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SATURATION FOR CESÀRO MEANS OF HIGHER ORDER

LAURIAN SUCIU

In the memory of
Cristin Lucian Suciu

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Abstract. The classical Cesàro means of higher order are investigated from the point of view of saturation theory. In this direction the Cesàro means for power bounded operators were studied by Butzer-Westphal [1, 2], and certain appropriate results were given by Lin-Sine [6]. In this paper we prove that the behavior of the Cesàro means of higher order concerning the saturation, are essentially of the same form, under the boundedness condition of Cesàro means. Also, some results of Lin-Sine are extended to Cesàro means of higher order.

1. Introduction and preliminaries

Let $\mathscr{X}$ be a complex Banach space, and $\mathscr{B}(\mathscr{X})$ be the Banach algebra of all bounded linear operators on $\mathscr{X}$ with the unit element I (the identity operator on $\mathscr{X}$). For $T \in \mathscr{B}(\mathscr{X})$ we denote by $\mathscr{R}(T)$, $\mathscr{N}(T)$ and $\sigma(T)$, the range, the null space and the spectrum of T, respectively.

If $\{T_n\}$ is a sequence in $\mathscr{B}(\mathscr{X})$, the notation $||T_n|| = O(1)$ as $n \to \infty$ means $\sup_{n\geqslant 1} ||T_n|| < \infty$. Also $||T_n x|| = o(1)$ as $n \to \infty$ means $T_n x \to 0$, and $||T_n x|| = o(n)$ as $n \to \infty$ stands for $\frac{1}{n} T_n x \to 0$, for $x \in \mathscr{X}$.

Recall ([8], [9, 10], [13]) that for $T \in \mathscr{B}(\mathscr{X})$ and $p \in \mathbb{N}$, *the Cesàro means of order* p of T are defined for $n \in \mathbb{N}$ by $M_0^{(p)}(T) = I$, $M_n^{(0)}(T) = T^n$ and in general,

$$M_n^{(p)}(T) := \frac{p}{(n+1)\ldots(n+p)} \sum_{j=0}^{n} \frac{(j+p-1)!}{j!} M_j^{(p-1)}(T) \tag{1.1}$$
$$= \frac{p}{(n+1)\ldots(n+p)} \sum_{j=0}^{n} (n+1-j)\ldots(n+p-1-j) T^j.$$

As usually, we put $M_n(T) := M_n^{(1)}(T) = \frac{1}{n+1} \sum_{j=0}^{n} T^j$.

We say that T is *Cesàro ergodic* if the sequence $\{M_n(T)\}$ converges strongly in $\mathscr{B}(\mathscr{X})$. Also, T is *Cesàro bounded* if $||M_n(T)|| = O(1)$ as $n \to \infty$. In particular, if T

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is power bounded that is $||T^n|| = O(1)$ as $n \to \infty$ then T is Cesàro bounded, but it is not necessarily Cesàro ergodic when $\mathscr{X}$ is not reflexive.

We shall frequently need two basic identities on $M_n^{(p)}(T)$ which are given, for completeness, in the following

LEMMA 1.1. *If $T \in \mathscr{B}(\mathscr{X})$ then for $p,n \geqslant 1$ we have*

$$M_n^{(p)}(T)(T-I) = \frac{p}{n+1}(M_{n+1}^{(p-1)}(T)-I), \tag{1.2}$$

$$TM_n^{(p)}(T) = \frac{n+p+1}{n+1}M_{n+1}^{(p)}(T) - \frac{p}{n+1}I. \tag{1.3}$$

Proof. For T,p,n as above one has

$$\begin{aligned}
&M_n^{(p)}(T)(T-I)\\
&= \frac{p}{(n+1)\dots(n+p)}\sum_{j=0}^{n}(n+1-j)\dots(n+p-1-j)(T^{j+1}-T^j)\\
&= \frac{p(p-1)}{(n+1)\dots(n+p)}\Big[\sum_{j=1}^{n+1}(n+2-j)\dots(n+p-j)T^j - \frac{(n+1)\dots(n+p-1)}{p-1}I\Big]\\
&= \frac{p}{n+1}\Big(M_{n+1}^{(p-1)}(T) - \frac{p-1}{n+p}I - \frac{n+1}{n+p}I\Big) = \frac{p}{n+1}(M_{n+1}^{(p-1)}(T)-I),
\end{aligned}$$

which gives (1.2). For (1.3) we have

$$\begin{aligned}
TM_n^{(p)}(T) &= \frac{p}{(n+1)\dots(n+p)}\sum_{j=1}^{n+1}(n+2-j)\dots(n+p-j)T^j\\
&= \frac{p}{(n+1)\dots(n+p)}\Big(\sum_{j=0}^{n+1}(n+2-j)\dots(n+p-j)T^j - (n+2)\dots(n+p)I\Big)\\
&= \frac{n+p+1}{n+1}M_{n+1}^{(p)}(T) - \frac{p}{n+1}I. \quad \square
\end{aligned}$$

The convergence of Cesàro means of operators (the case $p = 1$), in different topologies, was studied by many authors. More recently, such investigations refer also to the case $p > 1$ like in [8], where especially the uniform convergence is treated.

Concerning the strong convergence of $\{M_n(T)\}$, P.L. Butzer and U. Westphal described in [1, 2] the order of approximation of Px by $M_n^{(p)}(T)x$ for some $x \in \mathscr{X}$, when T is power bounded, P being the ergodic projection associated to T that is the bounded projection on the closed subspace $\mathscr{X}_0 = \mathscr{R}(P) \oplus \mathscr{N}(P)$ with $\mathscr{R}(P) = \mathscr{N}(I-T)$ and $\mathscr{N}(P) = \overline{\mathscr{R}(I-T)}$.

The results of [1] and [2] regarding the discrete case, can be connected to certain results of M. Lin and R. Sine [6] concerning some averages, naturally associated to

$M_n(T)$, and directly related to $M_n^{(2)}(T)$. For this reason, the purpose of this paper is to investigate the mean ergodic theorem for the Cesàro averages of higher order, from the point of view of saturation theory, under the more general condition of Cesàro boundedness. We obtain certain versions, or generalizations, of some results from [1, 2] and [6], which lead to the conclusion that all Cesàro means $\{M_n^{(p)}(T)\}_n$ for $p \geqslant 1$ are equivalent, in the sense that they have the same saturation class. Recall that a similar conclusion is also true for Cesàro averages of A-contractions in Hilbert spaces induced by positive operators A (see [12]). Some facts in this context related to results of [6] can be also found in [7, 11].

As in [1, 2] our method is based on the concept of relative completion of a subspace. More precisely, if $\mathscr{Y}$ is a Banach subspace of $\mathscr{X}$, then the completion of $\mathscr{Y}$ relative to $\mathscr{X}$, denoted by $\widetilde{\mathscr{Y}}_{\mathscr{X}}$, is the set of all elements $x \in \mathscr{X}$ for which there exists a sequence $\{y_n\} \subset \mathscr{Y}$ with $||y_n||_{\mathscr{Y}} = O(1)$ and $||y_n - x|| = o(1)$, as $n \to \infty$. Notice that $\widetilde{\mathscr{Y}}_{\mathscr{X}} = \mathscr{Y}$ if $\mathscr{X}$ is reflexive.

2. Ergodic theorem and saturation

The general ergodic theorem for Cesàro averages in the form quoted in [6], can be extended for the Cesàro averages of higher order as follows (see also [1] for power bounded operators).

THEOREM 2.1. *Let $T \in \mathscr{B}(\mathscr{X})$ be such that $||M_n^{(p)}(T)|| = O(1)$ and $||M_n^{(p-1)}(T)x|| = o(n)$, as $n \to \infty$, for $x \in \mathscr{X}$ and some positive integer p. Then the set of all elements $x \in \mathscr{X}$ for which the sequence $\{M_n^{(q)}(T)x\}$ converges in the norm of $\mathscr{X}$ for $q \geqslant p$, is precisely equal to the direct sum*

$$\mathscr{X}_0 := \overline{\mathscr{R}(I-T)} \oplus \mathscr{N}(I-T), \tag{2.1}$$

which is a closed subspace in $\mathscr{X}$. Here the limit itself is equal to Px, where $P \in \mathscr{B}(\mathscr{X}_0)$ is the projection onto the range $\mathscr{R}(P) = \mathscr{N}(I-T)$ parallel to the null space $\mathscr{N}(P) = \overline{\mathscr{R}(I-T)}$.

Moreover, if either T^m is weakly compact for some $m > 0$, or $\mathscr{X}$ is reflexive, then $\mathscr{X} = \mathscr{X}_0$, that is the above convergence holds for all $x \in \mathscr{X}$.

The proof of the former part can be obtained in the same way as for the mean ergodic theorem in Krengel [3] (Theorem 2.1.3), using the relation (1.2) and the Banach-Steinhauss theorem. If T^m is weakly compact for an integer $m \geqslant 1$, then the sequence $\{M_n^{(q)}(T)T^m x\}$ is weakly sequentially compact for $x \in \mathscr{X}$ and $q \geqslant p$, so this sequence has a subsequence which weakly converges to a limit belonging to $\mathscr{N}(I-T)$. Then as in [3] it follows that $x \in \mathscr{X}_0$, hence $\mathscr{X} = \mathscr{X}_0$. When $\mathscr{X}$ is reflexive, every bounded sequence posses a weakly convergent subsequence hence, as above, we infer $\mathscr{X} = \mathscr{X}_0$. We omit the details.

The question which arises now is how rapidly the Cesàro means of an order $p \geqslant 1$ of T on x converge to Px, when x belongs to some subspaces of $\mathscr{X}_0$. The case when

T is power bounded was studied by Butzer-Westphal in [1, 2], while we consider in this paper the general case $p \geqslant 1$ under the boundedness condition of $M_n^{(p)}(T)$ for $n \in \mathbb{N}$. In this order, we define the operators $S_n^{(p)}(T)$, for $n, p \geqslant 1$ by

$$S_n^{(p)}(T) := \frac{n+p}{p} M_n^{(p)}(T). \tag{2.2}$$

Clearly, $S_n^{(1)}(T) = \sum_{j=0}^n T^j$ and, for $p \geqslant 2$, we have

$$S_n^{(p)}(T) = \sum_{j=0}^{n} \Big(1 - \frac{j}{n+1}\Big) \dots \Big(1 - \frac{j}{n+p-1}\Big) T^j.$$

In particular, we get

$$\begin{aligned} S_n^{(2)}(T) &= \frac{1}{n+1} \sum_{j=0}^{n} (n+1-j) T^j \\ &= \frac{1}{n+1} ((n+1)I + nT + \dots + T^n) \\ &= \frac{1}{n+1} \sum_{j=1}^{n+1} (I + T + \dots + T^{j-1}) = \frac{1}{n+1} \sum_{j=0}^{n} \sum_{i=0}^{j} T^i. \end{aligned}$$

Since the relations (1.2) and (2.2) lead to

$$S_n^{(p+1)}(T)(I-T) = \frac{n+p+1}{n+1} (I - M_{n+1}^{(p)}(T)), \tag{2.3}$$

we infer that the behavior of $M_n^{(p)}(T)$ on $\mathscr{X}$ (as well convergence, or boundedness) is the same to that of $S_n^{(p+1)}(T)$ on $\mathscr{R}(I-T)$. This fact just motivates the hypotheses of Theorem 2.1 for the results below.

Notice that $\mathscr{X}_0$ from (2.1) is an invariant subspace for T, so we can define the operator $T_0 := T|_{\mathscr{X}_0} \in \mathscr{B}(\mathscr{X}_0)$.

We begin to characterize the norm convergence of $\{S_n^{(p+1)}(T)x\}$, for $x \in \mathscr{X}$.

THEOREM 2.2. *Let $T \in \mathscr{B}(\mathscr{X})$ be such that $||M_n^{(p)}(T)|| = O(1)$ and $||M_n^{(p-1)}(T)x|| = o(n)$, as $n \to \infty$, for any $x \in \mathscr{X}$ and some $p \geqslant 1$. The following statements are equivalent:*

(i) $x \in \mathscr{R}(I - T_0)$;

(ii) $\{S_n^{(p+1)}(T)x\}$ converges in the norm of $\mathscr{X}$;

(iii) $\{S_n^{(p+1)}(T)x\}$ has a weakly convergent subsequence.

In this case, if $x = (I - T_0)x_0$ with $x_0 \in \mathscr{X}_0$ then $(I-P)x_0$ is the limit in (ii), and $(I-T_0)(I-P)x_0 = x$.

Proof. If $x = (I-T_0)x_0$ with $x_0 \in \mathscr{X}_0$ then from relation (2.3) and Theorem 2.1 it follows that $S_n^{p+1}(T)x \to x_0 - Px_0$ as $n \to \infty$, and clearly $(I-T_0)(I-P)x_0 = (I-T_0)x_0 = x$. So, (i) implies (ii), and obviously (ii) implies (iii).

Suppose now (iii), that is a subsequence $\{S_{n_k}^{(p+1)}(T)x\}$ weakly converges to $y \in \mathscr{X}$. Then using the relation (2.3) we have (as in Theorem 2.1.3 [3])

$$\begin{aligned}(I-T)y &= w-\lim_{k\to\infty} S_{n_k}^{(p+1)}(T)(I-T)x\\ &= w-\lim_{k\to\infty}\frac{n_k+p+1}{n_k+1}(x-M_{n_k+1}^{(p)}(T)x)\\ &= x-Px,\end{aligned}$$

hence $x=(I-T)y+Px \in \mathscr{X}_0$. Since one has

$$\begin{aligned}\frac{n_k+p+1}{p+1}Px = \frac{n_k+p+1}{p+1}M_{n_k}^{(p+1)}(T)Px &= S_{n_k}^{(p+1)}(T)Px\\ &= S_{n_k}^{(p+1)}(T)x+S_{n_k}^{(p+1)}(T)(T-I)y,\end{aligned}$$

or equivalently

$$M_{n_k}^{(p+1)}(T)(T-I)y = Px-\frac{p+1}{n_k+p+1}S_{n_k}^{(p+1)}(T)x,$$

we infer (using our assumption) that

$$w-\lim_{k\to\infty} M_{n_k}^{(p+1)}(T)(T-I)y = Px.$$

On the other hand, by Theorem 2.1 we have $M_{n_k}^{(p+1)}(T)(T-I)y \to 0$ $(k\to\infty)$ in norm, therefore $Px=0$. Then $x=(I-T)y$ which gives

$$y = w-\lim_{k\to\infty} S_{n_k}^{(p+1)}(T)(I-T)y = w-\lim_{k\to\infty}\frac{n_k+p+1}{n_k+1}(y-M_{n_k+1}^{(p)}(T)y),$$

whence

$$w-\lim_{k\to\infty} M_{n_k+1}^{(p)}(T)y=0.$$

This yields (as in Theorem 2.1.3 [3]) that $y\in\overline{\mathscr{R}(I-T)}$, and finally we get $x=(I-T_0)y\in\mathscr{R}(I-T_0)$. In conclusion (iii) implies (i), which ends the proof. □

Since the hypotheses on T in this theorem ensures also $||M_n^{(q)}(T)||=O(1)$ as $n\to\infty$ and (by (1.2)) $||M_n^{(q-1)}(T)x||=o(n)$ as $n\to\infty$ for any $x\in\mathscr{X}$ and $q>p$, we can change p by such a q in the statements (ii) and (iii). So, we infer immediately the following

COROLLARY 2.3. *Let $T\in\mathscr{B}(\mathscr{X})$ be as in Theorem 2.2 relative to an integer $p\geqslant 1$. If $x\in\mathscr{X}$ and $||S_n^{(q)}(T)x||=o(1)$ as $n\to\infty$, for some $q>p$, then $x=0$.*

Proof. By Theorem 2.2 we have $x=(I-T_0)\lim_{n\to\infty} S_n^{(q)}(T)x=0$, if $\lim_{n\to\infty} S_n^{(q)}(T)x=0$ and $q>p$. □

Some particular cases of Theorem 2.2 will be discussed in Section 3, these being related to the results from Lin-Sine [6].

REMARK 2.4. From the proof of the previous theorem we obtain $\lim_{n\to\infty} S_n^{(p+1)}(T)x \in \overline{\mathscr{R}(I-T)}$ if $x \in \mathscr{R}(I-T_0)$. Thus, for T as in Theorem 2.2 we can define the linear operator $S : \mathscr{R}(I-T_0) \to \mathscr{X}_0$ by

$$Sx = \lim_{n\to\infty} S_n^{(p+1)}(T)x, \quad x \in \mathscr{R}(I-T_0) =: \mathscr{S}. \tag{2.4}$$

Then $PS = 0$, and by Theorem 2.2 we have $(I-T_0)Sx = x$ for $x \in \mathscr{S}$. In addition, S is a closed operator. Indeed, let $(x,y) \in \mathscr{S} \times \mathscr{X}_0$ and $\{x_k\} \subset \mathscr{S}$ be a sequence such that $x_k \to x$ and $Sx_k \to y$, $k \to \infty$. Therefore $x_k = (I-T_0)Sx_k \to (I-T_0)y$, and we get $x = (I-T_0)y$. This yields by (2.3)

$$Sx = \lim_{n\to\infty} S_n^{(p+1)}(T)(I-T)y = y - Py$$

and, on the other hand, one has $Sx = \lim_{k\to\infty} Sx_k = y$. So, $Py = 0$ that is $y = Sx$, hence S is closed.

This fact ensures that the domain $\mathscr{S}$ of the operator S is a Banach space with the norm

$$||x||_0 := ||x|| + ||Sx|| \quad (x \in \mathscr{S}), \tag{2.5}$$

and we shall use this remark in the sequel.

THEOREM 2.5. *Let $T \in \mathscr{B}(\mathscr{X})$ be such that $||M_n^{(p)}(T)|| = O(1)$ and $||M_n^{(p-1)}(T)x|| = o(n)$, as $n \to \infty$, for any $x \in \mathscr{X}$ and some $p \geqslant 1$. Then $||S_n^{(p+1)}(T)x|| = O(1)$ as $n \to \infty$ if and only if $x \in \widetilde{R(I-T_0)}_{\mathscr{X}_0}$ (the completion of $\mathscr{S}$ relative to $\mathscr{X}_0$).*

Proof. Suppose $||S_n^{(p+1)}(T)x|| = O(1)$ as $n \to \infty$, for some $x \in \mathscr{X}$. Then $M_n^{(p+1)}(T)x = \frac{p+1}{n+p+1} S_n^{(p+1)}(T)x \to 0$ $(n \to \infty)$ and by Theorem 2.1 it follows $x \in \overline{\mathscr{R}(I-T)}$.

We define $S_{n,0}^{(p+1)}(T)x_0 = S_n^{(p+1)}(T)(I-T_0)x_0$, for $x_0 \in \mathscr{X}_0$. So, $S_{n,0}^{(p+1)}(T) \in \mathscr{B}(\mathscr{X}_0)$, and by (2.3) and Theorem 2.1 we obtain immediately $S_{n,0}^{(p+1)}(T)x \to x$ $(n \to \infty)$, the convergence being in the norm of $\mathscr{X}$. Using also (2.3) and the hypothesis, we infer that there exists a constant $c > 1$ such that, for each $y \in \mathscr{X}_0$ and $n \geqslant p-1$, we have

$$\begin{aligned} ||S_{n,0}^{(p+1)}(T)y|| &\leqslant \frac{n+p+1}{n+1}(1 + ||M_{n+1}^{(p)}(T)||)||y|| \\ &\leqslant 2(1 + \sup_{j\geqslant 1} ||M_j^{(p)}(T)||)||y|| = c||y||. \end{aligned}$$

Thus, for x as above and $n \geqslant p-1$, we get

$$\begin{aligned}
||S_{n,0}^{(p+1)}(T)x||_0 &= ||S_{n,0}^{(p+1)}(T)x|| + ||SS_{n,0}^{(p+1)}(T)x|| \\
&\leqslant c||x|| + \lim_{k\to\infty} ||S_k^{(p+1)}(T)S_{n,0}^{(p+1)}(T)x|| \\
&= c||x|| + \lim_{k\to\infty} ||S_{n,0}^{(p+1)}(T)S_k^{(p+1)}(T)x|| \\
&\leqslant c(||x|| + \limsup_{k\to\infty} ||S_k^{(p+1)}(T)x||),
\end{aligned}$$

because $S_k^{(p+1)}(T)x \in \mathscr{X}_0$, for any k. Hence $\sup_{n\geqslant 1} ||S_{n,0}^{(p+1)}(T)x|| < \infty$. Now, since $||S_{n,0}^{(p+1)}(T)x - x|| \to 0$ $(n \to \infty)$, while $\mathscr{R}(S_{n,0}^{(p+1)}(T)) \subset \mathscr{S}$ and $(\mathscr{S}, ||\cdot||_0)$ is a Banach subspace of $\mathscr{X}_0$, we conclude that $x \in \widetilde{\mathscr{S}}_{\mathscr{X}_0} = \widetilde{\mathscr{R}(I-T_0)}_{\mathscr{X}_0}$.

Conversely, let $x \in \widetilde{\mathscr{R}(I-T_0)}_{\mathscr{X}_0}$ and let $x_k \in \mathscr{R}(I-T_0)$ be such that $\sup_{k\geqslant 1} ||x_k||_0 < \infty$ and $||x_k - x|| \to 0$ $(k \to \infty)$. Remark firstly that, for $y \in \mathscr{S}$ of the form $y = (I-T_0)z$ with $z \in \mathscr{X}_0$, we obtain by (2.3) and Theorem 2.1 that $S_n^{(p+1)}(T)y \to z - Pz$ $(n \to \infty)$. So, we have $Sy = z - Pz$ and also, for $n \geqslant p-1$,

$$\begin{aligned}
||S_n^{(p+1)}(T)y|| &\leqslant 2||z - M_{n+1}^{(p)}(T)z|| \\
&= 2||z - Pz + M_{n+1}^{(p)}(T)(Pz - z)|| \\
&\leqslant 2(1 + \sup_{j\geqslant 1} ||M_j^{(p)}(T)||)||Sy|| \leqslant c||y||_0
\end{aligned}$$

with $c > 1$ as above. Taking $y = x_k$, for $k \geqslant 1$, we find

$$||S_n^{(p+1)}(T)x_k|| \leqslant c||x_k||_0 \leqslant c \sup_{j\geqslant 1} ||x_j||_0 = cc_0,$$

and by passing to limit as $k \to \infty$ we obtain $||S_n^{(p+1)}(T)x|| \leqslant cc_0$, for $n \geqslant p-1$. This means $||S_n^{(p+1)}(T)x|| = O(1)$ as $n \to \infty$, which ends the proof. □

Notice that Theorem 2.5 is also valid in the case $p = 0$ under the hypothesis $||T^n|| = ||M_n^{(0)}(T)|| = O(1)$ as $n \to \infty$, that is T is power bounded. In fact, this result is just the version for arbitrary Banach spaces of Proposition 1 (b) from [2], obtained in the reflexive spaces. In addition, if T is conditionally weakly compact then $\mathscr{X}_0 = \mathscr{X}$, so $T_0 = T$ and $\widetilde{\mathscr{R}(I-T_0)}_{\mathscr{X}_0} = \widetilde{\mathscr{R}(I-T)}_{\mathscr{X}}$. When $\mathscr{X}$ is reflexive, every $T \in \mathscr{B}(\mathscr{X})$ is conditionally weakly compact and $\widetilde{\mathscr{R}(I-T)}_{\mathscr{X}} = \mathscr{R}(I-T)$ (see [1, 2]). Having in view these facts, we can give the general saturation theorem, as follows.

THEOREM 2.6. *Let* $T \in \mathscr{B}(\mathscr{X})$ *be such that* $||M_n^{(p)}(T)|| = O(1)$ *and* $||M_n^{(p-1)}(T)x|| = o(n)$, *as* $n \to \infty$, *for any* $x \in \mathscr{X}$ *and some* $p \geqslant 1$. *The following statements hold:*

(i) $||M_n^{(p+1)}(T)x - Px|| = o(\frac{1}{n})$ *as* $n \to \infty$ *if and only if* $x \in \mathscr{N}(I-T)$*;*

(ii) $||M_n^{(p+1)}(T)x - Px|| = O(\frac{1}{n})$ *as* $n \to \infty$ *if and only if*

(a) $x \in \widetilde{\mathscr{R}(I-T_0)}_{\mathscr{X}_0} \oplus \mathscr{N}(I-T)$, *or equivalently*

(b) $x \in \widetilde{\mathscr{R}(I-T)}_{\mathscr{X}} \oplus \mathscr{N}(I-T)$, *if in addition* T *is conditionally weakly compact, respectively*

(c) $x \in \mathscr{R}(I-T) \oplus \mathscr{N}(I-T)$, *if* $\mathscr{X}$ *is reflexive.*

Proof. (i) Suppose that $||M_n^{(p+1)}(T)x - Px|| = o(\frac{1}{n})$ as $n \to \infty$, for some $x \in \mathscr{X}$. So, $n(M_n^{(p+1)}(T)x - Px) \to 0$ $(n \to \infty)$ which implies

$$S_n^{(p+1)}(T)(x-Px) = \frac{n+p+1}{p+1}(M_n^{(p+1)}(T)x - Px) \to 0, \quad n \to \infty.$$

Then by Corollary 2.3 we have $x - Px = 0$, that is $x \in \mathscr{N}(I-T)$. The other implication in (i) is obvious.

(ii) Assume now that $||M_n^{(p+1)}(T)x - Px|| = O(\frac{1}{n})$ as $n \to \infty$. This yields that $S_n^{(p+1)}(T)(x-Px) = \frac{n+p+1}{n(p+1)}\, n(M_n^{(p+1)}(T)x - Px)$ is a bounded sequence, so by Theorem 2.5 we have $x - Px \in \widetilde{\mathscr{R}(I-T_0)}_{\mathscr{X}_0}$, hence $x \in \widetilde{\mathscr{R}(I-T_0)}_{\mathscr{X}_0} \oplus \mathscr{N}(I-T)$.

Conversely, let $x \in \widetilde{\mathscr{R}(I-T_0)}_{\mathscr{X}_0} \oplus \mathscr{N}(I-T)$. Then

$$n(M_n^{(p+1)}(T)x - Px) = \frac{n(p+1)}{n+p+1} S_n^{(p+1)}(x-Px),$$

and since $x - Px \in \widetilde{\mathscr{R}(I-T_0)}_{\mathscr{X}_0}$ by our assumption, from Theorem 2.5 the sequence $\{S_n^{(p+1)}(T)(x-Px)\}$ is bounded. So, $||M_n^{(p+1)}(T)x - Px|| = O(\frac{1}{n})$ as $n \to \infty$, and (ii) is established in the general case (a). The particular cases corresponding to (b) and (c) being discussed above, the proof is finished. $\square$

Remark that Theorem 2.6 is also true in the case $p = 0$, that is for Cesàro means of order 1. This is a generalized version (for arbitrary Banach spaces) of Theorem 1 [1, p. 1173] from reflexive case. Our saturation theorem for Cesàro means of order $p \geqslant 1$, has the same form like in the case $p = 1$ in [1] or the case $p > 1$ in [2], and we proved the above results by adapting the method from [1] concerning the saturation of Abel means. In fact, the above saturation theorems extend the similar results from [1, 2] obtained for power bounded operators.

It follows from Theorem 2.1 that, if $\{M_n^{(p)}(T)\}$ is strongly convergent for some $p \geqslant 1$, then $\{M_n^{(q)}(T)\}$ is also strongly convergent for $q > p$ (and the limits are equal). But the converse is not necessarily true. Since the hypotheses of Theorem 2.6 remain valid for $q > p$, we conclude that, from the point of view of saturation, both processes $\{M_n^{(p)}(T)\}$ and $\{M_n^{(q)}(T)\}$ are "equivalent". So, in the ergodic case, all Cesàro means (of arbitrary order) are "equivalent". Concretely, using the terminology of saturation theory like in [1], we have

COROLLARY 2.7. *Under the hypotheses of Theorem 2.6, the Cesàro process* $\{M_n^{(p+1)}(T)\}$ *is saturated with order* $O(\frac{1}{n})$ *as* $n\to\infty$, *and the Favard (or saturation) class is given by the statements* (a), (b) *or* (c) *of Theorem 2.6, depending upon the quoted hypotheses.*

REMARK 2.8. The hypotheses of Theorem 2.6 do not ensure the saturation of $\{M_n^{(p)}(T)\}$, that is the conclusion of this theorem is not necessary true with $M_n^{(p)}(T)$ instead of $M_n^{(p+1)}(T)$ in the conditions (i) and (ii). Indeed, for $x\in\overline{\mathscr{R}(I-T)}$ we have

$$\begin{aligned} n||M_n^{(p)}(T)(T-I)x|| &= \frac{np}{n+1}||M_{n+1}^{(p-1)}(T)x-x|| \\ &\geqslant \frac{p}{2}(||M_{n+1}^{(p-1)}(T)x||-||x||), \end{aligned}$$

and so the assertions (i) and (ii) are not true for $M_n^{(p)}(T)y$ with $y=(T-I)x\in\mathscr{R}(I-T_0)$, if $\{M_n^{(p-1)}(T)x\}$ is unbounded. We can see such an operator in the following

EXAMPLE 2.9. Let $T\in\mathscr{B}(\mathscr{X})$ be a Cesàro ergodic operator which is not power bounded. Consider the Banach space $\mathscr{Y}=\overline{\mathscr{R}(I-T)}\oplus\overline{\mathscr{R}(I-T)}$ with the norm of $y=x\oplus z\in\mathscr{Y}$ given by

$$||y||=\sqrt{||x||^2+||z||^2}.$$

Let $\widetilde{T}\in\mathscr{B}(\mathscr{Y})$ be the operator defined by the matrix

$$\widetilde{T}=\begin{pmatrix} T_0 & S\\ 0 & I\end{pmatrix},$$

where $T_0=T|_{\overline{\mathscr{R}(I-T)}}$, $S=(T_0-I)J$ and $J(0\oplus z)=z\oplus 0$ for $z\in\overline{\mathscr{R}(I-T)}$, I being the identity operator in the matrix of $\widetilde{T}$. We have

$$\widetilde{T}^n=\begin{pmatrix} T_0^n & S_{n-1}^{(1)}(T_0)S\\ 0 & I\end{pmatrix},\quad M_n(\widetilde{T})=\begin{pmatrix} M_n(T_0) & \frac{n}{n+1}S_{n-1}^{(2)}(T_0)S\\ 0 & I\end{pmatrix}.$$

Clearly, $\frac{1}{n}S_{n-1}^{(1)}(T_0)S=M_{n-1}(T_0)S\to 0$ strongly on $\overline{\mathscr{R}(I-T)}$, hence $\frac{1}{n}\widetilde{T}^n\to 0$ strongly on $\mathscr{Y}$, that is $||\widetilde{T}^n y||=o(n)$ as $n\to\infty$ for $y\in\mathscr{Y}$. On the other hand, since T_0 is Cesàro ergodic on $\overline{\mathscr{R}(I-T)}$ and $\mathscr{R}(S)=\mathscr{R}(I-T_0)$, we infer from Theorem 2.5 for T_0 (the case $p=1$) that $||S_n^{(2)}(T_0)S||=O(1)$ as $n\to\infty$, hence $||M_n(\widetilde{T})||=O(1)$ as $n\to\infty$.

By the choice T is not power bounded on $\mathscr{X}=\overline{\mathscr{R}(I-T)}\oplus\mathscr{N}(I-T)$, so there exists $x_0\in\overline{\mathscr{R}(I-T)}$ such that $\sup_{n\geqslant 1}||T_0^n x_0||=\infty$, which means that the sequence $\{\widetilde{T}^n x_0\}$ is unbounded. Since $\mathscr{R}(I-\widetilde{T})=\mathscr{R}(I-T_0)$ (by the definition of $\widetilde{T}$ and the fact that $\widetilde{T}$ is an extension of T_0), and T_0 is Cesàro ergodic with $\mathscr{N}(I-T_0)=\{0\}$, we have

$$\overline{\mathscr{R}(I-\widetilde{T})}=\overline{\mathscr{R}(I-T_0)}=\overline{\mathscr{R}(I-T)}.$$

Thus $x_0 \in \overline{\mathscr{R}(I-\widetilde{T})}$ and $\{M_n^{(0)}(\widetilde{T})x_0\}$ is unbounded, which by the last statement of Remark 2.8 implies $\{n||M_n(\widetilde{T})y - Py||\}$ unbounded, where $y = (I-\widetilde{T})x_0$.

In conclusion, $\widetilde{T}$ satisfies the hypotheses of Theorem 2.6 in the case $p = 1$, but the assertion (ii) concerning the order of saturation of the process $\{M_n(\widetilde{T})y\}$ is not true for some $y \in \mathscr{R}(I-\widetilde{T})$. In fact, it is easy to see that $\widetilde{T}$ is Cesàro ergodic on $\mathscr{Y}$, namely $M_n(\widetilde{T})(x \oplus z) \to -z \oplus z$ $(n \to \infty)$ for $x, z \in \overline{\mathscr{R}(I-T)}$.

This example gives also a negative answer to a question of Butzer-Westphal in [1, p.1173], namely Theorem 1 [1] is not valid, in general, under the hypothesis of Cesàro ergodicity (even in the reflexive case).

REMARK 2.10. In the above results the hypothesis $||M_n^{(p-1)}(T)x|| = o(n)$ as $n \to \infty$ for $x \in \mathscr{X}$ can be omitted for T with $\sigma(T) \cap \mathbb{T} \subset \{1\}$, this being a consequence of the condition $||M_n^{(p)}(T)|| = O(1)$ as $n \to \infty$. This fact was proved in Theorem 2.2 [13], where even a stronger conclusion is obtained, namely that $||M_n^{(p)}(T)|| = O(1)$ implies $||M_n^{(p-1)}(T)|| = o(n)$, as $n \to \infty$, if $\sigma(T) \cap \mathbb{T} \subset \{1\}$.

In particular, if we choose T in Example 2.9 with $\sigma(T) = \{1\}$ such that 1 is not an eigenvalue of T, then $T = T_0$ and $\sigma(\widetilde{T}) = \{1\}$. In addition, $\widetilde{T}$ is Cesàro ergodic on $\mathscr{Y} = \mathscr{X} \oplus \mathscr{X}$, hence $\{M_n^{(p)}(\widetilde{T})\}$ strongly converges in $\mathscr{B}(\mathscr{Y})$, for any $p \geqslant 1$.

We can conclude from the above results, roughly speaking, that all Cesàro means of any order $p \geqslant 1$, have the same behavior from the point of view of saturation theory.

3. Connections with some results of Lin-Sine

Remark now that Theorem 2.2 before is a generalized version for Cesàro means of higher order of Theorem 1 [6] (see also [4]), the last being obtained from our theorem for the order $p = 1$ and T Cesàro ergodic. In fact, for $p = 1$ we infer from Theorem 2.2 the following result of Lin-Sine [6] (Corollary 3).

COROLLARY 3.1. *Let $T \in \mathscr{B}(\mathscr{X})$ be Cesàro bounded with $\frac{T^n}{n} \to 0$ strongly. The following statements are equivalent for $x \in \mathscr{X}$:*

(i) $x \in \mathscr{R}(I-T_0)$;

(ii) $x_n := \frac{1}{n+1}\sum_{j=0}^{n}\sum_{i=0}^{j} T^i x$ converges strongly;

(iii) $\{x_n\}$ has a weakly convergent subsequence.

Moreover, if $x \in \mathscr{R}(I-T_0)$ then $x = (I-T_0)(\lim_{n\to\infty} x_n)$.

Proof. We know that $x_n = S_n^{(2)}(T)x$, so we can apply Theorem 2.2 in the case $p = 1$. □

Another interesting case of Theorem 2.2 is given below.

COROLLARY 3.2. *Let $T \in \mathscr{B}(\mathscr{X})$ be such that $||M_n^{(2)}(T)|| = O(1)$ and $||M_n(T)x|| = o(n)$, as $n \to \infty$, for $x \in \mathscr{X}$. The following statements are equivalent for $x \in \mathscr{X}$:*

(i) $x \in \mathscr{R}(I-T_0)$*;*
(ii) $\widetilde{x}_n := \frac{2}{(n+1)(n+2)}\sum_{j=0}^{n}\sum_{i=0}^{j}\sum_{k=0}^{i}T^k x$ *converges strongly;*
(iii) $\{\widetilde{x}_n\}$ *has a weakly convergent subsequence.*
Moreover, if $x \in \mathscr{R}(I-T_0)$ *then* $x = (I-T_0)(\lim_{n\to\infty}\widetilde{x}_n)$.

Proof. We only have to show that $\widetilde{x}_n = S_n^{(3)}(T)x$ and to apply Theorem 2.2 in the case $p=2$. For this one has

$$(n+1)(n+2)S_n^{(3)}(T)x = \frac{(n+1)(n+2)(n+3)}{3}M_n^{(3)}(T)x$$
$$= \sum_{j=0}^{n}(n+1-j)(n+2-j)T^j x = (n+2)(n+1)I + (n+1)nT + \ldots + 2\cdot 1T^n$$
$$= \sum_{j=1}^{n+1}\sum_{i=0}^{j-1}(n+2-i)T^i x = \sum_{j=1}^{n+1}[(n+2-j)\sum_{i=0}^{j-1}T^i x + \sum_{i=0}^{j-1}(j-i)T^i x]$$
$$= \sum_{j=1}^{n+1}[(n+2-j)S_{j-1}^{(1)}(T)x + jS_{j-1}^{(2)}(T)x]$$
$$= \sum_{j=0}^{n}(n+1-j)S_j^{(1)}(T)x + \sum_{j=1}^{n+1}\sum_{i=1}^{j}\sum_{k=0}^{i-1}T^k x$$
$$= \sum_{j=1}^{n+1}\sum_{i=0}^{j-1}S_i^{(1)}(T)x + \sum_{j=1}^{n+1}\sum_{i=0}^{j-1}\sum_{k=0}^{i}T^k x = 2\sum_{j=0}^{n}\sum_{i=0}^{j}\sum_{k=0}^{i}T^k x,$$

having in view the expressions of $S_j^{(1)}(T)$ and $S_j^{(2)}(T)$ quoted in the beginning of Section 2. Thus we have $\widetilde{x}_n = S_n^{(3)}(T)x$. $\square$

In Corollary 2 [6] was shown that, in certain cases, the conditions of Corollary 3.1 are equivalent to the boundedness of $\{S_n^{(1)}(T)x\}$. The general result corresponding to Theorem 2.2, which also generalizes Butzer-Westphal's result [1] (for Cesàro averages) is the following

PROPOSITION 3.3. *Let* $T \in \mathscr{B}(\mathscr{X})$ *be with* $||M_n(T)|| = O(1)$ *as* $n\to\infty$*, and assume that* T^m *is weakly compact, for some* $m \geqslant 1$*. Then the three conditions of Theorem 2.2, for some* $x \in \mathscr{X}$ *and* $p \geqslant 2$*, are equivalent to the condition*
(iv) $||S_n^{(2)}(T)x|| = O(1)$ *as* $n\to\infty$.

Proof. If $x = (I-T)y$ with $y \in \mathscr{X}$ then, for $n \geqslant 1$ and $p \geqslant 2$, we have

$$||S_n^{(p)}(T)x|| = \frac{n+p}{n+1}||y - M_{n+1}^{(p-1)}(T)y|| \leqslant c(1+\sup_{j\geqslant 1}||M_j^{(p-1)}(T)||)||y||,$$

for some constant $c>1$ (depending of p). Hence $||S_n^{(p)}(T)x|| = O(1)$ as $n\to\infty$, and this particularly holds for $p=2$.

Conversely, suppose now (iv), that is $\sup_{n\geqslant 1}||S_n^{(2)}(T)x||<\infty$. From the proof of Corollary 3.2 and the expression of $S_n^{(2)}(T)$ we have

$$S_n^{(3)}(T)=\frac{2}{(n+1)(n+2)}\sum_{j=0}^{n}(j+1)S_j^{(2)}(T), \tag{3.1}$$

whence we infer that $\sup_{n\geqslant 1}||S_n^{(3)}(T)x||\leqslant \sup_{n\geqslant 1}||S_n^{(2)}(T)x||<\infty$.

Having in view that T^m is weakly compact, we obtain that the sequence $\{T^mS_n^{(3)}(T)x\}=\{S_n^{(3)}(T)T^mx\}$ is weakly sequentially compact, hence it has a weakly convergent subsequence. Since the assumption $||M_n(T)||=O(1)$ as $n\to\infty$ ensures that $||M_n^{(p)}(T)||=O(1)$ and $||M_n^{(p-1)}(T)y||=o(n)$, as $n\to\infty$, for $y\in\mathscr{X}$ and $p\geqslant 2$, from Theorem 2.2 (the case $p=2$) it follows that $T^mx\in\mathscr{R}(I-T_0)$. So $T^mx=(I-T_0)z$ with $z\in\mathscr{X}_0$, and if $y=z+\sum_{j=0}^{m-1}T^jx$ then $(I-T)y=x$. Since $\sup_{n\geqslant 1}||S_n^{(3)}(T)x||<\infty$ we have $x\in\mathscr{X}$ by Theorem 2.5, and also $y\in\mathscr{X}_0$. Finally, $x=(I-T_0)y$ so $x\in\mathscr{R}(I-T_0)$, and using an above remark we conclude that (iv) implies the (equivalent) conditions of Theorem 2.2, for $p\geqslant 2$. This ends the proof. □

REMARK 3.4. From relation (3.1) we infer that the boundedness of $\{S_n^{(2)}(T)\}$ implies the boundedness of $\{S_n^{(3)}(T)\}$. But from Theorem 2.5 it follows that $\sup_{n\geqslant 1}||S_n^{(p)}(T)x||<\infty$, which ensures that $\sup_{n\geqslant 1}||S_n^{(p+1)}(T)x||<\infty$, for any $p\geqslant 1$ and $x\in\mathscr{X}$. So, Proposition 3.3 remains valid if the Cesàro boundedness of T is replaced by the weaker hypothesis $||M_n^{(p-1)}(T)||=O(1)$ as $n\to\infty$, and with $S_n^{(p)}(T)$ instead of $S_n^{(2)}(T)$ in the condition (iv).

As in [6] (in the case $p=1$) we mention that (iv) does not imply the conditions of Theorem 2.2, for $p\geqslant 2$, in general. We see this fact in the following example, which shows also that (iv) is not equivalent to the convergence of $\{S_n^{(2)}(T)x\}$.

EXAMPLE 3.5. Let $T\in\mathscr{B}(\mathscr{X})$ be Cesàro bounded which is not Cesàro ergodic, with $\sigma(T)=\{1\}$. Therefore $\mathscr{X}_0\subsetneqq\mathscr{X}$ and let $x_0\in\mathscr{X}$, $x_0\notin\mathscr{X}_0$, and $\widehat{\mathscr{X}}=\vee_{n\geqslant 0}T^nx$ (the closed linear manifold), where $x=(I-T)x_0$. We put $\widehat{T}=T|_{\widehat{\mathscr{X}}}$.

By Remark 2.10, one has $\frac{1}{n}T^n\to 0$ strongly on $\mathscr{X}$, so $M_n(T)x\to 0$ as $n\to\infty$, and since $\sup_{n\geqslant 1}||M_n(T)||<\infty$ we infer $M_n(\widehat{T})y\to 0$ as $n\to\infty$ for all $y\in\widehat{\mathscr{X}}$. Hence $\widehat{T}$ is Cesàro ergodic on $\widehat{\mathscr{X}}$ and we have $\widehat{\mathscr{X}}=\widehat{\mathscr{X}_0}=\overline{\mathscr{R}(I-\widehat{T})}$, $\widehat{T}=\widehat{T}_0$. Now, assuming $x\in\mathscr{R}(I-\widehat{T})$ we get $x_1\in\widehat{\mathscr{X}}$ with $x=(I-\widehat{T})x_1$. Then $(I-T)(x_0-x_1)=0$ and we obtain

$$M_n(T)x_0=M_n(T)(x_0-x_1)+M_n(T)x_1\to x_0-x_1, \quad n\to\infty,$$

and this contradicts the choice of x_0. Hence $x\notin\mathscr{R}(I-\widehat{T})$.

On the other hand, we have

$$S_n^{(2)}(\widehat{T})x=S_n^{(2)}(T)x=\frac{n+2}{n+1}(x_0-M_{n+1}(T)x_0),$$

and it follows that $||S_n^{(2)}(\widehat{T})x|| = O(1)$ as $n \to \infty$. Having in view the choice of x_0, we infer that $\{S_n^{(2)}(\widehat{T})x\}$ is not convergent. We conclude that $\widehat{T}$ satisfies the condition (iv) of Proposition 3.3, but the conditions of Theorem 2.2 are not satisfied for $\widehat{T}$ on x, for arbitrary $p \geqslant 2$.

Notice that Corollary 2.3 gives an information concerning the norm convergence of $\{S_n^{(p)}(T)x\}$, for $x \in \mathscr{X}$.

Next we refer to the weak convergence of this sequence. We have the following

THEOREM 3.6. *Let $T \in \mathscr{B}(\mathscr{X})$ be such that $||M_n^{(p-1)}(T)|| = O(1)$ as $n \to \infty$, and assume that T^m is weakly compact, for some $p, m \geqslant 1$. Then $\{S_n^{(p)}(T)x\}$ weakly converges for some $x \in \mathscr{X}$ if and only if $M_n^{(p-1)}(T)x \to 0$ $(n \to \infty)$ weakly and*

$$\limsup_{n\to\infty} ||S_n^{(p)}(T)x|| < \infty. \tag{3.2}$$

Moreover, if $S_n^{(p)}(T)x \to y$ $(n \to \infty)$ weakly, then $x = (I-T)y$.

Proof. Suppose $S_n^{(p)}(T)x \to y$ weakly. This leads to the relation

$$(I-T)y = w - \lim_{n\to\infty} S_n^{(p)}(T)(I-T)x = w - \lim_{n\to\infty} \frac{n+p}{n+1}(x - M_{n+1}^{(p-1)}(T)x)$$

and we infer that $M_n^{(p-1)}(T)x \to x - (I-T)y =: z$ weakly. Using (1.3) we get

$$(I-T)z = w - \lim_{n\to\infty} \left(M_n^{(p-1)}(T)x - \frac{n+p}{n+1} M_{n+1}^{(p-1)}(T)x + \frac{p-1}{n+1}x \right) = 0,$$

that is $z \in \mathscr{N}(I-T)$. So $x = (I-T)y + z \in \mathscr{X}_0$ and because $\{S_n^{(p)}(T)x\}$ is bounded (being weakly convergent), by Theorem 2.5 it follows that $x \in \mathscr{R}(I-T_0)_{\mathscr{X}_0}$. Thus $z \in \overline{\mathscr{R}(I-T)} \cap \mathscr{N}(I-T)$ hence $z = 0$. We conclude that $M_n^{(p-1)}(T)x \to 0$ weakly and the condition (3.2) holds by the boundedness of $\{S_n^{(p)}(T)x\}$.

Conversely, assume that $M_n^{(p-1)}(T)x \to 0$ weakly and that (3.2) is satisfied. This implies that there exists a subsequence $\{n_k\}$ of positive integers such that $\sup_{k\geqslant 1} ||S_{n_k}^{(p)}(T)x|| < \infty$. Since T^m is weakly compact, the sequence $\{S_{n_k}^{(p)}(T)T^m x\}$ is sequentially weakly compact, so there exists a subsequence $\{k_i\} \subset \{n_k\}$ such that $S_{k_i}^{(p)}(T)T^m x \to y$ weakly as $i \to \infty$. Then the limit y will be a norm limit of convex combinations y_q of the form

$$y_q = \sum_{j=1}^{m_q} \lambda_j^{(q)} S_{k_j}^{(p)}(T)T^m x, \quad \lambda_j^{(q)} \geqslant 0, \quad \sum_{j=1}^{m_q} \lambda_j^{(q)} = 1.$$

Let $\varepsilon > 0$ and $q_0 \geqslant 1$ such that $||y_{q_0} - y|| < \varepsilon$. For any $f \in \mathscr{X}^*$ and $n \geqslant 1$ we obtain

$$
\begin{aligned}
|f(M_n^{(p-1)}(T)y)| &\leqslant |f(M_n^{(p-1)}(T)(y-y_{q_0}))| + |f(M_n^{(p-1)}(T)y_{q_0})| \\
&\leqslant \varepsilon||f|| \sup_{j\geqslant 1} ||M_j^{(p-1)}(T)|| + \sum_{j=1}^{m_{q_0}} \lambda_j^{(q_0)} |(f \circ S_{k_j}^{(p)}(T)T^m)(M_n^{(p-1)}(T)x)|.
\end{aligned}
$$

So, using the fact that $M_n^{(p-1)}(T)x \to 0$ weakly, we infer

$$
\limsup_{n\to\infty} |f(M_n^{(p-1)}(T)y)| \leqslant \varepsilon||f|| \sup_{j\geqslant 1} ||M_j^{(p-1)}(T)||.
$$

Hence $\lim_{n\to\infty} f(M_n^{(p-1)}(T)y) = 0$ for $f \in \mathscr{X}^*$, that is $M_n^{(p-1)}(T)y \to 0$ weakly.

Let now $z = y + \sum_{j=0}^{m-1} T^j x$, therefore $(I-T)z = x$. We have

$$
\begin{aligned}
S_n^{(p)}(T)x &= \frac{n+p}{n+1}(z - M_{n+1}^{(p-1)}(T)z) \\
&= \frac{n+p}{n+1}(z - M_{n+1}^{(p-1)}(T)y - \sum_{j=0}^{m-1} T^j M_{n+1}^{(p-1)}(T)x),
\end{aligned}
$$

whence it follows that $S_n^{(p)}(T)x \to z$ weakly as $n \to \infty$. This ends the proof. $\square$

The assumption that T^m is weakly compact was used only in the "if" part of Theorem 3.6.

The case $p = 1$ of this theorem gives Corollary 4 [6], and the cases $p = 2$ and $p = 3$ are mentioned below.

COROLLARY 3.7. *Let $T \in \mathscr{B}(\mathscr{X})$ be Cesàro bounded with T^m weakly compact for some $m \geqslant 1$. The following are equivalent for $x \in \mathscr{X}$:*

(i) $x_n = \frac{1}{n+1}\sum_{j=0}^{n}\sum_{i=0}^{j} T^i x$ converges weakly;

(ii) $M_n(T)x \to 0$ weakly, and $\limsup_{n\to\infty} ||x_n|| < \infty$.

In this case, we have $x = (I-T)z$, z being the limit of (i).

COROLLARY 3.8. *Let $T \in \mathscr{B}(\mathscr{X})$ be with $||M_n^{(2)}(T)|| = O(1)$, $n \to \infty$ and with T^m weakly compact, for some $m \geqslant 1$. The following are equivalent, for $x \in \mathscr{X}$:*

(i) $\widetilde{x}_n = \frac{2}{(n+1)(n+2)}\sum_{j=0}^{n}\sum_{i=0}^{j}\sum_{k=0}^{i} T^i x$ converges weakly;

(ii) $M_n^{(2)}(T)x \to 0$ weakly, and $\limsup_{n\to\infty} ||\widetilde{x}_n|| < \infty$.

In this case, we have $x = (I-T)z$, z being the limit of (i).

Concerning the remark before Example 3.5, we can see now that if $\mathscr{X}$ is a dual space then the condition (iv) of Proposition 3.3 on some T and x is equivalent to $x \in \mathscr{R}(I-T)$, a weaker condition than $x \in \mathscr{R}(I-T_0)$ from Theorem 2.2.

In fact, we have the following general result. As usually, for $T \in \mathscr{B}(\mathscr{X})$ we denote by $T^* \in \mathscr{B}(\mathscr{X}^*)$ the adjoint of T on the dual space $\mathscr{X}^*$ of $\mathscr{X}$.

THEOREM 3.9. *Let* $T \in \mathscr{B}(\mathscr{X})$ *be such that* $||M_n^{(p-1)}(T)|| = O(1)$ *as* $n \to \infty$, *for some* $p \geqslant 1$. *The following statements are equivalent, for* $f \in \mathscr{X}^*$:

(*i*) $f \in \mathscr{R}(I - T^*)$;

(*ii*) $||S_n^{(p)}(T^*)f|| = O(1)$ *as* $n \to \infty$;

(*iii*) $||S_n^{(q)}(T^*)f|| = O(1)$ *as* $n \to \infty$, *for all* $q \geqslant p$.

Moreover, if the sequence $\{M_n^{(p-1)}(T^*)\}$ *converges strongly in* $\mathscr{B}(\mathscr{X}^*)$, *for some* $p \geqslant 2$, *then the conditions* $(i) - (iii)$ *are also equivalent to*

(*iv*) $\{S_n^{(q)}(T^*)f\}$ *converges in the norm of* $\mathscr{X}^*$, *for* $q \geqslant p$.

In this last case, we have $f = (I - T^*)g$, *where* g *is the limit of* (iv).

Proof. The hypothesis on T implies $||M_n^{(q)}(T^*)|| = O(1)$ as $n \to \infty$, for $q \geqslant p - 1$. Suppose (i), so $f = (I - T^*)g$ with $g \in \mathscr{X}^*$. Then, for $q \geqslant p$ we have

$$S_n^{(q)}(T^*)f = \frac{n+q}{n+1}(g - M_{n+1}^{(q-1)}(T^*)g),$$

and it follows that $||S_n^{(q)}(T^*)f|| = O(1)$ as $n \to \infty$, that is (iii).

Assume now (ii). Then $\{S_n^{(p)}(T^*)f\}$ is bounded and, because of weak*-compactness, this sequence has a subnet $S_{n_k}^{(p)}(T^*)f \to g$ in the weak*-topology of $\mathscr{X}^*$, for some $g \in \mathscr{X}^*$. So, for $x \in \mathscr{X}$ we have

$$\begin{aligned}[(I - T^*)g](x) &= g((I - T)x) = \lim[S_{n_k}^{(p)}(T^*)f]((I - T)x) \\ &= \lim(S_{n_k}^{(p)}(T^*)(I - T^*)f)(x) = \lim \frac{n_k + p}{n_k + 1}(f - M_{n_k+1}^{(p-1)}(T^*)f)(x),\end{aligned}$$

whence we infer that $M_{n_k+1}^{(p-1)}(T^*)f \to f - (I - T^*)g =: h$ in the weak*-topology. But using (1.3) we get $(I - T^*)M_{n_k+1}^{(p-1)}(T^*)f \to 0$ in the weak*-topology, hence $(I - T^*)h = 0$, that is $h \in \mathscr{N}(I - T^*)$. But the assumption (ii) implies by Theorem 2.5 that $f \in \overline{\mathscr{R}(I - T^*)}$, hence $h = f - (I - T^*)g \in \overline{\mathscr{R}(I - T^*)}$, and so $h = 0$. Thus $f = (I - T^*)g$ which means (i). Trivially (iii) implies (ii), and we conclude that all conditions (i), (ii) and (iii) are equivalent.

Now, if $\{M_n^{(p-1)}(T^*)\}$ converges strongly in $\mathscr{B}(\mathscr{X}^*)$, for some $p \geqslant 2$, then $\mathscr{X}^* = \mathscr{X}_0^*$ and $T = T_0^*$. We have also $\{\frac{1}{n}M_n^{(p-2)}(T^*)\}$ converges strongly, and applying Theorem 2.2 to T^* we conclude that (i) is equivalent to (iv), in this case. □

The case $p = 1$ of Theorem 3.9 is just Theorem 5 [6] (see also [5]) obtained for power bounded operators. Our version in the case of Cesàro bounded operators, that is the case $p = 2$, is the following

COROLLARY 3.10. *Let* $T \in \mathscr{B}(\mathscr{X})$ *be Cesàro bounded. The following are equivalent for* $f \in \mathscr{X}^*$:

(*i*) $f \in \mathscr{R}(I - T^*)$;

(*ii*) $\sup_{n \geqslant 1} ||\frac{1}{n+1}\sum_{j=0}^{n}\sum_{i=0}^{j} T^{*j} f|| < \infty$.

If T is Cesàro ergodic then these conditions are equivalent to the norm convergence of the sequence from (ii) and, in this case, the limit g of this sequence satisfies $(I-T^*)g = f$.

Since Cesàro boundedness (respectively, ergodicity) of T^* ensures the boundedness (the strong convergence) of $\{M_n^{(p-1)}(T^*)\}$ for all $p \geqslant 2$, we have, in fact, that the conditions of Corollary 3.10 are equivalent to those of Theorem 3.9, for all $p \geqslant 2$ (respectively).

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Laurian Suciu
Department of Mathematics and Informatics
"Lucian Blaga" University of Sibiu
Dr. Ion Raţiu 5–7
Sibiu, 550012, Romania
e-mail: laurians2002@yahoo.com