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OPERATOR–VALUED FRAME GENERATORS FOR GROUP–LIKE UNITARY SYSTEMS

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Abstract. We investigate operator-valued frames with the structure of group-like unitary systems. We show that the commutant of a group-like unitary system can be characterized in terms of the analysis operators associated with all the operator-valued Bessel generators and give some sufficient and necessary conditions to describe when an operator-valued frame generator admits a Parseval dual. This extends work of J.Gabardo and D.Han but the proofs turn out to be more complicated. Then we characterize the Parseval operator-valued frame generators for certain unitary systems on a finite dimensional Hilbert space.

1. Introduction

Frames in a Hilbert space have been used to capture significant signal characteristics, provide numerical stability of reconstruction, and enhance resilience to additive noise etc. Motivated by these applications the theory has developed rapidly in the past decade. Important examples of frames are Gabor frames and wavelet frames [7]. Recently, many generalized versions of frames have appeared, e.g. g-frames [21], modular frames [8], fusion frames [5, 6] and operator-valued frames [12, 17]. Among these, operator-valued frames can be used in quantum communication and packets encoding theory [1]. So operator-valued frame theory becomes attractive. In fact, designing various quantum channels is an essential issue in quantum communication theory and we found that this issue is equivalent to the design of operator-valued frames [1, 12].

Frames with special structures are very important since most of the useful frames in theory and in applications are of this kind, including Gabor frames and wavelet frames [10, 13]. Motivated by Gabor analysis, one often considers group-like unitary systems or projective unitary representation for a countable group. On the other hand, operator-valued frames are the dual maps of quantum channels. In applications one would require the quantum channels to have some additional structures (e.g. parameterized quantum channels). So it is natural to consider operator-valued frames with the structure of group-like unitary systems. Some results about structured operator-valued frames have appeared in [12, 17], such as the Dilation Theorem, dual frame generators and orthogonality, etc. However in the present paper, we give a more systematic investigation and obtain more results which generalize their counterpart in classical frame

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theory [10, 13, 15]. Although most of the results of the vector-valued case can be generalized to the operator-valued case, there are some essential differences between them. For examples, the local commutant of an operator-valued orthonormal generator is in general not the commutant of the corresponding unitary system. The commutant of a unitary system which admits an operator-valued frame generator may not be a finite von Neumann algebra.

Finite frames, that is, frames in finite dimensional Hilbert spaces, play a fundamental role in a variety of important areas including multiple antenna coding, perfect reconstruction filter banks, etc.. In this paper we will consider structured finite operator-valued frames. The main results are the characterization of Parseval operator-valued frame generators for a unitary system generated by one or more unitary operators.

The paper is organized as follows:

In section 2 we revisit operator-valued frames and obtain some results about Parseval dual frames and the excess of an operator-valued frame.

In section 3 we study the operator-valued frame generators for a group-like unitary system and describe the set of all Bessel generators.

In section 4 we study when a frame generator has a Parseval dual under the same unitary system.

In section 5 we study operator-valued frame generators for unitary systems on a finite dimensional Hilbert space. We show that the Parseval operator-valued frame generators for a unitary system generated by one unitary operator are isometric up to a constant. Then we characterize them for a unitary system generated by two or more generators using projections.

2. (OPV)-frames

In this section we review some notions about operator-valued frames ((OPV)-frames) and point out when an (OPV)-frame admits a Parseval dual. This problem for the case of vector-valued frames has been considered in [13, 15] and here, although the proofs are similar, we need to deal with some details carefully. Throughout this paper the Hilbert spaces are at most countably dimensional and J is a finite or countable index set. We use $B(H,H_0)$ to denote the set of all bounded operators from H into H_0, where H,H_0 are Hilbert spaces.

DEFINITION 1. [17] Let H and H_0 be Hilbert spaces, and let $V_j \in B(H,H_0)$. If there exist positive constants a and b such that

$$aI \leqslant \sum_{j\in J} V_j^* V_j \leqslant bI,$$

then $\{V_j\}_{j\in J}$ is called an operator-valued frame ((OPV)-frame) for H with range in H_0. The optimal a,b are called the lower frame bound and upper frame bound respectively. $\{V_j\}_{j\in J}$ is called Parseval if $a=b=1$ and Bessel if we only require the right side inequality.

Throughout this paper, when we speak of an (OPV)-frame for a Hilbert space, we always suppose its range is in H_0.

In the study of frame theory, operator theoretic methods are the main tools. Analysis operators and frame operators are the most important operators in frame theory. Let $V_j \in B(H,H_0)\,(j \in J)$ such that $\{V_j\}_{j\in J}$ is a Bessel (OPV)-frame for H. The *analysis operator* θ_V for $\{V_j\}_{j\in J}$ is an operator from H to $l^2 \otimes H_0$ defined by

$$\theta_V(x) = \sum_{j\in J} e_j \otimes V_j(x), \forall x \in H$$

where $\{e_j\}_{j\in J}$ is the standard orthonormal basis for l^2. One can check

$$\theta_V^*(e_j \otimes h) = V_j^*(h), \forall j \in J, h \in H_0.$$

$S := \theta_V^*\theta_V = \sum_{j\in J} V_j^*V_j$ will be called the *frame operator* for $\{V_j\}_{j\in J}$. Obviously, when $\{V_j\}_{j\in J}$ is an (OPV)-frame, θ_V is bounded invertible (not necessarily onto). If $\{V_j\}_{j\in J}$ is Parseval, then θ_V is isometric, i.e. $\theta_V^*\theta_V = I$. We call two (OPV)-frames $\{V_j\}_{j\in J}$, $\{W_j\}_{j\in J}$ *similar* if there is an onto invertible operator $T \in B(H)$ such that $W_j = V_jT, \forall j \in J$. Letting θ_V, θ_W be their analysis operators respectively then $Range(\theta_V) = Range(\theta_W)$.

DEFINITION 2. [12, 17] Let $\{V_j\}_{j\in J}$ be an (OPV)-frame for H with $V_j \in B(H,H_0)$ $(j \in J)$. If $Range(\theta_V) = l^2 \otimes H_0$ then $\{V_j\}_{j\in J}$ will be called a Riesz (OPV)-frame. A Parseval Riesz (OPV)-frame will be called an orthonormal (OPV)-frame.

If $\{V_j\}_{j\in J}$ is an orthonormal (OPV)-frame, then θ_V is unitary. When $\{V_j\}_{j\in J}$, $\{W_j\}_{j\in J} \subseteq B(H,H_0)$ are both orthonormal (OPV)-frames, there exists a unitary operator U such that $W_j = V_jU, \forall j \in J$.

DEFINITION 3. [17] Let $\{V_j\}_{j\in J} \subseteq B(H,H_0)$, $\{W_j\}_{j\in J} \subseteq B(K,H_0)$ be two Bessel (OPV)-frames for H,K respectively. If $\theta_V^*\theta_W = 0$, we call $\{V_j\}_{j\in J}$ orthogonal to $\{W_j\}_{j\in J}$. If $\{V_j\}_{j\in J}$, $\{W_j\}_{j\in J} \subseteq B(H,H_0)$ are Bessel (OPV)-frames satisfying $\theta_V^*\theta_W = I$, we call $\{V_j\}_{j\in J}$ dual to $\{W_j\}_{j\in J}$.

Obviously, $\{V_jS^{-1}\}_{j\in J}$ is dual to $\{V_j\}_{j\in J}$ where S is the frame operator for $\{V_j\}_{j\in J}$. It is easy to observe that $\{W_j\}_{j\in J}$ is dual to $\{V_j\}_{j\in J}$ if and only if $W_j = V_jS^{-1} + U_j$, $j \in J$ for some Bessel (OPV)-frame $\{U_j\}_{j\in J}$ orthogonal to $\{V_j\}_{j\in J}$.

The following theorem is the Dilation Theorem for (OPV)-frames which has been given in [12, 17].

THEOREM 4. (Dilation Theorem) [12, 17] *Let $V_j \in B(H,H_0)$ such that $\{V_j\}_{j\in J}$ is a Parseval (OPV)-frame for H. Then there exist a Hilbert space $K \supseteq H$ and $W_j : K \to H_0$, $j \in J$ such that $\{W_j\}_{j\in J}$ is an orthonormal (OPV)-frame for K and $V_j = W_j|_H$, $j \in J$.*

In fact, in the above theorem, we can take $K = l^2 \otimes H_0$.

The following result is well known in the vector-valued frame case and for the (OPV)-case it has appeared in [17]. Here we give a quite direct proof.

PROPOSITION 5. *Let $\{V_j\}_{j\in J} \subseteq B(H,H_0)$ be an (OPV)-frame for H. Then $\{V_j\}_{j\in J}$ admits only one dual if and only if $\{V_j\}_{j\in J}$ is a Riesz (OPV)-frame.*

Proof. Let $\{V_j\}_{j\in J}$ be a Riesz (OPV)-frame. Let $W_j := V_jS^{-1} + T_j$, $j \in J$ be a dual for $\{V_j\}_{j\in J}$ where S is the frame operator for $\{V_j\}_{j\in J}$ and $\{T_j\}_{j\in J}$ is a Bessel (OPV)-frame orthogonal to $\{V_j\}_{j\in J}$. Since $\{V_j\}_{j\in J}$ is a Riesz (OPV)-frame, we know $\theta_V(H) = l^2 \otimes H_0$. By $\theta_T^*\theta_V = 0$, where θ_T is the analysis operator for $\{T_j\}_{j\in J}$, we get $\theta_T = 0$ and thus $T_j = 0, j \in J$.

For the converse, we first consider the case of $\{V_j\}_{j\in J}$ Parseval. Assuming $\{V_j\}_{j\in J}$ admits only one dual but that $\{V_j\}_{j\in J}$ is not a Riesz (OPV)-frame we deduce a contradiction. In fact, since $\{V_j\}_{j\in J}$ is not a Riesz (OPV)-frame, $Range(\theta_V)$ is a proper subspace of $l^2 \otimes H_0$. Then we have the orthogonal decomposition

$$l^2 \otimes H_0 = Range(\theta_V) \oplus M,$$

for a certain nonzero subspace M. By Theorem 4, we know there is an orthonormal (OPV)-frame $\{U_j\}_{j\in J}$ for $l^2 \otimes H_0$ such that $V_j = U_j|_P$, where P is the orthogonal projection from $l^2 \otimes H_0$ onto $Range(\theta_V)$. $\{U_jP^{\perp}\}_{j\in J}$ can be viewed as a Parseval (OPV)-frame for M. We show $\{V_j\}_{j\in J}$ is orthogonal to $\{U_jP^{\perp}\}$. In fact,

$$\begin{aligned}
&\sum_{j\in J} V_j^* U_j P^{\perp} \\
&= \sum_{j\in J} (U_jP)^* U_j P^{\perp} \\
&= \sum_{j\in J} P U_j^* U_j P^{\perp} \\
&= PP^{\perp} = 0.
\end{aligned}$$

Thus $\{V_j + U_jP^{\perp}\}_{j\in J}$ is dual to $\{V_j\}_{j\in J}$. Therefore $\{V_j\}_{j\in J}$ admits two dual frames $\{V_j\}_{j\in J}$ and $\{V_j + U_jP^{\perp}\}_{j\in J}$ which contradicts to the assumption.

Now suppose $\{V_j\}_{j\in J}$ is a general (OPV)-frame, let S be the frame operator for $\{V_j\}_{j\in J}$ and then $\{V_jS^{-\frac{1}{2}}\}_{j\in J}$ is Parseval. Assuming $\{T_j\}_{j\in J}$ is orthogonal to $\{V_jS^{-\frac{1}{2}}\}_{j\in J}$, then $\sum_{j\in J} T_j^* V_j S^{-\frac{1}{2}} = 0$. Since S is onto invertible, we get $\sum_{j\in J} T_j^* V_j = 0$. On the other hand, since $\{V_j\}_{j\in J}$ has only one dual, we know $T_j = 0, j \in J$. So $\{V_jS^{-\frac{1}{2}}\}_{j\in J}$ has only one dual and thus $\{V_jS^{-\frac{1}{2}}\}_{j\in J}$ is a Riesz (OPV)-frame. Hence $\{V_j\}_{j\in J}$ is a Riesz (OPV)-frame. $\square$

In the following, we study when an (OPV)-frame admits Parseval dual (OPV)-frames. The vector-valued frame case has been treated in [13, 15]. The proofs are slightly different here.

PROPOSITION 6. *Let $\{V_j\}_{j\in J}$ be an (OPV)-frame for H with range in H and suppose there exists a $\{W_j\}_{j\in J} \subseteq B(H)$ which is a Parseval dual (OPV)-frame for $\{V_j\}_{j\in J}$. Then the lower frame bound of $\{V_j\}_{j\in J}$ is greater than or equal 1.*

Proof. Obviously, since $\{V_j\}_{j\in J}$, $\{W_j\}_{j\in J}$ are (OPV)-frames, we can regard $\{V_j\}_{j\in J}$, $\{W_j\}_{j\in J}$ as vectors in $l^2\otimes B(H)$. We define an operator-valued inner product on $l^2\otimes B(H)$ by

$$\langle X,Y\rangle := \sum_{j\in J} X_j^* Y_j,$$

where $X=\{X_j\}_{j\in J}$, $Y=\{Y_j\}_{j\in J}\in l^2\otimes B(H)$. Then $\langle\,,\rangle$ is a $B(H)$-valued inner product and $l^2\otimes B(H)$ becomes a inner product module equipped with this inner product [20]. For such an operator-valued inner product, Cauchy-Schwarz inequality still holds [20]. So we have

$$\begin{aligned} I &= \sum_{j\in J} V_j W_j^* \leqslant (\sum_{j\in J} V_j^* V_j)^{\frac{1}{2}} (\sum_{j\in J} W_j^* W_j)^{\frac{1}{2}} \\ &= (\sum_{j\in J} V_j^* V_j)^{\frac{1}{2}}. \end{aligned}$$

Therefore, the lower frame bound of $\{V_j\}_{j\in J}$ is greater than or equal 1. $\square$

The following theorem is well known in vector-valued frame theory [11].

THEOREM 7. *Let $V_j\in B(H,H_0)$ be such that $\{V_j\}_{j\in J}$ is an (OPV)-frame with the frame operator $S>I$. Then $\{V_j\}_{j\in J}$ admits a Parseval dual if and only if $dim(\theta_V(H)^\perp)\geqslant dim H$ where θ_V is the analysis operator for $\{V_j\}_{j\in J}$.*

Proof. For the necessity, we refer to [12].

For the sufficiency, we suppose $\{V_j\}_{j\in J}$ admits a Parseval dual $\{W_j\}_{j\in J}$ and let $Z_j = V_j S^{-\frac{1}{2}}, j\in J$. Then $\{Z_j\}_{j\in J}$ is Parseval. By the Dilation Theorem, we know there is an orthonormal (OPV)-frame $\{U_j\}_{j\in J}$ for $H\oplus M$ such that $U_j|_H = Z_j, j\in J$ where $M=\theta_Z(H)^\perp$. It follows that $\{Z_j\oplus U_j|_M\}_{j\in J}$ is an orthonormal (OPV)-frame for $H\oplus M$.

Let $T: H\to H\oplus M$ defined by $T := \sum_{j\in J}(Z_j\oplus U_j|_M)^* W_j$. It is easy to see T is isometric since $\{W_j\}_{j\in J}$ is Parseval. On the other hand,

$$\begin{aligned} T &= \sum_{j\in J}(Z_j\oplus U_j|_M)^* W_j \\ &= \sum_{j\in J} Z_j^* W_j \oplus U_j|_M^* W_j \\ &= \sum_{j\in J} S^{-\frac{1}{2}} V_j^* W_j \oplus U_j|_M^* W_j \\ &= S^{-\frac{1}{2}} \oplus \sum_{j\in J} U_j|_M^* W_j, \end{aligned}$$

and for $x\in H, y\in M$,

$$\begin{aligned} T^*(x\oplus y) &= \sum_{j\in J} W_j^*(Z_j\oplus U_j|_M)(x\oplus y) \\ &= \sum_{j\in J} W_j^*[Z_j x + U_j|_M y] \end{aligned}$$

$$= \sum_{j\in J} W_j^* V_j S^{-\frac{1}{2}} x + \sum_{j\in J} W_j^* U_j|_M y$$
$$= S^{-\frac{1}{2}} x + \sum_{j\in J} W_j^* U_j|_M y.$$

Therefore we get

$$I = T^*T = S^{-1} + (\sum_{i\in J} W_i^* U_i|_M)(\sum_{j\in J} U_j|_M^* W_j).$$

We denote $\sum_{j\in J} U_j|_M^* W_j$ by D. Since $\|S^{-1}\| < 1$, we know D^*D is invertible and we infer $D : H \to M$ is injective. Hence $dimH \leqslant dimM$. $\square$

In the following, we call $dim(\theta_V(H)^\perp)$ the *excess* of $\{V_j\}_{j\in J}$.

PROPOSITION 8. *Let $V_j \in B(H,H_0)$ such that $\{V_j\}_{j\in J}$ is an (OPV)-frame for H and let θ_V be its analysis operator with $(\theta_V^*\theta_V)^{-1} \leqslant I$. If the excess of $\{V_j\}_{j\in J}$ is infinite, then $\{V_j\}_{j\in J}$ admits a Parseval dual.*

Proof. Since $dim(\theta_V(H)^\perp) = \infty \geqslant dimH$, there is a subspace $N \subseteq \theta_V(H)^\perp$ such that $dimN = dimH$. Let $W : N \to H$ be the unitary operator identifying N with H and let $P : l^2 \otimes H_0 \to N$ be the orthogonal projection onto N. By the Dilation Theorem, we know there is an orthonormal (OPV)-frame $\{U_j\}_{j\in J}$ for some Hilbert space $K \supseteq H$ such that $U_j|_H = V_j, j \in J$. It is easy to check that $\{U_j|_N W^{-1}\}_{j\in J}$ is a Parseval (OPV)-frame for H which is orthogonal to $\{V_j\}_{j\in J}$. Let $B := \sqrt{I - S^{-1}}$. Then $\{U_j|_N W^{-1}B\}$ is still orthogonal to $\{V_j\}_{j\in J}$ and so $\{V_jS^{-1} + U_j|_N W^{-1}B\}_{j\in J}$ is dual to $\{V_j\}_{j\in J}$. Simultaneously, $\{V_jS^{-1} + U_j|_N W^{-1}B\}_{j\in J}$ is Parseval since

$$\sum_{j\in J} (V_jS^{-1} + U_j|_N W^{-1}B)^*(V_jS^{-1} + U_j|_N W^{-1}B)$$
$$= \sum_{j\in J} S^{-1}V_j^*V_jS^{-1} + \sum_{j\in J} B^*WU_j|_N^* U_j|_N W^{-1}B$$
$$= S^{-1} + B^*B = I. \quad \square$$

3. group-like unitary systems

In this section, we consider (OPV)-frames with group-like structures. This is the theoretical foundations for structured quantum channels.

DEFINITION 9. [10] Let $\mathscr{U}$ be a countable set of unitary operators acting on a separable Hilbert space H containing the identity operator. Then $\mathscr{U}$ is called a unitary system. Letting $group(\mathscr{U})$ denote the group generated by $\mathscr{U}$ and $\mathbb{T} := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$, we call $\mathscr{U}$ a group-like unitary system if

$$group(\mathscr{U}) \subseteq \mathbb{T}\mathscr{U} := \{\lambda U : \lambda \in \mathbb{T}, U \in \mathscr{U}\}$$

and $\mathscr{U}$ is linearly independent in the sense that $\mathbb{T}U \neq \mathbb{T}V$ whenever U and V are different elements of $\mathscr{U}$.

If $\mathscr{U}$ is a group-like unitary system, then there exist a function $f : group(\mathscr{U}) \to \mathbb{T}$ and a mapping $\sigma : group(\mathscr{U}) \to \mathscr{U}$ such that $W = f(W)\sigma(W)$ for all $W \in group(\mathscr{U})$.

For a unitary system $\mathscr{U}$, $A \in B(H,H_0)$ is called a *frame generator* (respectively a *Parseval frame generator* or a *Bessel generator*) for $\mathscr{U}$ if $\{AU\}_{U\in\mathscr{U}}$ is an (OPV)-frame (respectively a Parseval (OPV)-frame or a Bessel (OPV)-frame) for H. If $\{AU\}_{U\in\mathscr{U}}$ is an orthonormal (OPV)-frame (respectively a Riesz (OPV)-frame) A will be called an orthonormal generator (respectively a Riesz generator).

PROPOSITION 10. [10] *Let $\mathscr{U}$, f and σ be as the above.*
(i) $f(U\sigma(VW))f(VW) = f(\sigma(UV)W)f(UV)$, $U,V,W \in group(\mathscr{U})$;
(ii) $\sigma(U\sigma(VW)) = \sigma(\sigma(UV)W), U,V,W \in group(\mathscr{U})$;
(iii) $\sigma(U) = U$ and $f(U) = 1, \forall U \in \mathscr{U}$;
(iv) $\forall V,W \in group(\mathscr{U})$, we have

$$\begin{aligned}\mathscr{U} &= \{\sigma(UV) : U \in \mathscr{U}\} = \{\sigma(VU^{-1}) : U \in \mathscr{U}\} \\ &= \{\sigma(VU^{-1}W) : U \in \mathscr{U}\} = \{\sigma(V^{-1}U) : U \in \mathscr{U}\}.\end{aligned}$$

A *unitary representation* π of a group-like unitary system $\mathscr{U}$ is a one-to-one mapping from $\mathscr{U}$ into the set of all unitary operators on some Hilbert space K such that

$$\pi(U)\pi(V) = f(UV)\pi(\sigma(UV)), \pi(U)^{-1} = f(U^{-1})\pi(\sigma(U^{-1})),$$

where f and σ are the corresponding mappings associated with $\mathscr{U}$. If $\pi(\mathscr{U})$ admits a frame generator, then π will be called an *(OPV)-frame representation*.

Note that $\pi(\mathscr{U}) := \{\pi(U) : U \in \mathscr{U}\}$ is also a group-like unitary system. Let e_U be the element in $l^2(\mathscr{U})$ which takes value 1 at U and zero everywhere else. Then $\{e_U : U \in \mathscr{U}\}$ is the standard orthonormal basis for $l^2(\mathscr{U})$. For each fixed $U \in \mathscr{U}$, we define $L_U, R_U \in B(l^2(\mathscr{U}))$ by $L_U e_V = f(UV)e_{\sigma(UV)}$ and $R_U e_V = f(VU^{-1})e_{\sigma(VU^{-1})}, \forall V \in \mathscr{U}$. We also introduce unitary representations $\widetilde{L}$ and $\widetilde{R}$ for $\mathscr{U}$ on $l^2(\mathscr{U}) \otimes H_0$ by $\widetilde{L_U} := L_U \otimes I_0$ and $\widetilde{R_V} := R_V \otimes I_0$, where I_0 is the identity on H_0. In the following, let $\widetilde{L_\mathscr{U}} := \{\widetilde{L_U} : U \in \mathscr{U}\}$ and $\widetilde{R_\mathscr{U}} := \{\widetilde{R_U} : U \in \mathscr{U}\}$.

The following proposition has appeared in [17]. However here we give a more explicit proof.

PROPOSITION 11. [17] *$\{\widetilde{L_U}\}_{U\in\mathscr{U}}$ and $\{\widetilde{R_V}\}_{V\in\mathscr{U}}$ admit orthonormal generators.*

Proof. For any $V \in \mathscr{U}$, let $P_V : l^2(\mathscr{U}) \otimes H_0 \to H_0$ defined by

$$P_V(e_W \otimes h) = \begin{cases} h, & V = W \\ 0 & V \neq W \end{cases}, \forall W \in \mathscr{U},$$

where $\{e_W\}_{W\in\mathscr{U}}$ is the standard orthonormal basis for $l^2(\mathscr{U})$. We show P_V is an orthonormal generator for both $\widetilde{L_\mathscr{U}}$ and $\widetilde{R_\mathscr{U}}$.

First, we show P_V is a Parseval frame generator for $\widetilde{L_{\mathscr{U}}}$. For any $W \in \mathscr{U}, h \in H_0$, we have

$$\begin{aligned}
&\sum_{U\in\mathscr{U}} (P_V\widetilde{L_U})^*(P_V\widetilde{L_U})(e_W\otimes h)\\
&=\sum_{U\in\mathscr{U}} \widetilde{L_U}^* P_V^* P_V \widetilde{L_U}(e_W\otimes h)\\
&=\widetilde{L_U}^* P_V^* f(UW)h \,(\text{where } V=\sigma(UW))\\
&=f(UW)\widetilde{L_U}^*(e_V\otimes h)\\
&=f(UW)f(U^{-1})\widetilde{L_{\sigma(U^{-1})}}(e_V\otimes h)\\
&=e_W\otimes h.
\end{aligned}$$

It is easy to check $(P_V\widetilde{L_U})(P_V\widetilde{L_W})^* = \delta_{UW}I_0$ and thus $\{P_V\widetilde{L_U}\}_{U\in\mathscr{U}}$ is an orthonormal (OPV)-frame.

For the case of $\widetilde{R_{\mathscr{U}}}$, we can prove it similarly. $\square$

For $A \in B(H)$, we denote the set $\{T \in B(H) : ATU = AUT, \forall U \in \mathscr{U}\}$ by $C_A(\mathscr{U})$. Obviously $\mathscr{U}' \subseteq C_A(\mathscr{U})$ where $\mathscr{U}'$ is the commutant of $\mathscr{U}$. In the following $(\cdot)'$ always denotes the commutant of a set in $B(H)$.

PROPOSITION 12. *Let $\mathscr{U}$ be a unitary system on H and let $A \in B(H)$ satisfy $Range(A^*) = H$. Then $C_A(\mathscr{U}) = \mathscr{U}'$.*

Proof. We only need to show $C_A(\mathscr{U}) \subseteq \mathscr{U}'$. For any $T \in C_A(\mathscr{U})$, we have $ATU = AUT, \forall U \in \mathscr{U}$, i.e.

$$U^*T^*A^* = T^*U^*A^*.$$

For all $W \in \mathscr{U}$ and $y \in H$, we get $A^*x = y$ for some $x \in H$ and

$$W^*T^*(y) = W^*T^*A^*(x) = T^*W^*A^*(x) = T^*W^*y.$$

Thus $W^*T^* = T^*W^*$, i.e. $TW = WT$, that is, $T \in \mathscr{U}'$. $\square$

Obviously, $C_A(\mathscr{U})$ is an analogue of the local commutant in vector-valued frame theory [11]. In that case, the local commutant at a wandering vector is just the commutant $\mathscr{U}'$. However, in the (OPV)-case, $C_A(\mathscr{U})$ is in general not equal to $\mathscr{U}'$, even if A is an orthonormal generator. On the other hand, supposing $\mathscr{U}$ admits an (OPV)-frame generator A, $\mathscr{U}'$ is in general not a finite von Neumann algebra unless A is of finite rank. These lead to more complicated proofs in the (OPV)-case than those in the vector-valued case.

PROPOSITION 13. *Let $\mathscr{U}$ be a group-like unitary system on H and let $W_1 \in B(H,H_0)$ be an orthonormal generator for $\mathscr{U}$. Then $W_2 \in B(H,H_0)$ is an orthonormal generator for $\mathscr{U}$ if and only if there is a unitary operator $T \in \mathscr{U}'$ such that $W_2 = W_1T$.*

Proof. Suppose $W_2 \in B(H,H_0)$ is an orthonormal generator for $\mathscr{U}$. Let $\theta_{W_1}, \theta_{W_2}$ be the analysis operators for $\{W_1U\}_{U\in\mathscr{U}}, \{W_2U\}_{U\in\mathscr{U}}$ respectively. Then

$$Range(\theta_{W_1}) = Range(\theta_{W_2}) = l^2(\mathscr{U}) \otimes H_0$$

and there exists an onto bounded invertible T such that $W_2U = W_1UT$, $\forall U \in \mathscr{U}$. Observing $\theta_{W_1}, \theta_{W_2}$ are isometric, we know T is unitary. Choosing $U = I$, we get $W_2 = W_1T$ and so $W_1TU = W_1UT$. Now we show $T \in \mathscr{U}'$. For any $U,V \in \mathscr{U}$,

$$W_1VUT = f(VU)W_1\sigma(VU)T = f(VU)W_1T\sigma(VU) = W_1TVU = W_1VTU.$$

and thus

$$\sum_{V\in\mathscr{U}} V^*W_1^*W_1VUT = \sum_{V\in\mathscr{U}} V^*W_1^*W_1VTU.$$

It follows $UT = TU$, that is $T \in \mathscr{U}'$.

The converse is obvious. $\square$

PROPOSITION 14. *Let $\mathscr{U}$ be a group-like unitary system on H. π_1, π_2 are unitary representations of $\mathscr{U}$ which admit orthonormal generators in $B(H,H_0)$. Then π_1, π_2 are unitarily equivalent.*

Proof. Let $A_1, A_2 \in B(H,H_0)$ be the orthonormal generators for $\pi_1(\mathscr{U}), \pi_2(\mathscr{U})$ respectively. Then there exists a unitary operator T such that

$$A_2\pi_2(U) = A_1\pi_1(U)T, \forall U \in \mathscr{U}.$$

Taking any $V \in \mathscr{U}$, we have

$$\begin{aligned}
&A_1\pi_1(V)\pi_1(U)T \\
&= A_1 f(VU)\pi_1(\sigma(VU))T \\
&= f(VU)A_2\pi_2(\sigma(VU)) \\
&= A_2\pi_2(V)\pi_2(U) \\
&= A_1\pi_1(V)T\pi_2(U).
\end{aligned}$$

So $\pi_1(U)T = T\pi_2(U), \forall U \in \mathscr{U}$. $\square$

PROPOSITION 15. *Let $\mathscr{U}$ be a group-like unitary system on H which admits a Bessel generator B. Then for any $U \in \mathscr{U}$, $\widetilde{L_u}\theta_B = \theta_{BU^*}$, where θ_B, θ_{BU^*} are the analysis operators for $B\mathscr{U}, BU^*\mathscr{U}$ respectively.*

Proof. Let $x \in H$. Then

$$\begin{aligned}
\widetilde{L_U}\theta_B(x) &= \widetilde{L_U}\Big(\sum_{V\in\mathscr{U}} e_V \otimes BV(x)\Big) \\
&= \sum_{V\in\mathscr{U}} f(UV)e_{\sigma(UV)} \otimes BV(x).
\end{aligned}$$

Letting $\sigma(UV) = V'$, one gets $V = f(UV)U^*V'$. Then

$$\begin{aligned}\widetilde{L_U}\theta_B(x) &= \sum_{V\in\mathscr{U}} f(UV)e_{V'} \otimes f(UV)BU^*V'(x)\\ &= \sum_{V'\in\mathscr{U}} e_{V'} \otimes BU^*V'(x)\\ &= \theta_{BU^*} \quad \square\end{aligned}$$

LEMMA 16. [17] *Let A be a frame generator for $\mathscr{U}$ and let S_A be the frame operator for $\{AU\}_{U\in\mathscr{U}}$. Then $S_A \in \mathscr{U}'$.*

The following theorem tells us that any (OPV)-frame representation can be viewed as a sub-representation of $\widetilde{R}$.

THEOREM 17. *Let $\mathscr{U}$ be a group-like unitary system on H which admits frame generators. Then there is a unitary operator W such that*

$$WV = \widetilde{R_V}|_P W, \ \ \forall V \in \mathscr{U},$$

where P is an orthogonal projection from $l^2(\mathscr{U})$ onto a certain subspace isomorphic to H.

Proof. Let A be a frame generator for $\mathscr{U}$. By Lemma 16, we know that $\{AS_A^{-\frac{1}{2}}U\}_{U\in\mathscr{U}} = \{AUS_A^{-\frac{1}{2}}\}_{U\in\mathscr{U}}$ is a Parseval (OPV)-frame. Let θ be a mapping from H to $l^2(\mathscr{U})\otimes H$ defined by $\theta(x) = \sum_{U\in\mathscr{U}} e_U \otimes AS_A^{-\frac{1}{2}}Ux$. Then θ is isometric and

$$\theta V(x) = \sum_{U\in\mathscr{U}} e_U \otimes AS_A^{-\frac{1}{2}}UVx.$$

On the other hand,

$$\begin{aligned}\widetilde{R_V}\theta(x) &= \widetilde{R_V}\Big(\sum_{U\in\mathscr{U}} e_U \otimes AUS_A^{-\frac{1}{2}}x\Big)\\ &= \sum_{U\in\mathscr{U}} f(UV^{-1})e_{\sigma(UV^{-1})} \otimes AUS_A^{-\frac{1}{2}}x\\ &= \sum_{U'\in\mathscr{U}} e_{U'} \otimes AS_A^{-\frac{1}{2}}U'Vx (\text{letting } U' = \sigma(UV^{-1})).\end{aligned}$$

So we get $\theta V(x) = \widetilde{R_V}\theta(x)$.

Let W be a mapping from H onto $\theta(H)$ satisfying $W(x) = \theta(x), \forall x \in H$. Then W is unitary and $WVW^* = \widetilde{R_V}|_P$ where P is the orthogonal projection from $l^2(\mathscr{U})\otimes H$ onto $\theta(H)$. $\square$

THEOREM 18. (Dilation Theorem for unitary system)[12] *Let $\mathscr{U}$ be a unitary system on H. Let π be a unitary representation of $\mathscr{U}$ on H and A be a Parseval frame generator for $\pi(\mathscr{U})$. Then there exist a unitary representation σ of $\mathscr{U}$ on a Hilbert space K and an orthonormal generator B for $\sigma(\mathscr{U})$ such that*

(i) $K \supseteq H$;

(ii) H is invariant under σ and $\pi = \sigma|_H$;

(iii) $A = B|_H$

In fact we can choose $K = l^2(\mathscr{U}) \otimes H$, $\sigma = \widetilde{R}$.

PROPOSITION 19. *Let $\mathscr{U}$ be a group-like unitary system on H. Let π be a unitary representation of $\mathscr{U}$ on H and A be a Bessel generator for $\pi(\mathscr{U})$. Then*

(1) for any $V \in \mathscr{U}$, $A\pi(V)$ is a Bessel generator for $\pi(\mathscr{U})$;

(2) for any $T \in \pi(\mathscr{U})'$, AT is a Bessel generator for $\pi(\mathscr{U})$.

Proof. (1) Let $V \in \mathscr{U}$. The result follows from

$$\begin{aligned}
&\sum_{U\in\mathscr{U}} [A\pi(V)\pi(U)]^*[A\pi(V)\pi(U)] \\
&= \sum_{U\in\mathscr{U}} \pi(U)^*\pi(V)^*A^*A\pi(V)\pi(U) \\
&= \sum_{U\in\mathscr{U}} [\pi(V)\pi(U)]^*A^*A\pi(V)\pi(U) \\
&= \sum_{U\in\mathscr{U}} [f(VU)\pi(\sigma(VU))]^*A^*A[f(VU)\pi(\sigma(VU))] \\
&= \sum_{U\in\mathscr{U}} \pi(\sigma(VU))^*A^*A\pi(\sigma(VU)) \\
&= \sum_{U'\in\mathscr{U}} \pi(U')^*A^*A\pi(U') \\
&\leqslant bI
\end{aligned}$$

where b is the upper frame bound of $\{A\pi(U)\}_{U\in\mathscr{U}}$.

(2) is proved similarly. $\square$

In the following, we denote the set of all Bessel generators for $\pi(\mathscr{U})$ by B_π.

THEOREM 20. *Let $\mathscr{U}$ be a group-like unitary system on H. Let π be a unitary representation of $\mathscr{U}$. Then π is an (OPV)-frame representation if and only if*

$$\pi(\mathscr{U})' = \{\theta_A^*\theta_B : A, B \in B_\pi\},$$

where θ_A, θ_B are the analysis operators for $A\pi(\mathscr{U}), B\pi(\mathscr{U})$ respectively.

Proof. Suppose π is an (OPV)-frame representation. We show $\theta_A^*\theta_B \in \pi(\mathscr{U})'$ for any Bessel generators A,B. In fact, for arbitrary $\pi(V) \in \pi(\mathscr{U})$ and $x \in H$, we have

$$\begin{aligned}
&\theta_A^*\theta_B\pi(V)x \\
&= \theta_A^*[\sum_{U\in\mathscr{U}} e_U \otimes B\pi(U)\pi(V)x] \\
&= \sum_{U\in\mathscr{U}} \theta_A^*[e_U \otimes B\pi(U)\pi(V)x] \\
&= \sum_{U\in\mathscr{U}} (A\pi(U))^*(B\pi(U)\pi(V)x) \\
&= \sum_{U\in\mathscr{U}} \pi(U)^*A^*B\pi(U)\pi(V)x \\
&= \sum_{U\in\mathscr{U}} \pi(U)^*A^*Bf(UV)\pi(\sigma(UV))x \\
&= \sum_{U\in\mathscr{U}} f(U^{-1})\pi(\sigma(U^{-1}))A^*Bf(UV)\pi(\sigma(UV))x \\
&= \sum_{U\in\mathscr{U}} f(U^{-1})\pi(\sigma(f(UV)^{-1}VU'^*))A^*Bf(UV)\pi(U')x \ \text{(letting } \sigma(UV)=U'\text{)} \\
&= \sum_{U\in\mathscr{U}} f(U^{-1})\pi(\sigma(VU'^*))A^*Bf(UV)\pi(U')x \\
&= \sum_{U\in\mathscr{U}} f(U^{-1})f(VU'^*)^{-1}\pi(V)\pi(U'^*)A^*Bf(UV)\pi(U')x \\
&= \sum_{U\in\mathscr{U}} f(U^{-1})f(VU'^*)^{-1}f(UV)\pi(V)\pi(U')^*A^*B\pi(U')x \\
&= \sum_{U'\in\mathscr{U}} \pi(V)\pi(U')^*A^*B\pi(U')x \\
&= \pi(V)\theta_A^*\theta_B x.
\end{aligned}$$

So $\{\theta_A^*\theta_B : A,B \in B_\pi\} \subseteq \pi(\mathscr{U})'$.

On the other hand, let $T \in \pi(\mathscr{U})'$. Since π is an (OPV)-frame representation, we know there is a Parseval frame generator S for $\pi(\mathscr{U})$. It is easy to see ST^* is a Bessel generator. We have

$$\begin{aligned}
&\theta_{ST^*}^*\theta_S(x) \\
&= \theta_{ST^*}^*(\sum_{U\in\mathscr{U}} e_U \otimes S\pi(U)x) \\
&= \sum_{U\in\mathscr{U}} \theta_{ST^*}^*(e_U \otimes S\pi(U)x) \\
&= \sum_{U\in\mathscr{U}} [ST^*\pi(U)]^*[S\pi(U)x] \\
&= \sum_{U\in\mathscr{U}} \pi(U)^*TS^*S\pi(U)x \\
&= T(\sum_{U\in\mathscr{U}} \pi(U)^*S^*S\pi(U)x) \\
&= Tx.
\end{aligned}$$

So $\pi(\mathscr{U})' \subseteq \{\theta_A^*\theta_B : A, B \in B_\pi\}$.

Thus $\pi(\mathscr{U})' = \{\theta_A^*\theta_B : A, B \in B_\pi\}$.

For the sufficiency, since $I \in \pi(\mathscr{U})'$ and $I = \theta_A^*\theta_B$ for some Bessel generators A, B, we have for any $x \in H$:

$$\begin{aligned} x &= \theta_A^*\theta_B(x) \\ &= \theta_A^*(\sum_{U\in\mathscr{U}} e_U \otimes B\pi(U)x) \\ &= \sum_{U\in\mathscr{U}} \theta_A^*(e_U \otimes B\pi(U)x) \\ &= \sum_{U\in\mathscr{U}} (A\pi(U))^*(B\pi(U)x) \\ &= \sum_{U\in\mathscr{U}} \pi(U)^*A^*B\pi(U)x. \end{aligned}$$

From this, one knows A^*B is a frame generator for $\pi(\mathscr{U})$ and thus π is an (OPV)-frame representation. $\square$

PROPOSITION 21. *Let $\mathscr{U}$ be a group-like unitary system on H and let π be a unitary representation of $\mathscr{U}$. Then*

$$span\{\theta_A^*\theta_B : A, B \in B_\pi\}$$

is a two-sided ideal in $\pi(\mathscr{U})'$, where θ_A, θ_B are the analysis operators for $A\pi(\mathscr{U})$, $B\pi(\mathscr{U})$ respectively.

Proof. By the proof of Theorem 20, $\theta_A^*\theta_B \in \pi(\mathscr{U})'$ for any Bessel generators A, B and thus

$$span\{\theta_A^*\theta_B : A, B \in B_\pi\} \subseteq \pi(\mathscr{U})'.$$

For arbitrary $T \in \pi(\mathscr{U})'$, we have

$$\begin{aligned} & T\theta_A^*\theta_B(x) \\ &= T\theta_A^*(\sum_{U\in\mathscr{U}} e_U \otimes B\pi(U)x) \\ &= T\sum_{U\in\mathscr{U}} \theta_A^*(e_U \otimes B\pi(U)x) \\ &= T\sum_{U\in\mathscr{U}} (A\pi(U))^*(B\pi(U)x) \\ &= T\sum_{U\in\mathscr{U}} \pi(U)^*A^*B\pi(U)x \\ &= \sum_{U\in\mathscr{U}} \pi(U)^*TA^*B\pi(U)x \\ &= \sum_{U\in\mathscr{U}} \pi(U)^*(AT^*)^*B\pi(U)x \\ &= \theta_{AT^*}\theta_B x. \end{aligned}$$

Since AT^* is a Bessel generator, we know $span\{\theta_A^*\theta_B : A,B \in B_\pi\}$ is a left ideal in $\pi(\mathscr{U})'$. Similarly, we can prove that $span\{\theta_A^*\theta_B : A,B \in B_\pi\}$ is also a right ideal in $\pi(\mathscr{U})'$. $\square$

COROLLARY 22. *Let $\mathscr{U}$ be a group-like unitary system on H. Assume that $\pi_i, i = 1,2,\cdots,n$ are (OPV)-frame representations of $\mathscr{U}$ and $\pi := \bigoplus_{i=1}^{n} \pi_i$. Then*

$$\pi(\mathscr{U})' = \{\theta_{A_1}^*\theta_{B_1} + \cdots + \theta_{A_n}^*\theta_{B_n} : A_i, B_i \in B_{\pi_i}\}.$$

Proof. $\pi(\mathscr{U})' \supseteq \{\theta_{A_1}^*\theta_{B_1} + \cdots + \theta_{A_n}^*\theta_{B_n} : A_i, B_i \in B_{\pi_i}\}$ is obvious.

Conversely, let A_i be a Parseval frame generator for $\pi_i(\mathscr{U})$, $i = 1,2,\cdots,n$. Then for any $T \in \pi(\mathscr{U})'$,

$$\theta_{A_1}^*\theta_{A_1} + \theta_{A_2}^*\theta_{A_2} + \cdots + \theta_{A_n}^*\theta_{A_n} = I_{H_1\oplus\cdots\oplus H_n},$$

and thus

$$\begin{aligned} T &= T(\theta_{A_1}^*\theta_{A_1} + \theta_{A_2}^*\theta_{A_2} + \cdots + \theta_{A_n}^*\theta_{A_n}) \\ &= \theta_{A_1T^*}^*\theta_{A_1} + \cdots + \theta_{A_nT^*}^*\theta_{A_n} \end{aligned}$$

and we get the result. $\square$

4. Dual frame generators

We know a "structured" frame $A\mathscr{U}$, in most cases, admits Parseval dual (OPV)-frames. However the dual frames may not preserve the structure. In this section we mainly study when a frame generators admits a Parseval dual generator, that is, the dual frame is Parseval and preserves the same structure.

THEOREM 23. *Let $\mathscr{U}$ be a group-like unitary system on H with $f : \mathscr{U} \to \mathbb{R}, \sigma : group(\mathscr{U}) \to \mathscr{U}$ satisfying $W = f(W)\sigma(W), \forall W \in group(\mathscr{U})$. Suppose there exists $U_0 \in \mathscr{U}$ such that $\sigma(U_0^k) \neq I, \forall k \in \mathbb{N}$. Let π be a unitary representation of $\mathscr{U}$ on H. If $\{A\pi(U)\}_{U\in\mathscr{U}}$ is an (OPV)-frame, then its excess is infinite.*

Proof. Define $\theta_A : H \to l^2(\mathscr{U}) \otimes H$ by $\theta_A(x) = \sum_{V\in\mathscr{U}} e_V \otimes AV(x)$ for any $x \in H$. We show that $\theta_A(H)$ is invariant under $\widetilde{R_U}, U \in \mathscr{U}$. In fact,

$$\begin{aligned} \widetilde{R_U}\theta_A &= \widetilde{R_U}(\sum_{V\in\mathscr{U}} e_V \otimes AV) \\ &= \sum_{V\in\mathscr{U}} \widetilde{R_U}(e_V \otimes AV) \\ &= \sum_{V\in\mathscr{U}} f(VU^{-1})e_{\sigma(VU^{-1})} \otimes AV \end{aligned}$$

$$= \sum_{V\in\mathscr{U}} e_{\sigma(VU^{-1})} \otimes \overline{f(VU^{-1})}AV$$
$$= \sum_{V\in\mathscr{U}} e_V \otimes AVU.$$

At the same time $\theta_A(H)^{\perp}$ is invariant under $\pi(U)$. Since $\sigma(U_0^k) \neq I$ for all $k \in \mathbb{N}$, we see that $\widetilde{R_{U_0}}$ has no eigenvalues. Thus $dimM = \infty$, since otherwise the restriction of $\pi(U_0)$ to M would have an eigenvalue. $\square$

In the following, denote by $\mathscr{M}$ the von Neumann algebra generated by $\{\widetilde{R_U}\}_{u\in\mathscr{U}}$.

LEMMA 24. *Let $\mathscr{U}$ be a group-like unitary system and let π be a unitary representation for $\mathscr{U}$ which admits a Parseval frame generator A. Denote the orthogonal projection from $l^2(\mathscr{U})\otimes H$ onto $\theta_A(H)$ by P_A, where θ_A is the analysis operator for $A\pi(\mathscr{U})$. Then $P_A \in \mathscr{M}'$.*

Proof. Since A is a Parseval frame generator, we have $\theta_A^*\theta_A = I$ and $\theta_A\theta_A^* = P_A$. For any $e_W \otimes h \in l^2(\mathscr{U})\otimes H$, we have

$$\begin{aligned}
&\widetilde{R_U}P_A(e_W\otimes h)\\
&= \widetilde{R_U}\theta_A\theta_A^*(e_W\otimes h)\\
&= \widetilde{R_U}\theta_A[(A\pi(W))^*(h)]\\
&= \widetilde{R_U}[\sum_{V\in\mathscr{U}} e_V\otimes A\pi(V)\pi(W)^*A^*h]\\
&= \sum_{V\in\mathscr{U}} f(VU^{-1})e_{\sigma(VU^{-1})}\otimes A\pi(V)\pi(W)^*A^*h\\
&= \sum_{V\in\mathscr{U}} f(VU^{-1})\overline{f(V\sigma(W^{-1}))}e_{\sigma(VU^{-1})}\otimes A\pi(\sigma(VW^{-1}))A^*h\\
&= \sum_{V\in\mathscr{U}} f(VU^{-1})\overline{f(V\sigma(W^{-1}))}e_{V'}\otimes A\pi(\sigma(V'UW^{-1}))A^*h \quad (\text{letting } V' = \sigma(VU^{-1}))\\
&= \sum_{V'\in\mathscr{U}} \overline{f(V'U\sigma(W^{-1}))}e_{V'}\otimes A\pi(\sigma(V'UW^{-1}))A^*h\\
&= \sum_{V'\in\mathscr{U}} \overline{f(V'UW^{-1})}e_{V'}\otimes A\pi(\sigma(V'UW^{-1}))A^*h.
\end{aligned}$$

On the other hand

$$\begin{aligned}
&P_A\widetilde{R_U}(e_W\otimes h) = \theta_A\theta_A^*[f(WU^{-1})e_{\sigma(WU^{-1})}\otimes h]\\
&= f(WU^{-1})\theta_A\theta_A^*(e_{\sigma(WU^{-1})}\otimes h)\\
&= f(WU^{-1})\theta_A[\pi(\sigma(WU^{-1}))^*A^*h]\\
&= f(WU^{-1})\sum_{V\in\mathscr{U}} e_V\otimes A\pi(V)\pi(\sigma(WU^{-1}))^*A^*h
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{V\in\mathscr{U}} f(WU^{-1})\overline{f(V\sigma(WU^{-1})^{-1})}e_V \otimes A\pi(\sigma(V\sigma(WU^{-1})^{-1})A^*h \\
&= \sum_{V\in\mathscr{U}} f(WU^{-1})\overline{f(V\sigma(WU^{-1})^{-1})}e_V \otimes A\pi(\sigma(VUW^{-1}))A^*h \\
&= \sum_{V\in\mathscr{U}} \overline{f(VUW^{-1})}e_V \otimes A\pi(\sigma(VUW^{-1}))A^*h. \quad \square
\end{aligned}
$$

THEOREM 25. *Let $\mathscr{U}$ be a group-like unitary system which admits a Parseval frame generator B. Denote the set of all Bessel generators for $\mathscr{U}$ by $B_{\mathscr{U}}$. Then $B_{\mathscr{U}} = \{BA : A \in w^*(\mathscr{U})\}$, where $w^*(\mathscr{U})$ is the von Neumann algebra generated by $\mathscr{U}$.*

Proof. Let θ_B be the analysis operator for $\{BU\}_{U\in\mathscr{U}}$ and let P be the orthogonal projection from $l^2(\mathscr{U})\otimes H$ onto $\theta_B(H)$. Then by Theorem 17, we have $\{\widetilde{R_U|_P} : U \in \mathscr{U}\} \approx \mathscr{U}$. Letting $W_I : l^2(\mathscr{U})\otimes H \to H_I := span\{e_I\}\otimes H$ be the orthogonal projection, we infer W_I is an orthonormal generator for $\widetilde{R_{\mathscr{U}}}$. Without loss of generality, we let $B = W_I|_P$, where P is the orthogonal projection from $l^2(\mathscr{U})\otimes H$ onto $\theta_B(H)$. Then $P \in \mathscr{M}'$. For any $A \in w^*(\mathscr{U})$, we have $A = PTP$ for some $T \in \mathscr{M}$ and so $BA = W_I|_P PTP = W_I|_P TP$. Following we show that BA is a Bessel generator. In fact,

$$
\begin{aligned}
&\sum_{U\in\mathscr{U}} (BA\widetilde{R_U|_P})^*(BA\widetilde{R_U|_P}) \\
&= \sum_{U\in\mathscr{U}} \widetilde{R_U|}_P^* A^*B^*BA\widetilde{R_U|_P} \\
&= \sum_{U\in\mathscr{U}} \widetilde{R_U|}_P^*(W_I|_P TP)^* W_I|_P TP\widetilde{R_U|_P} \\
&= \sum_{U\in\mathscr{U}} \widetilde{R_U|}_P^* PT^* W_I|_P TP\widetilde{R_U|_P} \\
&= PT^*TP \\
&\leqslant \|T\|^2 I
\end{aligned}
$$

Conversely, let $V \in B_{\mathscr{U}}$. Then $\{VU\}_{U\in\mathscr{U}}$ is a Bessel (OPV)-frame. We can view $\{VU\}_{U\in\mathscr{U}}$ as an (OPV)-frame for $H_1 := Range(\theta_V^*)$. Without loss of generality, we may suppose $\{VU\}_{U\in\mathscr{U}}$ is Parseval. Then there are a Hilbert space K and a group-like unitary system $\mathscr{W}$ on K, such that $\mathscr{W}|_{H_1} = \mathscr{U}$. Let P_{H_1} be the orthogonal projection from K onto H_1. By Theorem 18, we know there are two orthonormal generators W_1, W_2 such that $V = W_1|_{H_1}, B = W_2|_{H_1}$. There is a unitary operator T such that $W_1\mathscr{W} = W_2\mathscr{W}T$ and thus $W_1 = W_2T$. It follows that

$$
\begin{aligned}
V &= W_1|_{H_1} = W_2T|_{H_1} = W_2TP_{H_1} \\
&= W_2P_{H_1}T|_{H_1} \\
&= W_2P_{H_1}P_{H_1}T|_{H_1} \\
&= BP_{H_1}T|_{H_1}P_{H_1}.
\end{aligned}
$$

Since $P_{H_1}T|_{H_1}P_{H_1} \in w^*(\mathscr{U})$, we get the desired result. $\square$

THEOREM 26. *Let $\mathscr{U}$ be a group-like unitary system on H and π be a unitary representation of $\mathscr{U}$. Let A be a Parseval generator for $\pi(\mathscr{U})$ and P_A be the orthogonal projection from $l^2(\mathscr{U})\otimes H$ onto $\theta_A(H)$. Then the following are equivalent:*

(1) For any Parseval frame generator $B\in B(H)$ for $\pi(\mathscr{U})$, there is a unitary operator $V\in\pi(\mathscr{U})'$ such that $B=AV$;

(2) A admits an unique dual (OPV)-frame generator;

(3) $P_A\in\mathscr{M}\bigcap\mathscr{M}'$.

Proof. $(1)\Rightarrow(2)$. Suppose $A_1\in B(H)$ is a dual frame generator to A. Then

$$\sum_{U\in\mathscr{U}}\pi(U)^*A_1^*A\pi(U)=I.$$

Since $A_1S_1^{-\frac{1}{2}}$ is a Parseval frame generator for $\pi(\mathscr{U})$, there exists a unitary operator $V\in\pi(\mathscr{U})'$, such that $A_1S_1^{-\frac{1}{2}}=AV$, where S_1 is the frame operator for $A_1\pi(\mathscr{U})$. Thus

$$\sum_{U\in\mathscr{U}}\pi(U)^*S_1^{-\frac{1}{2}}A_1^*A\pi(U)=\sum_{U\in\mathscr{U}}\pi(U)^*V^*A^*A\pi(U)=V^*.$$

On the other hand,

$$\sum_{U\in\mathscr{U}}\pi(U)^*S_1^{-\frac{1}{2}}A_1^*A\pi(U)=S_1^{-\frac{1}{2}}.$$

So we get $V^*=S_1^{-\frac{1}{2}}$. Observing that V is unitary and S_1 is positive, we know $V=S_1=I$. Hence A admits only one dual frame generator, itself.

$(2)\Rightarrow(3)$. Assuming P_A is not in $\mathscr{M}$, then there is a unitary operator $V\in\mathscr{M}'$ such that $VP_A\neq P_AV$, that is, H is not an invariant subspace for V. It is easy to see that $(I-P_A)V|_H\neq 0$ and

$$\sum_{U\in\mathscr{U}}\pi(U)^*V^*(I-P_A)A\pi(U)=0,$$

that is, $(I-P_A)V|_H\pi(\mathscr{U})$ is orthogonal to $A\pi(\mathscr{U})$ and so $A+(I-P_A)V|_H$ is a dual frame generator for A which contradicts (2).

$(3)\Rightarrow(1)$. By the Dilation Theorem, there are $W_1,W_2\in B(l^2(\mathscr{U})\otimes H)$ which are orthonormal generators for $\sigma_1(\mathscr{U}),\sigma_2(\mathscr{U})$ respectively, where σ_1,σ_2 are two unitary representation for $\mathscr{U}$ which are equivalent to $\widetilde{R}$. Then there exists a unitary operator $V\in\mathscr{M}'$ such that $W_2=W_1V$. Since $W_1V=AP_A, W_2=BP_B$ and $P_A\in\mathscr{M}$, we have $AVP_A=BP_B$ and so $AV|_H=B$. We infer $V|_H$ is unitary on H from $P_A\in\mathscr{M}$. □

LEMMA 27. *Let $\mathscr{U}$ be a group-like unitary system on H. Let π be a unitary representation of $\mathscr{U}$ and $Q\in\widetilde{R_{\mathscr{U}}}'$ be an orthogonal projection. Suppose $\{A\pi(U)\}_{U\in\mathscr{U}}$ is an (OPV)-frame and $P: l^2(\mathscr{U})\otimes H\to\theta_A(H)$ is an orthogonal projection. Then the following are equivalent*

(1) $Q\sim P$ in $\widetilde{R_{\mathscr{U}}}'$;

(2) there exists a Parseval frame generator B such that Q is the orthogonal projection from $l^2(\mathscr{U})$ onto $\theta_B(H)$.

Proof. Since $\theta_A(H)=\theta_{AS^{-\frac{1}{2}}}(H)$, we may suppose A is a Parseval frame generator for $\pi(\mathscr{U})$.

$(1)\Rightarrow(2)$. Without loss of generality, we assume $\pi=\widetilde{R}|_P$, $H=P(l^2(\mathscr{U})\otimes H)$. Then there exists $W\in B(l^2(\mathscr{U})\otimes H)$ such that $A=W|_H$. Since $Q\sim P$, there exist $V\in\widetilde{R_{\mathscr{U}}}'$ such that $V^*V=P$, $VV^*=Q$ and a unitary operator $U:H\to Range(Q)$ in $R'_{\mathscr{U}}$. Letting $B=W|_HU^{-1}V|_H$, we have

$$\begin{aligned}\sum_{U\in\mathscr{U}}(BR_U|_P)^*(BR_U|_P)&=\sum_{U\in\mathscr{U}}(R_U|_P)^*B^*BR_U|_P\\&=\sum_{U\in\mathscr{U}}(R_U|_P)^*(V|_H)^*U(W|_H)^*W|_HU^{-1}V|_HR_U|_P\\&=(V|_H)^*V|_H=P.\end{aligned}$$

Thus B is a Parseval frame generator for $\pi(\mathscr{U})$.

Therefore for any $x\in H$,

$$\begin{aligned}\theta_B(x)&=\sum_{U\in\mathscr{U}}e_U\otimes B\pi(U)(x)\\&=\sum_{U\in\mathscr{U}}e_U\otimes W|_HU^{-1}V|_H\pi(U)(x)\\&=\sum_{U\in\mathscr{U}}e_U\otimes W|_H\pi(U)U^{-1}V|_H(x)\\&=U^{-1}V|_H,\end{aligned}$$

so $\theta_B(H)=U^{-1}V(H)=Q(l^2(\mathscr{U})\otimes H)$.

$(2)\Rightarrow(1)$. By the decomposition $l^2(\mathscr{U}\otimes H)=H\oplus H^\perp$, we can define a partial isometry V on $l^2(\mathscr{U}\otimes H)$ such that $V|_H=\theta_B$ and $V(H^\perp)=0$. Then $VV^*=Q$, $V^*V=P$ and it is easy to see $V\in\widetilde{R'_{\mathscr{U}}}$. $\square$

A unitary representation π is said to have *frame multiplicity* n if n is the supremum of all natural numbers k with the property that there are frame generators A_i, $i=1,2,\cdots,k$ such that $\{A_i\pi(\mathscr{U})\}_{i=1}^k$ are mutual orthogonal.

LEMMA 28. *Let $\mathscr{U}$ be a group-like unitary system and let π be a unitary representation of $\mathscr{U}$ with frame multiplicity greater than or equal 2. Then for any (OPV)-frame $\{A\pi(U)\}_{U\in\mathscr{U}}$, there exists a Parseval (OPV)-frame $\{B\pi(U)\}_{U\in\mathscr{U}}$ which is orthogonal to $\{A\pi(U)\}_{U\in\mathscr{U}}$.*

Proof. Since the frame multiplicity is greater than or equal 2, there exist $X,Y\in B(H)$ such that $X\pi(\mathscr{U})$, $Y\pi(\mathscr{U})$ are orthogonal Parseval (OPV)-frames. Let P,Q be the orthogonal projections from $l^2(\mathscr{U})\otimes H$ onto $\theta_X(H)$, $\theta_Y(H)$ respectively. Obviously, $Range(P)\perp Range(Q)$ and by Lemma 27, $P\sim Q$.

Let $A\pi(\mathscr{U})$ be an (OPV)-frame and let R be the orthogonal projection from $l^2(\mathscr{U})\otimes H$ onto $\theta_A(H)$. Then $R\sim P\sim Q$ in $\mathscr{M}'$ and thus $R^\perp\sim P^\perp$. Since Q is a sub-projection of $P^\perp$, there is a sub-projection D of $R^\perp$ such that $D\sim Q\in\mathscr{M}'$.

Using Lemma 27 again, there is a Parseval (OPV)-frame $B\pi(\mathscr{U})$ such that D is the orthogonal projection from $l^2(\mathscr{U})\otimes H$ onto $\theta_B(H)$. Since $Range(D)\perp Range(R)$, we get that $A\pi(\mathscr{U})$ is orthogonal to $B\pi(\mathscr{U})$. $\square$

THEOREM 29. *Let $\mathscr{U}$ be a group-like unitary system on H and let π be a unitary representation of $\mathscr{U}$. Then the following are equivalent*

(1) π has frame multiplicity greater than or equal 2;

(2) any (OPV)-frame $A\pi(\mathscr{U})$ with lower frame bound greater than or equal 1 admits a Parseval dual $B\pi(\mathscr{U})$.

Proof. (1) $\Rightarrow$ (2). Suppose π has frame multiplicity greater than or equal 2 and $A\pi(\mathscr{U})$ is an (OPV)-frame with lower frame bound greater than or equal 1. Let $S=\theta_A^*\theta_A$ be the frame operator and $\|S^{-1}\|\leqslant 1$. Thus $I-S^{-1}$ is positive and $T=\sqrt{I-S^{-1}}\in\pi(\mathscr{U})'$. By Lemma 28, there exists a Parsveal (OPV)-frame $B\pi(\mathscr{U})$ orthogonal to $AS^{-1}\pi(\mathscr{U})$. Then $\theta_B^*\theta_B=I$, $\theta_{AS^{-1}}^*=\theta_A^*\theta_B=0$.

Let $C=AS^{-1}+BT$. Since T is onto invertible in $\pi(\mathscr{U})'$ and $\theta_A^*\theta_B=0$, we get $\theta_A\theta_{BT}^*=0$. Thus $C\pi(\mathscr{U})$ is dual to $A\pi(\mathscr{U})$. We claim that $C\pi(\mathscr{U})$ is Parseval. In fact,

$$\begin{aligned}
&\sum_{U\in\mathscr{U}}(C\pi(U))^*(C\pi(U))\\
&=\sum_{U\in\mathscr{U}}[AS^{-1}\pi(U)+BT\pi(U)]^*[AS^{-1}\pi(\mathscr{U})+BT\pi(U)]\\
&=\sum_{U\in\mathscr{U}}\pi(U)^*S^{-1}A^*AS^{-1}\pi(U)+\pi(U)^*S^{-1}A^*BT\pi(U)\\
&\quad+\pi(U)^*TB^*AS^{-1}\pi(U)+\pi(U)^*T^*B^*BT\pi(U)\\
&=S^{-1}+0+0+T^2=I.
\end{aligned}$$

The desired result follows.

(2) $\Rightarrow$ (1). Choosing an (OPV)-frame $A\pi(\mathscr{U})$ with frame operator S satisfying $\|S^{-1}\|<1$, then $A\pi(\mathscr{U})$ admits a Parseval dual $B\pi(\mathscr{U})$. Let $C=B-AS^{-1}$. Then

$$\begin{aligned}
&\sum_{U\in\mathscr{U}}[C\pi(U)]^*[A\pi(U)]\\
&=\sum_{U\in\mathscr{U}}[\pi(U)^*B^*-\pi(U)^*S^{-1}A^*][A\pi(U)]\\
&=\sum_{U\in\mathscr{U}}\pi(U)^*B^*A\pi(U)-\pi(U)^*S^{-1}A^*A\pi(U)\\
&=I-I=0
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\sum_{U\in\mathscr{U}}[C\pi(U)]^*[C\pi(U)]\\
&=\sum_{U\in\mathscr{U}}[(B-AS^{-1})\pi(U)]^*[(B-AS^{-1})\pi(U)]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{U\in\mathscr{U}} [\pi(U)^*(B^* - S^{-1}A^*)][B\pi(U) - AS^{-1}\pi(U)] \\
&= \sum_{U\in\mathscr{U}} [\pi(U)^*B^* - \pi(U)^*S^{-1}A^*][B\pi(U) - AS^{-1}\pi(U)] \\
&= \sum_{U\in\mathscr{U}} \pi(U)^*B^*B\pi(U) - \pi(U)^*B^*AS^{-1}\pi(U) \\
&\quad -\pi(U)^*S^{-1}A^*B\pi(U) + \pi(U)^*S^{-1}A^*AS^{-1}\pi(U) \\
&= I - 0 - 0 + S^{-1}.
\end{aligned}
$$

Since $I+S^{-1}$ is onto invertible, we know C is an (OPV)-frame generator for $\pi(\mathscr{U})$. □

5. Unitary systems on finite dimensional Hilbert spaces

Finite frame theory has developed almost as a separate theory in itself. This theory has applications on several areas including multiple antenna coding, perfect reconstruction filter banks, and quantum theory. It is also useful to consider structured finite frames when one wants to get a parameterized frame. In this section we want to describe the (OPV)-frame generators for the unitary systems generated by one or more unitary operators which acting on a finite dimensional Hilbert space H.

Since the dimension of H is finite, we identify H with $\mathbb{C}^n$. We often fix an orthonormal basis and regard vectors as columns and operators as matrices. Let $\{V_j\}_{j=1}^m$ be a finite (OPV)-frame for H, that is, $\{V_j\}_{j=1}^m$ is an (OPV)-frame with $dim(H) < \infty$, $dim(H_0) < \infty$.

For $V_j : H \to H_0$, $j = 1,2,\cdots,m$, we write the analysis operator for $\{V_j\}_{j=1}^m$ as

$$
\theta_V = \begin{pmatrix} V_1 \\ V_2 \\ \vdots \\ V_m \end{pmatrix}.
$$

Obviously, $\{V_j\}_{j=1}^m$ is an (OPV)-frame if and only if θ_V has full column rank. For a general $\{V_j\}_{j=1}^m \subseteq B(H,H_0)$, supposing the column rank of θ_V is $k \leqslant n$, then $\{V_j\}_{j=1}^m$ is an (OPV)-frame for some subspace of H with dimension k. This subspace will be called the *spanning subspace* for $\{V_j\}_{j=1}^m$.

Let U be a unitary operator on H. Then $\mathscr{U} = \{U^k\}_{k=0}^{m-1}$ is a unitary system. We will describe the (OPV)-frame generators for $\mathscr{U}$.

THEOREM 30. *Let $U \in B(H)$ be unitary. Suppose U^m has pairwise distinct eigenvalues and $\{AU^k\}_{k=1}^{m-1}$ is a Parseval (OPV)-frame. Then $\frac{1}{\sqrt{m}}A$ is an isometry.*

Proof. Since $\{AU^k\}_{k=0}^{m-1}$ is Parseval, we have

$$
I = \sum_{k=0}^{m-1} (AU^k)^*(AU^k) = \sum_{k=0}^{m-1} (U^k)^*A^*AU^k.
$$

So,

$$\begin{aligned} U &= \sum_{k=0}^{m-1} (U^*)^{k-1} A^* A U^k \\ &= \sum_{k=0}^{m-1} (U^*)^{k-1} A^* A U^{k-1} U \\ &= \sum_{k=-1}^{m-2} (U^*)^{k} A^* A U^{k}) U \end{aligned}$$

It follows that $\sum\limits_{k=-1}^{m-2} (U^*)^k A^* A U^k = I$ and hence

$$\sum_{k=0}^{m-1} (U^*)^k A^* A U^k = \sum_{k=-1}^{m-2} (U^*)^k A^* A U^k.$$

We get $UA^*AU^{-1} = (U^*)^{m-1} A^* A U^{m-1}$, therefore

$$A^*A = (U^*)^m A^* A U^m. \tag{1}$$

Since A^*A is self-adjoint, there exists a unitary matrix V such that

$$A^*A = V^* diag(a_1, a_2, \cdots, a_n) V. \tag{2}$$

From this and (1), we get

$$V^* diag(a_1, a_2, \cdots, a_n) V = (U^*)^m V^* diag(a_1, a_2, \cdots, a_n) V U^m,$$

i.e.

$$diag(a_1, a_2, \cdots, a_n) = W^* diag(a_1, a_2, \cdots, a_n) W \tag{3}$$

where $W = VU^mV^* := (w_{ij})_{i,j=1}^n$. So, for any $i \neq j$, we get $(a_i - a_j) w_{ij} = 0$, i.e. $a_i = a_j$ or $w_{ij} = 0$.

Case 1. There exist i, j with $i \neq j$ satisfying $a_i = a_j$.

From (3) we have $a_i = a_j$, for any $i \neq j \in \{1, 2, \cdots, n\}$ and thus from (2), $A^*A = aI$, for some $a \geqslant 0$.

Since $\{AU^k\}_{k=0}^{m-1}$ is Parseval, we have

$$\sum_{k=0}^{m-1} (U^k)^* A^* A U^k = I,$$

and thus $a = \dfrac{1}{m}$. Hence $A^*A = \dfrac{1}{m} I$, that is $\dfrac{1}{\sqrt{m}} A$ is isometric.

Case 2. For any $i \neq j$, $a_i \neq a_j$.

In this case, $w_{ij} = 0, \forall i \neq j$ and $W = diag(w_{11}, w_{22}, \cdots, w_{nn})$.

So $U^m = V^* diag(w_{11}, w_{22}, \cdots, w_{nn})V$ i.e.

$$(VUV^*)^m = diag(w_{11}, w_{22}, \cdots, w_{nn})$$

Since U^m has pairwise distinct eigenvalues, from matrix theory (see for instance [14, p. 60] or [9, p. 232]), we have

$$VUV^* = T^* diag(w_{11}^{\frac{1}{m}}, w_{22}^{\frac{1}{m}}, \cdots, w_{nn}^{\frac{1}{m}})T \tag{4}$$

for some unitary matrix T with $T diag(w_{11}, w_{22}, \cdots, w_{nn}) = diag(w_{11}, w_{22}, \cdots, w_{nn})T$. From [19, Proposition 1], we know T is a polynomials in $diag(w_{11}, w_{22}, \cdots, w_{nn})$. Thus T commutes with $diag(w_{11}^{\frac{1}{m}}, w_{22}^{\frac{1}{m}}, \cdots, w_{nn}^{\frac{1}{m}})$ and

$$U = V^* diag(w_{11}^{\frac{1}{m}}, w_{22}^{\frac{1}{m}}, \cdots, w_{nn}^{\frac{1}{m}})V.$$

Again since $\{AU^k\}_{k=0}^{m-1}$ is Parseval, we have

$$\begin{aligned}
I &= \sum_{k=0}^{m-1} U^{*k} A^* A U^k \\
&= \sum_{k=0}^{m-1} U^{*k} V^* diag(a_1, a_2, \cdots, a_n) V U^k \\
&= \sum_{k=0}^{m-1} V^* diag(\overline{w_{11}^{\frac{k}{m}}}, \overline{w_{22}^{\frac{k}{m}}} \cdots, \overline{w_{nn}^{\frac{k}{m}}}) \\
&\quad VV^* diag(a_1, a_2, \cdots, a_n) VV^* diag(w_{11}^{\frac{k}{m}}, w_{22}^{\frac{k}{m}}, \cdots, w_{nn}^{\frac{k}{m}})V \\
&= \sum_{k=0}^{m-1} V^* diag(\overline{w_{11}^{\frac{k}{m}}}, \overline{w_{22}^{\frac{k}{m}}} \cdots, \overline{w_{nn}^{\frac{k}{m}}}) diag(a_1, a_2, \cdots, a_n) diag(w_{11}^{\frac{k}{m}}, w_{22}^{\frac{k}{m}}, \cdots, w_{nn}^{\frac{k}{m}})V \\
&= \sum_{k=0}^{m-1} V^* diag(a_1, a_2, \cdots, a_n)V = mA^*A
\end{aligned}$$

Thus $\frac{1}{\sqrt{m}}A$ is isometric. $\square$

The following proposition can be inferred quickly.

PROPOSITION 31. *Let U be a unitary operator on H and let A be a Parseval frame generator for $\{V^i\}_{i=0}^{m-1}$. Then for any $l \in \mathbb{N}$, A is a Parseval frame generator for the unitary system $\{V^i U^k\}_{i=0,k=0}^{m-1,l-1}$.*

COROLLARY 32. *Let A be a Parseval frame generator for $\{V^i U^k\}_{i=0,k=0}^{m-1,l-1}$. Then for any $M \in \mathbb{N}$, A is a Parseval frame generator for $\{V^i U^{k+M}\}_{i=0,k=0}^{m-1,l-1}$.*

THEOREM 33. *Let* $A \in B(H,H_0)$ *and let* $U,V \in B(H)$ *be unitary. Suppose* $\{AV^j\}_{j=1}^{l-1}$ *is a Parseval (OPV)-frame for its spanning subspace* H_A*,* P *is the orthogonal projection from* H *onto* H_A *and* P_i *is the orthogonal projection from* H *onto* $U^{-i}H_A$*,* $i=0,1,\cdots,m-1$*. Then* $\{AV^jU^i\}_{i=0,j=0}^{m-1,l-1}$ *is a Parseval (OPV)-frame for* H *if and only if* $\sum\limits_{i=0}^{m-1} P_i = I$*.*

Proof. Supposing $\{AV^jU^i\}_{i=0,j=0}^{m-1,l-1}$ is a Parseval (OPV)-frame for H then $P_i = U^{-i}PU^i$, $i=0,1,\cdots,m-1$. We have

$$\begin{aligned}
\sum_{i=0}^{m-1} P_i &= \sum_{i=0}^{m-1} U^{-i}PU^i \\
&= \sum_{i=0}^{m-1} U^{-i}\Big(\sum_{j=0}^{l-1}(V^j)^*A^*AV^j\Big)U^i \\
&= \sum_{i=0}^{m-1}\sum_{j=0}^{l-1}(U^*)^i(V^j)^*A^*AV^jU^i \\
&= I.
\end{aligned}$$

Conversely, suppose $\sum\limits_{i=0}^{m-1} P_i = I$. We get

$$\begin{aligned}
&\sum_{i=0}^{m-1}\sum_{j=0}^{l-1}(U^*)i(V^*)^jA^*AV^jU^i \\
&= \sum_{i=0}^{m-1}(U^*)^iPU^i \\
&= \sum_{i=0}^{m-1} P_i \\
&= I.
\end{aligned}$$

Thus $\{AV^jU^i\}_{i=0,j=0}^{m-1,l-1}$ is a Parseval (OPV)-frame for H. $\square$

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