 \\ 0 & 0 & I_s & 0 \end{bmatrix}$$

and XJ^kPX^T satisfies (35). As $\mathscr{S}$ is $H_k(\Theta)$-neutral we have $XJ^kPX^T = 0$ for $k = 1, \ldots 2m-2$ and, since $\mathscr{S}$ is $H_{2m-1}(\Theta)$-nondegenerate,

$$XJ^{2m-1}X^T = \begin{bmatrix} X_+ \; U \end{bmatrix} H_{2m-1}(\Theta) \begin{bmatrix} X_+^T \\ U^T \end{bmatrix}$$

is invertible (Lemma 4). Thus (X,J,PX^T) is a real selfadjoint Jordan triple of a symmetric matrix polynomial of degree $2m$ with nonsingular leading coefficient. In addition, if $\mathscr{S}$ is $H_{2m-1}(\Theta)$-positive, then the leading coefficient of that matrix polynomial is positive definite. □

REMARK 3.

(i) Since $H_0(\Theta) = 0$ for any orthogonal matrix Θ, all subspaces are $H_0(\Theta)$-neutral. In other words, the condition $XPX^T = 0$ imposes no further restriction on Θ, X_+ and U.

(ii) The existence of positive and neutral subspaces with respect to a real symmetric matrix H can be characterized in terms of the number of positive, negative and zero eigenvalues of H. In fact, the proof of Theorem 2.3.2 in [7] can be slightly modified to show that the maximal dimension of a positive subspace with respect to the indefinite inner product defined by a symmetric matrix H (that may be singular) is the number of positive eigenvalues of H (counting multiplicities). Also the proof of Theorem 2.3.4 in [7] can be adapted to admit singular symmetric matrices. It can be seen that, in that case, the maximal dimension of an H-neutral subspace is $\min(i_+,i_-)+i_0$ where i_+, i_- and i_0 are the number of positive, negative and zero eigenvalues of H, respectively (i.e. its inertia).

7. Distribution of the real eigenvalues

Since neutral subspaces for $H_k(\Theta)$ are closely related to the eigenvalues of this symmetric matrix, we consider the distribution of the real eigenvalues on the real line. If we define

$$G_k := \begin{bmatrix} R_+^k & 0 & 0 & 0 \\ 0 & M_k & 0 & -N_k \\ 0 & 0 & -R_-^k & 0 \\ 0 & -N_k & 0 & -M_k \end{bmatrix}, \quad k \in \mathbb{Z}, \tag{37}$$

then (32) takes the form

$$H_k(\Theta) = \begin{bmatrix} I_{nm} \ \Theta \end{bmatrix} G_k \begin{bmatrix} I_{nm} \ \Theta \end{bmatrix}^T \qquad \text{with} \quad \Theta^T\Theta = I. \tag{38}$$

PROPOSITION 6. *For $k = 1,2,\ldots$, let $\lambda_1(G_k) \geqslant \cdots \geqslant \lambda_{2nm}(G_k)$ denote the eigenvalues of G_k and let $H_k(\Theta)$ be defined as in (38). Then for any $nm \times nm$ orthogonal matrix Θ the following inequalities hold:*

$$2\lambda_i(G_k) \geqslant \lambda_i(H_k(\Theta)) \geqslant 2\lambda_{i+nm}(G_k), \quad 1 \leqslant i \leqslant nm \tag{39}$$

where $\lambda_1(H_k(\Theta)) \geqslant \cdots \geqslant \lambda_{nm}(H_k(\Theta))$ denote the eigenvalues of $H_k(\Theta)$.

Proof. For any $nm \times nm$ orthogonal matrix Θ

$$\Omega = \frac{1}{\sqrt{2}} \begin{bmatrix} I_{nm} & \Theta \\ -\Theta^T & I_{nm} \end{bmatrix}$$

is also orthogonal. Clearly, $H_k(\Theta)$ is just the leading $nm\times nm$ principal submatrix of $2\Omega G_k\Omega^T$. Since this matrix and $2G_k$ have the same eigenvalues, property (39) follows at once from the Cauchy interlacing inequalities relating the eigenvalues of a symmetric matrix to those of any of its principal submatrices (see [14, Section 10.1], for example)). $\square$

It is clear that, for any k, the eigenvalues of G_k are easily computed from the eigenvalues of the prescribed Jordan form J. In fact, if J is a Jordan matrix as in (14), then the eigenvalues of G_k are:

$$r_1^k,\dots,r_q^k,-r_{q+1}^k,\dots,-r_{2q}^k,|\beta_1|^k,-|\beta_1|^k,|\beta_3|^k,-|\beta_3|^k,\dots,|\beta_{2s-1}|^k,-|\beta_{2s-1}|^k,$$

where $\beta_j=\mu_j+i\nu_j$, $j=1,3,\dots,2s-1$.

Let $i_+(\cdot)$ and $i_-(\cdot)$ denote the number of positive and negative eigenvalues, respectively. Then it is an immediate consequence of (39) that, for any orthogonal matrix Θ,

$$i_+(H_k(\Theta))\leqslant i_+(G_k) \quad \text{and} \quad s\leqslant i_-(G_k)\leqslant i_-(H_k(\Theta)). \tag{40}$$

These simple conditions impose restrictions on admissible Jordan structures for real symmetric quadratic matrix polynomials with positive definite leading coefficient. Stronger constraints on J and P are specified in the following theorem:

THEOREM 7. *Let $L(\lambda)$ be a semisimple Hermitian matrix polynomial with $L_\ell>0$ and a maximal real eigenvalue λ_{max}. For any $\alpha\leqslant\lambda_{max}$, let $p(\alpha)$, $n(\alpha)$ denote the number of real eigenvalues (counting multiplicites) of $L(\lambda)$ of positive and negative types (respectively) in the interval $(\alpha,\lambda_{max}]$. Then $n(\alpha)\leqslant p(\alpha)$ for all $\alpha\in[\lambda_{min},\lambda_{max}]$.*

This theorem can be proved with the help of the real eigenfunctions $\mu_1(\lambda),\dots,\mu_n(\lambda)$ of the $n\times n$ matrix function $L(\lambda)$, $\lambda\in\mathbb{R}$, and the fact that their zeros are just the real eigenvalues of $L(\lambda)$ (see Chapter 12 of [6]). The proof is postponed to a subsequent paper dedicated to the inverse quadratic eigenvalue problem.

To illustrate the scope of this result, there is *no* 3×3 quadratic symmetric matrix polynomial with $L_2>0$ and all real eigenvalues as follows:

Eigenvalue	-3	-2	-1	1	2	3
Sign-characteristic	$-$	$+$	$+$	$-$	$-$	$+$

In fact, $n(\alpha)>p(\alpha)$ on the interval $(-1,3]$.

As the real eigenvalues are displayed on the diagonal of G_1, the properties of Theorem 7 required to ensure $L_\ell>0$ are easily confirmed in special cases and incorporated in (37).

8. The quadratic case

We summarize results of the previous sections for the important special case of semisimple quadratic matrix polynomials. This is already a significant generalization of results in [10] and [9] since we admit mixed real/non-real spectrum and indefinite leading coefficient. Now (1) becomes

$$L(\lambda) = L_2\lambda^2 + L_1\lambda + L_0$$

and $L(\lambda)$ admits a standard triple (X, J, PX^T) as in Theorem 1, where $\begin{bmatrix} X_+ & U \end{bmatrix}$ and $\begin{bmatrix} X_- & V \end{bmatrix}$ are nonsingular $n \times n$ real matrices. According to Theorem 2 there is a real orthogonal matrix Θ such that

$$\begin{bmatrix} X_- & V \end{bmatrix} = \begin{bmatrix} X_+ & U \end{bmatrix} \Theta, \tag{41}$$

and this is the fundamental extension of the results of [10] and [9].

EXAMPLE 2. For $L(\lambda)$ of Example 1 there is mixed real/non-real spectrum. Furthermore,

$$\begin{bmatrix} X_+ & U \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} X_- & V \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

so that $\Theta = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$; an orthogonal matrix.

Now consider the inverse problem: To construct a family of real symmetric quadratic matrix polynomials (possibly with some definiteness constraints on the coefficients) with an admissible set of spectral data; namely, real and complex eigenvalues together with their partial multiplicities and sign characteristics for the real spectrum (an admissible Jordan structure). This is a difficult problem even in the semisimple case and we provide some partial insights in this section.

Recall first that a real selfadjoint Jordan triple, (X, J, PX^T), uniquely defines a matrix polynomial $L(\lambda)$ with this Jordan triple. Indeed, the coefficients can be given explicitly in terms of the triple. Specifically, following a lead given in [10], define the *moment functions* P_k acting on matrices $X \in \mathbb{R}^{n\times 2n}$ as follows:

$$P_k(X) := XJ^kPX^T \tag{42}$$

for integers k. (Recall equations (18) and (19), and note that k is *any* integer if zero is not in the spectrum and, otherwise, $k \geqslant 0$.).

Then $P_0(X) = 0$ because $XPX^T = 0$ and the coefficients are defined by the moments in the form:

$$\begin{aligned} L_2^{-1} &= P_1(X), \\ L_1 &= -L_2P_2(X)L_2 = -P_1(X)^{-1}P_2(X)P_1(X)^{-1}, \\ L_0 &= -L_2P_3(X)L_2 + L_1P_1(X)L_1, \\ &= -P_1(X)^{-1}\left[P_3(X) + P_2(X)P_1(X)^{-1}P_2(X)\right]P_1(X)^{-1}. \end{aligned} \tag{43}$$

Alternatively, if $0 \notin \sigma(L)$ then

$$L_0 = -P_{-1}(X)^{-1}. \tag{44}$$

We can also use Theorem 14.7.1 of [11] to obtain L_0 and L_1 in the form:

$$\left[L_0\ L_1\right] = -L_2XJ^2\begin{bmatrix} X \\ XJ \end{bmatrix}^{-1}.$$

As a consequence we have:

PROPOSITION 8. *If $m=1$, all eigenvalues of $L(\lambda)$ are semisimple and L_2, L_0 are both nonsingular, then the coefficients L_2, L_1, L_0 have the inertias of $P_1(X)$, $-P_2(X)$, and $-P_{-1}(X)$, respectively, where (X,J,PX^T) is a real selfadjoint Jordan triple for $L(\lambda)$.*

Hence, a real symmetric quadratic matrix polynomial with desirable definiteness conditions imposed on the coefficients can be constructed from an appropriate real selfadjoint Jordan triple. If the spectral data has been prescribed, the goal is to construct such a real selfadjoint Jordan triple with prescribed J and P as in (14) and (15), and Theorem 5 can be useful for this purpose. The following result is an immediate consequence of that theorem:

COROLLARY 9. *Let J and P be matrices of the form* (14) *and* (15) *with $\ell=2$. Then there is a full rank matrix $X\in\mathbb{R}^{n\times 2n}$ such that (X,J,PX^T) is a real selfadjoint Jordan triple if and only if there is an orthogonal matrix $\Theta\in\mathbb{R}^{n\times n}$ such that*

$$H_1(\Theta) = \left[I_n\ \Theta\right]\begin{bmatrix} R_+ & 0 & 0 & 0 \\ 0 & M & 0 & -N \\ 0 & 0 & -R_- & 0 \\ 0 & -N & 0 & -M \end{bmatrix}\begin{bmatrix} I_n \\ \Theta^T \end{bmatrix} \tag{45}$$

is nonsingular.

If $H_1(\Theta)$ is invertible for some orthogonal Θ, $Q\in\mathbb{R}^{n\times n}$ is any invertible matrix and we define $X = Q\left[I_n\ \Theta\right]$, then (X,J,PX^T) is a real selfadjoint Jordan triple as in Theorem 1, and so $XJPX^T$ is invertible. With this real selfadjoint Jordan triple, a real symmetric quadratic matrix polynomial is constructed with

$$L_2^{-1} = P_1(X) = XJPX^T = QH_1(\Theta)Q^T. \tag{46}$$

Thus, L_2 and $H_1(\Theta)$ must have the same inertia. In particular, if L_2 is to be *positive definite*, then an orthogonal matrix Θ must exist such that all eigenvalues of $H_1(\Theta)$ are positive.

It is easily seen that when *all* eigenvalues are in conjugate pairs, we recover results of [10] (see equation (22) and Theorem 5 of that paper).

EXAMPLE 3. Assume that $2q$ real eigenvalues for J are prescribed with $r_i > r_{q+i}$, $i=1,\dots,q$. Then (see (10)) $R_+ - R_- > 0$. Choosing $\Theta = \pm I_n$ in Corollary 9 and J, P as in (13), (14) we obtain

$$H_1(\Theta) = H_1(\pm I_n) = \begin{bmatrix} R_+ - R_- & 0 \\ 0 & \mp 2N \end{bmatrix}.$$

Since $N > 0$ (see (13)), $H_1(\pm I_n)$ is certainly nonsingular and, with $X = Q\left[\, I_n \ \pm I_n \,\right]$ (and Q nonsingular), a broad family of symmetric quadratic matrix polynomials is constructed with the Jordan structure (J,P) in common.

If $L_2 > 0$ is required then the choice $\Theta = I_n$ would not be admissible (see (46)).

APPENDICES

A. Symmetric matrix polynomials with real coefficients

In the main body of this paper (Sections 3 to 9) it has been found necessary to restrict the analysis to the *semisimple* case. Here, we do not make this assumption.

Theorem 10.7 of [6] shows how non-real spectral data for Hermitian polynomials can be organized in complex conjugate pairs. Here, we prove an analogous result in the *real symmetric* case with the spectral data organized in conjugate pairs (where appropriate). This is in contrast with the forms of equations (14) - (16), and *without the semisimple hypothesis*. The generalization of Theorem 10.7 of [6] (when applied to the real symmetric case) lies in the fact that the leading coefficient is not assumed to be positive definite.

THEOREM 10. *Let $L(\lambda)$ be an $n\times n$ real symmetric matrix polynomial with nonsingular leading coefficient. Then there exists a selfadjoint Jordan triple $(X,J,P_{\varepsilon,J}X^*)$ of $L(\lambda)$ with the following form: J and $P_{\varepsilon,J}$ are as in (6) and (7), respectively, and*

$$X = \left[\, X_0 \ U_1 - iV_1 \ U_1 + iV_1 \ \cdots, U_s - iV_s \ U_s + iV_s \,\right] \tag{47}$$

$X_0 \in \mathbb{R}^{n\times(l_1+\cdots+l_r)}$, $U_j, V_j \in \mathbb{R}^{n\times m_j}$, $j = 1,\ldots,s$.

Proof. Let $(\widetilde{X},J,P_{\varepsilon,J}\widetilde{X}^T)$ be a real selfadjoint Jordan triple of $L(\lambda)$ with $P_{\varepsilon,J}$ and J given by (7) and (8) respectively. Partition $\widetilde{X}$ according to the block diagonal structure of J:

$$\widetilde{X} = \left[\, \widetilde{X}_1 \cdots \widetilde{X}_r \ \widetilde{X}_{r+1} \cdots \widetilde{X}_{r+s} \,\right]$$

where $\widetilde{X}_j \in \mathbb{R}^{n\times l_j}$, $1 \leqslant j \leqslant r$, and $\widetilde{X}_{r+j} \in \mathbb{R}^{n\times 2m_j}$, $1 \leqslant j \leqslant s$. Put

$$X_0 = \left[\, \widetilde{X}_1 \cdots \widetilde{X}_r \,\right]$$

and, for $j = 1,\ldots,s$, consider the submatrix $\widetilde{X}_{r+j}$ and blocks K_{2m_j} and F_{2mj} appearing in the expression of J and $P_{\varepsilon,J}$ (cf. (7) and (8)). We are to find a unitary matrix W_j such that

$$\widetilde{X}_{r+j}W_j = \left[\, U_k - iV_j \ U_j + iV_j \,\right] \tag{48}$$

$$W_j^* F_{2m_j} W_j = F_{2m_j} \tag{49}$$

$$W_j^* K_{2m_j} W_j = \begin{bmatrix} J_{m_j}(\bar{\beta}_j) & 0 \\ 0 & J_{m_j}(\beta_j) \end{bmatrix}. \tag{50}$$

With these matrices we construct the unitary matrix

$$W = \mathrm{Diag}(I_{l_1+\cdots+l_r}, W_1, \ldots, W_s)$$

and define $X = \widetilde{X}W$, $J_c = W^*JW$ and $P_{\varepsilon,J_c} = W^*P_{\varepsilon,J}W$. Then $(\widetilde{X}, J, P_{\varepsilon,J}X^T)$ is (unitarily) similar to $(X, J_c, P_{\varepsilon,J_c}X^*)$ and so this triple is a selfadjoint Jordan triple of $L(\lambda)$ with the desired form.

In order to find the matrix W_j satisfying (48)-(50) the unitary matrix

$$Z = \frac{1}{2}\begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix}.$$

will be helpful. This matrix has the following properties:

(a) If $\widetilde{x}_k$ denotes the k-th columns of $\widetilde{X}_{r+j}$ then, for $k = 1, \ldots, m_j$

$$\begin{bmatrix} \widetilde{x}_{2k-1} & \widetilde{x}_{2k} \end{bmatrix} Z = \begin{bmatrix} u_{jk} - iv_{jk} & u_{jk} + iv_{jk} \end{bmatrix},$$

where $u_{jk} = \frac{1}{2}(\widetilde{x}_{2k-1} + \widetilde{x}_{2k})$ and $v_{jk} = \frac{1}{2}(\widetilde{x}_{2k-1} - \widetilde{x}_{2k})$.

(b) If $P_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ then $Z^*P_2Z = P_2$.

(c) $Z^*\begin{bmatrix} \mu_j & -\nu_j \\ \nu_j & \mu_j \end{bmatrix} Z = \begin{bmatrix} \mu_j - i\nu_j & 0 \\ 0 & \mu_j + i\nu_j \end{bmatrix} = \begin{bmatrix} \bar{\beta}_j & 0 \\ 0 & \beta_j \end{bmatrix}.$

Hence if $\widetilde{W}_j = \mathrm{Diag}(Z, Z, \ldots, Z)$ (m_j times) then $\widetilde{W}_j$ is unitary and

(a) $\widetilde{X}_{r+j}\widetilde{W}_j = \begin{bmatrix} u_{j1} - iv_{j1} & u_{j1} + iv_{j1} & \cdots & u_{jm_j} - iv_{jm_j} & u_{jm_j} + iv_{jm_j} \end{bmatrix},$

(b) $\widetilde{W}_j^* F_{2m_j} \widetilde{W}_j = F_{2m_j}$,

(c) $\widetilde{W}_j^* K_{2m_j} \widetilde{W}_j = \begin{bmatrix} B_j & & & \\ I_2 & B_j & & \\ & \ddots & \ddots & \\ & & I_2 & B_j \end{bmatrix}$, with $B_j = \begin{bmatrix} \bar{\beta}_j & 0 \\ 0 & \beta_j \end{bmatrix}.$

Now, let $Q \in \mathbb{R}^{2m_j \times 2m_j}$ be the permutation matrix

$$Q = \begin{bmatrix} e_1 & e_3 & \cdots & e_{2m_j-1} & e_2 & e_4 & \cdots & e_{2m_j} \end{bmatrix}$$

where e_i is the i-th column of I_{2m_j}. Then $Q^T F_{2m_j} Q = F_{2m_j}$,

$$Q^T \begin{bmatrix} B_j & & & \\ I_2 & B_j & & \\ & \ddots & \ddots & \\ & & I_2 & B_j \end{bmatrix} Q = \begin{bmatrix} J_{m_j}(\bar{\beta}_j) & 0 \\ 0 & J_{m_j}(\beta_j) \end{bmatrix},$$

and

$$\begin{array}{l}\left[\,u_{j1}-iv_{j1}\;u_{j1}+iv_{j1}\;\cdots\;u_{jm_j}-iv_{jm_j}\;u_{jm_j}+iv_{jm_j}\,\right]Q=\\ \left[\,u_{j1}-iv_{j1}\;\cdots\;u_{jm_j}-iv_{jm_j}\;\cdots\;u_{j1}+iv_{j1}\;\cdots u_{jm_j}+iv_{jm_j}\,\right]=\\ \left[\,U_j-iV_j\;U_j+iV_j\,\right],\end{array}$$

where $U_j=\left[\,u_{j1}\;\cdots\;u_{jm_j}\,\right]$ and $V_j=\left[\,v_{j1}\;\cdots\;v_{jm_j}\,\right]$.

Therefore if $W_j=\widetilde{W}_jQ$ then W_j is a unitary matrix and, with this matrix, conditions (48)-(50) are satisfied. $\square$

The above theorem is, of course, a result on the *existence* of desirable selfadjoint Jordan triples of real symmetric matrix polynomials rather than a constructive procedure to obtain them. In order to obtain Jordan chains of real or complex matrix polynomials one has to solve for each (real or non-real) eigenvalue λ_j a system of the form (see [6])

$$\begin{aligned}L(\lambda_j)x_{jl_j}&=0,\\ L(\lambda_j)x_{jl_j-1}+L^{(1)}(\lambda_j)x_{jl_j}&=0,\\ &\vdots\\ L(\lambda_j)x_{j1}+L^{(1)}(\lambda_j)x_{j2}+\cdots+\tfrac{1}{(l_j-1)!}L^{(l_j-1)}(\lambda_j)x_{jl_j}&=0,\end{aligned}$$

where $L^{(k)}(\lambda_j)$ is the kth derivative of $L(\lambda)$ at λ_j. Then x_{jl_j}, $x_{j,l_j-1},\ldots,x_{j1}$ is a real or non-real Jordan chain of $L(\lambda)$ associated with the real or non-real eigenvalue λ_j, respectively. Now, for general (not necessarily symmetric) real matrix polynomials, if $x_{j,l_j-1},\ldots,x_{j1}$ is a Jordan chain for a non-real eigenvalue λ_j then $\bar{x}_{j,l_j-1},\ldots,\bar{x}_{j1}$ is a Jordan chain for $\bar{\lambda}_j$. Theorem 10 says that the same is still true for selfadjoint Jordan chains of real symmetric matrix polynomials with nonsingular leading coefficients.

On the other hand, although real Jordan chains for a pair of non-real conjugate eigenvalues can be obtained by solving a similar but more complicated system (see [12]), this should be seen as a result of a rather theoretical nature. A more practical procedure is to obtain first a selfadjoint Jordan triple of $L(\lambda)$ with X as in (47) and reverse the procedure developed in the proof of the above theorem to obtain the corresponding real selfadjoint Jordan triple. This is the approach taken in Section 3.

B. The orthogonal matrix

Recall that $\mathbb{F}$ denotes either the real or complex number field.

LEMMA 11. *Assume that $n\geqslant p\geqslant m$. Then matrices $A_1\in\mathbb{F}^{m\times n}$ and $A_2\in\mathbb{F}^{m\times p}$ satisfy the equation $A_1A_1^*=A_2A_2^*$ if and only if $A_1=A_2\Theta$ for some matrix $\Theta\in\mathbb{F}^{p\times n}$ with orthonormal rows. In particular, if $n=p$ then Θ is a unitary matrix.*
(Note that, when $\mathbb{F}=\mathbb{R}$, Θ is, in fact, a real matrix.)

Proof. If $A_1=A_2\Theta$ and the rows of Θ form a system of orthonormal vectors then, obviously, $A_1A_1^*=A_2A_2^*$. Conversely, consider the polar decompositions, $A_1=H_1U_1$, $A_2=H_2U_2$ where $H_1,H_2\in\mathbb{F}^{m\times m}$, $U_1\in\mathbb{F}^{m\times n}$ and $U_2\in\mathbb{F}^{m\times p}$, H_1 and H_2 are positive semi-definite Hermitian or symmetric matrices, and U_1, U_2 are matrices with orthonormal rows (see Theorem 3.1.9 of [8], for example). Then $A_1A_1^* = A_2A_2^*$ implies $H_1^2 = H_2^2$ and, since the (semi-definite) square root is unique, $H_2 = H_1 =: H \geqslant 0$. Now $A_1 = HU_1$ and $A_2 = HU_2$. If $n \geqslant p > m$ then we can write $A_1 = \begin{bmatrix} H & 0 \end{bmatrix} V_1$ and $A_2 = \begin{bmatrix} H & 0 \end{bmatrix} V_2$ with $\begin{bmatrix} H & 0 \end{bmatrix} \in \mathbb{F}^{m\times p}$, $V_2 \in \mathbb{F}^{p\times p}$ unitary and $V_1 \in \mathbb{F}^{n\times p}$ with orthonormal rows. Then $A_1 = A_2V_2^*V_1$. If we put $\Theta = V_2^*V_1$ it follows that Θ has orthonormal rows and $A_1 = A_2\Theta$. If $p = n$ then V_1 is unitary and so is Θ. $\square$

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