Operators and Matrices
Volume 7, Number 2 (2013), 431–439
doi:10.7153/oam-07-26

AUTOMORPHISMS OF STRUCTURAL MATRIX ALGEBRAS

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(*Communicated by Richard A. Brualdi*)

Abstract. The topic considered in this paper falls under the general heading of automorphisms of structural matrix algebras. Herein we wish to give an answer to an open question given in [4]. We also would like to reprove of Theorem A and Theorem C in [6], a version of the principal results, by using the structure of the algebra in the block upper triangular case.

1. Introduction

Automorphisms of certain subalgebras of matrix algebras have been studied in several papers. In 1980, Isaacs [8] showed that the automorphisms of an $n \times n$ matrix algebra over a commutative ring can fail to be inner. The extent of this failure, however, is under control. For instance, the commutator of any two automorphisms and the n^{th} power of each of them are necessarily inner. In 1987, Barker and Kezlan [3] proved that every R-automorphism of the algebra of upper triangular matrices with the entries from an integral domain is inner. In the same year, Jondrup [9] showed that if a finite dimensional algebra $\mathscr{A}$ over its center K is simple, then all K-automorphisms of the algebra of upper triangular matrices over the algebra $\mathscr{A}$ are inner.

In 1989, Barker continued his work [4] on automorphism groups of the algebra of upper triangular matrices and he considered an algebra $\mathscr{A}$ of $n \times n$ matrices over an integral domain. He associated with $\mathscr{A}$ a graph whose edges are the pairs (i,j) such that the (i,j) entry of every element of $\mathscr{A}$ is zero. The graph in turn defines a group of permutations and automorphisms of $\mathscr{A}$ which are conjugations by permutation matrices. He showed that, for a suitably restricted class of algebras $\mathscr{A}$, the automorphism group of $\mathscr{A}$ is the semidirect product of this group of permutation matrices with the subgroup of inner automorphism. Following Barker's work, in 1993 Coelho [6], using graph theory, characterized the group of K-automorphisms of certain subalgebras of matrix algebra over the field K, which is known as the structural matrix algebra. These include the algebra of upper triangular matrices. Coelho also gave necessary and sufficient conditions for every K-automorphism of a subalgebra to be inner.

At the end of Barker's paper [4], he left an interesting open question which lead us to consider this problem as a starting point.

Open Question of Barker: Let $\mathscr{A}$ be an algebra of matrices over a field. Using Jordan-Holder theorem, or (if $\mathscr{A}$ contains a matrix with distinct diagonal entries) using

Mathematics subject classification (2010): 15A30, 16S50, 08A35.
Keywords and phrases: structural matrix algebras, block triangular matrices, automorphisms.

Paper OaM-07-26

the results of Laffey [10], we may choose a basis so that $\mathscr{A}$ is in block triangular form. To what extent can Theorem 2 of [4] be generalized to the block case and combined with knowledge of the automorphism groups of the diagonal blocks, which blocks are isomorphic to full matrix algebras, to obtain structure of $Aut(\mathscr{A})$?

Let $M_n(F)$ be the algebra of all $n\times n$ matrices over a field F and let $(\{1,\ldots,n\},\rho)$ be a quasi-ordered set (i.e. ρ is reflexive and transitive relation on the set $\{1,\ldots,n\}$). The set

$$M_n(F,\rho)=\{A\in M_n(F): a_{ij}=0 \text{ whenever } (i,j)\notin\rho\}$$

is a subalgebra of $M_n(F)$ (see [2]) and we call $M_n(F,\rho)$ the algebra of $n\times n$ structural matrices over F (with identity I).

Let $M_n(F,\rho)$ be a structural matrix algebra for the quasi-order ρ where F is a field. We wish to study the group $\mathrm{Aut}(M_n(F,\rho))$ of automorphism of $M_n(F,\rho)$. For simplicity, we write M_n when the order ρ and the field F are clear. The group of inner automorphism is a normal subgroup of $\mathrm{Aut}(M_n)$, but in general it is a proper subgroup. For some special cases such as upper triangular matrices (see [4, 5]) we know that an automorphism is a composition of an inner automorphism with a permutation similarity. Coelho [6] shows that any automorphism is a composition of an inner automorphism, a permutation similarity, and an automorphism generated by a transitive function g on ρ (definition will be given shortly). Since a structural matrix algebra is isomorphic with a block upper triangular matrix algebra which we shall describe below and this similarity is a conjugation by a permutation matrix, a block triangular algebra is also a structural algebra. We would like to reprove a version of the Coelho's principal results by using the structure of the algebra in the block upper triangular case. We begin by describing the block triangular form.

2. Block triangular form of structural matrix algebras and automorphisms

For a structural matrix algebra $M_n(F,\rho)$ we define an equivalence relation $\overline{\rho}$ by

$$(i,j)\in\overline{\rho} \text{ if and only if } (i,j),(j,i)\in\rho.$$

Let $[r_1],[r_2],\ldots,[r_p]$ denote the distinct equivalence classes of $\overline{\rho}$ with representatives $r_1,r_2,\ldots,r_p$. Construct a permutation π as follows. Note that π is not unique would possibly be in order. For

$$[r_1]=\{r_{11},r_{12},\ldots,r_{1m_1}\}$$

let

$$\pi(1)=\pi(r_{11})=1,\pi(r_{12})=2,\ldots,\pi(r_{1m_1})=m_1$$

and in general if

$$[r_k]=\left\{r_{k1},r_{k2},\ldots,r_{km_k}\right\}$$

then

$$\pi\left(r_{kj}\right)=m_1+m_2+\cdots+m_{k-1}+j.$$

If we apply this permutation similarity to $M_n(F,\rho)$ we have the following relation ρ' where

$$(i,j)\in\rho' \iff \left(\pi^{-1}(i),\pi^{-1}(j)\right)\in\rho$$

and $M_n(F,\rho')$ consists of all matrices of the form

$$\begin{bmatrix} M_{m_1}(F) & M_{m_1\times m_2}(R) & \cdots & M_{m_1\times m_p}(R) \\ 0 & M_{m_2}(F) & \cdots & M_{m_2\times m_p}(R) \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & M_{m_p}(F) \end{bmatrix},$$

where R is either F or 0.

Now, we define

$$[i]_{\rho'} \leqslant [j]_{\rho'} \tag{1}$$

to mean $(i,j)\in\rho'$. Then $\forall p_1\in[i], p_2\in[j]$,

$$(p_1,p_2)\in\rho' \text{ if and only if } (i,j)\in\rho'.$$

Let $[t_1],[t_2],\ldots,[t_q]$ be the classes which are incomparable with any other class, that is if $r_j\notin[t_k]$ then neither $(r_j,t_k)\in\rho'$ nor $(t_k,r_j)\in\rho'$. We now relable indices (a permutation similarity of $M_n(F,\rho)$) so that the classes comparable to another class are $[r_1],[r_2],\ldots,[r_l]$ so $l+q=p$, and

(i) each r_j and each t_k is minimal in its class,

(ii) $r_1<\ldots<r_l$ and $t_1<\ldots<t_q$,

(iii) for each class $[r_j]=\{r_{j1},r_{j2},\ldots,r_{jm_j}\}$ and $[t_k]=\{t_{k1},t_{k2},\ldots,t_{km_k}\}$

we have $r_j=r_{j1}<r_{j2}<\ldots<r_{jm_j}$ and $t_k=t_{k1}<t_{k2}<\ldots<t_{km_k}$. Note: $r_{js}+1=r_{js+1}$ etc.

If ρ' is the quasi-order corresponding to the this relabeling, then $M_n(F,\rho')$ consists of all matrices of the form

$$\begin{bmatrix} M_{m_1}(F) & M_{m_1\text{x}m_2}(R) & \cdots & M_{m_1\text{x}m_l}(R) & 0 & \cdots & 0 \\ 0 & M_{m_2}(F) & \cdots & M_{m_2\text{x}m_l}(R) & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & M_{m_l}(F) & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & M_{m_{l+1}}(F) & & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & M_{m_p}(F) \end{bmatrix} \tag{2}$$

where R is 0 or F. Specifically if we consider a block $M_{m_a\text{x}m_b}(R)$ we know that the row indices are a ρ' class and likewise for the column indices. If these classes are $[r_a]$ and $[r_b]$ respectively with $a<b\leqslant l$, then if $[r_a]\leqslant[r_b]$ we have $R=F$, otherwise $R=0$.

From this point on, we shall assume that the structural matrix algebra is in the form of (2). For notational simplicity, we may write M_i and M_{ij} for M_{m_i} and $M_{m_i\times m_j}$ respectively. The subset of $M_n(F,\rho)$ which has elements of M_{ij} (or M_i) in the i-th block row and j-th block column (or i-th block diagonal) and zero elsewhere will be denoted by $\overline{M}_{ij}$. Analogously we let A_{ij} denote an $m_i \times m_j$ matrix in M_{ij} while $\overline{A}_{ij}$ is the corresponding element of $\overline{M}_{ij}$. If $i = j$ we write M_i, A_i, $\overline{M}_i$, $\overline{A}_i$, for the corresponding sets and matrices.

We have two special subsets of $M_n(F,\rho)$. Let $\mathscr{D}$ denote the subset of all block diagonal elements. That is

$$\mathscr{D} = \left\{ diag\left(A_1, \dots, A_p\right) : A_j \in M_j \right\}.$$

Next, let J be the set of block strictly upper triangular matrices in $M_n(F,\rho)$. Any $A \in J$ is properly nilpotent (see Farenick, [7, p. 120]), and conversely we can show that any properly nilpotent element in $M_n(F,\rho)$ is actually an element of J. But the radical consists of all properly nilpotent elements, whence J is a radical of $M_n(F,\rho)$. If Φ is any automorphism of $M_n(F,\rho)$ and $T \in J$, then $\Phi(T)$ is properly nilpotent. Thus $\Phi(J) \subseteq J$. Since Φ is one to one, it is necessarily onto so that $\Phi(J) = J$.

Next, note that $M_n(F,\rho) = \mathscr{D}$ when and only when the radical J is zero, that is when and only when $M_n(F,\rho)$ is semisimple. However, $M_n(F,\rho) = \mathscr{D}$ if and only if ρ is symmetric. Thus ρ is symmetric if and only if $M_n(F,\rho)$ is semisimple.

First, let us consider the case $M_n(F,\rho)$ is semisimple and $\Phi \in \mathrm{Aut}(M_n)$.

THEOREM 2.1. *If $M_n(F,\rho)$ is semisimple and if $\Phi \in Aut(M_n)$, then we can write*

$$\Phi = \Psi \circ P_\tau,$$

where Ψ is an inner automorphism, and P_τ is a permutation similarity which is in $Aut(M_n)$.

Proof. Note that

$$M_n(F,\rho) = \overline{M}_1 \oplus \cdots \oplus \overline{M}_p$$

and each $\overline{M}_j$ is an ideal. Thus $\Phi\left(\overline{M}_j\right) \cap \overline{M}_t$ is an ideal in $\overline{M}_t$ which is simple. If the intersection is not $\{0\}$, then $\Phi\left(\overline{M}_j\right) \supseteq \overline{M}_t$. By considering $\Phi^{-1}\left(\overline{M}_t\right) \cap \overline{M}_j$ we conclude that the equality holds. This is possible if and only if $m_j = m_t$. This gives a bijection f on $\{1, \dots, p\}$ where

$$\Phi\left(\overline{M}_i\right) = \overline{M}_{f(i)}.$$

Now, extend f to a permutation π on $\{1, \dots, n\}$ by the way of (ii) above. If $\overline{M}_i$ corresponds to $[r_i]$ and $\overline{M}_{f(i)}$ corresponds to $\left[r_{f(i)}\right]$, then we set

$$\pi\left(r_{js}\right) = r_{\pi(j)s} \tag{3}$$

and similarly for $[t_k]$. If P_π is the corresponding permutation matrix then

$$\pi(A) = P_{\pi^{-1}} A P_\pi$$

is an automorphism of $M_n(F,\rho)$ while $\Psi = \Phi \circ \pi^{-1}$ is an automorphism of $M_n(F,\rho)$ for which

$$\Psi\left(\overline{M}_j\right) = \overline{M}_j, \;\; j = 1, \dots, p.$$

This induces an automorphism ψ_j of M_j such that

$$\Psi(M_n(F,\rho)) = \operatorname{diag}(\psi_1(M_1),\ldots\psi_p(M_p)).$$

At this point, one could appeal to the Skolem-Noether theorem. However, there is an elementary proof in [1, p. 90], that any automorphism of the full algebra of matrices over a field is inner. Hence there are nonsingular matrices $A_j \in M_j$ such that for every $B \in M_n(F,\rho)$, $B = \operatorname{diag}(B_1,\ldots,B_p)$, we have

$$\Psi(B) = \operatorname{diag}\left(A_1^{-1}B_1A_1,\ldots,A_p^{-1}B_pA_p\right)$$

so Ψ is inner. □

We can now utilize the lemma in [4, p. 210] or the argument in [6] to obtain the factorization in [6, theorem A]. Compare this also with [4, theorem 2].

As a final note, suppose that $M_n(F,\rho)$ is simple. It is therefore semisimple whence it is block diagonal. If there were more than one block, the algebra could have a non-trivial ideal contrary to the hypothesis of simplicity. Thus, the algebra is the full matrix algebra over F.

We now consider the general case of a structural matrix algebra $M_n = M_n(F,\rho)$ which we take to be in block upper triangular form of (2) and an automorphism $\Phi \in \operatorname{Aut}(M_n)$. We shall see in the course of the factorization theorem that a special type of function arises which generates a special type of automorphism. For the next definition we follow Coelho [6]. Let $F^* = F \setminus \{0\}$.

DEFINITION 1. A function $g : \rho \to F^*$ is transitive if and only if

$$g(i,j)\,g(j,k) = g(i,k)$$

for all $(i,j),(j,k) \in \rho$.

Every transitive function $g : \rho \to F^*$ determines an automorphism $G \in Aut(M_n(F,\rho))$ by defining

$$G\left(E^{ij}\right) = g(i,j)\,E^{ij}, \quad (i,j) \in \rho,$$

where E^{ij} is the $n \times n$ matrix with a 1 in position (i,j) and zeros elsewhere, if we consider $i = j$ then the matrix E^{ii} is written simple E^i.

THEOREM 2.2. (Factorization Theorem) *If $\Phi \in Aut(M_n)$ then we can write*

$$\Phi = \Psi_A \circ G \circ P_\tau,$$

where Ψ_A is an inner automorphism induced by A, G is an automorphism defined by transitive function g on ρ and P_τ is a permutation similarity which is in $Aut(M_n)$.

In the course of the proof of the Factorization Theorem we shall need some lemmas and corollaries from Coelho.

LEMMA 2.3. *Let $E = E^i$, $\eta \in J$, and let $\mathscr{E} = E + \eta$ be idempotent of $M_n(F,\rho)$. Then for all $i \geqslant 3$ we have that*

$$\mathscr{E} = E + \left(\eta + \eta^2 + \cdots + \eta^{i-1}\right) E + E\left(\eta + \eta^2 + \cdots + \eta^{i-1}\right) + \sum_{k=1}^{i-2} \left(\eta + \eta^2 + \cdots + \eta^{i-1-k}\right) E\eta^k + \eta^i.$$

COROLLARY 2.4. *With the notation of Lemma 2.3, we have that:*

(i) *if the index of the nilpotency of η is 2, then $\mathscr{E} = E + \eta E + E\eta$;*

(ii) *if the index of the nilpotency of η is $s > 2$, then*

$$\mathscr{E} = E + \left(\eta + \eta^2 + \cdots + \eta^{s-1}\right) E + E\left(\eta + \eta^2 + \cdots + \eta^{s-1}\right) + \sum_{k=1}^{s-2} \left(\eta + \eta^2 + \cdots + \eta^{s-1-k}\right) E\eta^k.$$

COROLLARY 2.5. *Let $\mathscr{E}$ be an idempotent of $M_n(F,\rho)$ under the conditions of Lemma 2.3. Then there exists $\theta \in J$ such that*

$$\mathscr{E} = E + E\theta + \theta E + \theta E\theta.$$

Conversely, if $\theta \in J$ and $E = E^j$ $(j \in \{1,2,\ldots,n\})$, then

$$\mathscr{E} = E + E\theta + \theta E + \theta E\theta$$

is an idempotent of $M_n(F,\rho)$.

LEMMA 2.6. *Let $\theta \in J$, $E = E^j$ $(j \in \{1,2,\ldots,n\})$, and let $\mathscr{E}$ be idempotent*

$$\mathscr{E} = E + E\theta + \theta E + \theta E\theta.$$

Then

$$\mathscr{U} = (I_n + E\theta)(I_n - \theta E) \in M_n(F,\rho)$$

is invertible, and

$$\mathscr{U}\mathscr{E}\mathscr{U}^{-1} = E.$$

Now, for each $j \in \{1,2,\ldots,n\}$, pick $\theta_j \in J$ and consider

$$\mathscr{E}_j = E^j + E^j\theta_j + \theta_j E^j + \theta_j E^j \theta_j,$$
$$\mathscr{U}_j = \left(I_n + E^j\theta_j\right)\left(I_n - \theta_j E^j\right).$$

Let A be an invertible matrix in $M_n(F,\rho)$, we denote by C_A the inner automorphism of $M_n(F,\rho)$ induced by A.

LEMMA 2.7. *With the notation above, we have*

(i) $\overline{C_{\mathscr{U}_j}} = 1$ *for all* $(j \in \{1,2,\ldots,n\})$, *where* $\overline{C_{\mathscr{U}_j}}$ *is the inner automorphism of* $M_n(F,\rho)/J$ *induced by the invertible matrix* $\mathscr{U}_j$.

(ii) *If* $E^i\mathscr{E}_j = \mathscr{E}_jE^i = 0$, *where* $i,j \in \{1,2,\ldots,n\}$, $i \neq j$ *then* $E^i\mathscr{U}_j = \mathscr{U}_jE^i = E^i$.

LEMMA 2.8. *Let* φ *be an automorphism of* $M_n(F,\rho)$ *such that there exits a permutation* σ *of* $\{1,2,\ldots,n\}$ *satisfying*

$$\overline{\varphi}\left(E^j + J\right) = E^{\sigma(j)} + J \quad \text{for all} \quad j \in \{1,2,\ldots,n\}.$$

Then there exists an invertible element $\mathscr{U}$ *of* $M_n(F,\rho)$ *such that* $\overline{C_{\mathscr{U}}} = 1$ *and*

$$(C_{\mathscr{U}} \circ \varphi)\left(E^j\right) = E^{\sigma(j)} \quad \text{for all} \quad j \in \{1,2,\ldots,n\}.$$

For the proofs of above lemmas and corrolaries see [6].

Proof of Factorization Theorem. Let Φ be an automorphism of M_n. The automorphism Φ determines an automorphism $\overline{\Phi}$ on the equivalence classes $\left[\overline{A}\right] = \overline{A} + J$ by

$$\overline{\Phi}\left(\left[\overline{A}\right]\right) = \left[\Phi\left(\overline{A}\right)\right] = \Phi\left(\overline{A}\right) + J.$$

Since J is the radical, M_n/J is semisimple. Recall that $\Phi(J) = J$. Note also that each equivalence class $\left[\overline{A}\right]$ contains a unique element of $\mathscr{D}$. If $\left[\overline{A}\right] \in M_n/J$ with $\overline{A} \in \mathscr{D}$ then the map

$$\varphi : M_n/J \to \mathscr{D}$$

defined by $\varphi\left(\left[\overline{A}\right]\right) = \overline{A}$ is an isomorphism and

$$\overline{\varphi}\left(\overline{A}\right) = \left(\varphi \circ \overline{\Phi}\right)\left(\left[\overline{A}\right]\right)$$

is an automorphism of $\mathscr{D}$. If $f = \varphi^{-1}$ then

$$\overline{\Phi}\left(\left[\overline{A}\right]\right) = (f \circ \overline{\varphi})\left(\overline{A}\right).$$

But $\overline{\varphi}$ factors as

$$\overline{\varphi} = \Psi_{\overline{D}} \circ P_\tau,$$

where $\overline{D} \in \mathscr{D}$, $\Psi_{\left(\overline{D}\right)}$ is the inner automorphism determined by $\overline{D}$ and P_τ is a permutation similarity of $\mathscr{D}$ which permutes diagonal blocks. Observe that

$$f \circ \Psi_{\overline{D}} = \Psi_{\left[\overline{D}\right]},$$

that is

$$f \circ \Psi_{\overline{D}} = \Psi_{f\left(\overline{D}\right)},$$

where $\Psi_{\left[\overline{D}\right]}$ is the inner automorphism of M_n/J determined by $\left[\overline{D}\right]$. Note that τ must take an equivalence class of ρ to an equivalence class of ρ so that $(\tau(i), \tau(j)) \in \overline{\rho}$ for every $(i,j) \in \overline{\rho}$. Let

$$P = f \circ P_\tau.$$

Then

$$P\left(E^{ij}+J\right)=E^{\tau(i)\tau(j)}+J$$

for all $(i,j)\in\overline{\rho}$.

Recall that E^{ij} is the element of $M_n(F,\rho)$ with a 1 in position (i,j) and zero elsewhere for $(i,j)\in\overline{\rho}$. Recall also the construction of τ in the semisimple case. On each equivalence class by (3) if $(a,b)\in\overline{\rho}$ and $a<b$, then $\tau(a)<\tau(b)$. Thus

$$\overline{\Phi}=\Psi_{\left[\overline{D}\right]}\circ P.$$

Next, we have

$$\left[\Psi_{\overline{D}^{-1}}\circ\Phi\right]=\Psi_{\left[\overline{D}^{-1}\right]}\circ\overline{\Phi}=\Psi_{\left[\overline{D}^{-1}\right]}\circ\Psi_{\left[\overline{D}\right]}\circ P=P.$$

Consequently

$$\left(\Psi_{\left[\overline{D}^{-1}\right]}\circ\overline{\Phi}\right)\left(E^{j}+J\right)=P\left(E^{j}+J\right)=E^{\tau(j)}+J,\ \ j=1,2,\ldots,n$$

so $\Psi_{\overline{D}^{-1}}\circ\Phi=\Theta$ satisfies the conditions of lemma 2.8. Hence there exist an invertible matrix $U\in M_n(F,\rho)$ such that

$$\left(\Psi_U\circ\Theta\right)\left(E^{j}\right)=E^{\tau(j)}\text{ for }j=1,2,\ldots,n.$$

Then we have $\Psi_{[U]}=1$. Take

$$\Gamma=\Psi_U\circ\Theta=\Psi_{U\overline{D}^{-1}}\circ\Phi.$$

So $\Gamma\left(E^{ij}\right)=\Gamma\left(E^{i}\right)\Gamma\left(E^{ij}\right)\Gamma\left(E^{j}\right)=E^{\tau(i)}\Gamma\left(E^{ij}\right)E^{\tau(j)}=c_{\tau(i)\tau(j)}E^{\tau(i)\tau(j)}$, where $c_{\tau(i)\tau(j)}\in F^*$. Then $(\tau(i),\tau(j))\in\rho$ for all $(i,j)\in\rho$. Hence, this gives us an automorphism of ρ which we also denote by τ.

Let $g:\rho\to F^*$ be a function defined by

$$g(\tau(i),\tau(j))=c_{\tau(i)\tau(j)}$$

since Γ is an automorphism, it follows that g is transitive. Now define G by

$$G\left(E^{ij}\right)=g(i,j)E^{ij}\ ,\ \ (i,j)\in\rho.$$

We extend G by linearity to obtain an automorphism of $M_n(F,\rho)$.

$$\Gamma\left(E^{ij}\right)=G\left(E^{\tau(i)\tau(j)}\right)=\left(G\circ P_\tau\right)\left(E^{ij}\right)\Longrightarrow\Gamma=G\circ P_\tau$$

so $\Psi_{U\overline{D}^{-1}}\circ\Phi=G\circ P_\tau\Longrightarrow\Phi=\Psi_{\overline{D}\,U^{-1}}\circ G\circ P_\tau$ as we desired. $\square$

The heart of the matter is the proof of the factorization theorem (theorem C of Coelho [6] and Theorem 2.2 of this paper). Both approaches start by dealing with the semisimple case. These approaches are via graph theory in Coelho and via the block diagonal form in this paper. Then the question is how to lift the factorization in the semisimple case to the general case. The resolution involves neither graph theory nor the block triangular form, so from this point forward there is no further contrast between the two approaches. What is done is to find a suitable inner automorphism (see Lemma 2.8) so that the resulting composition takes each of the unit matrices E^{ij} onto a multiple of another unit matrix. This yields that the composition is an automorphism determined by a transitive function, and thus the proof is complete. This last part which is certainly clever is independent of the approach used upto this point.

The purpose of the second approach is to give an alternative and more intuitive proof to the theorems. The proof via graph theory is also interesting and relates to some important work in linear algebra currently being done. Both have value for understanding the factorization theorems.

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(Received August 10, 2012)

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