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HYPONORMAL TRIGONOMETRIC TOEPLITZ OPERATORS

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Abstract. We investigate hyponormal Toeplitz operators T_ϕ with trigonometric polynomial symbols ϕ via the Carathéodory-Schur Interpolation Problem. We present several formulae for computing the rank of the selfcommutator $[T_\phi^*, T_\phi]$ in the cases where T_ϕ is a hyponormal operator. In addition we consider the hyponormal extension problem of Toeplitz operators.

Introduction

In 1988, C. Cowen [3] characterized the hyponormality of Toeplitz operators T_ϕ on the Hardy space $H^2(\mathbb{T})$ of the unit circle $\mathbb{T}$. This theorem makes it possible to answer an algebraic question coming from operator theory by studying the function ϕ itself. K. Zhu [22] noticed that for the cases of trigonometric polynomials ϕ, this algebraic question is exactly the Carathéodory–Schur Interpolation Problem. Indeed the Carathéodory–Schur Interpolation Problem can be carried out to obtain substancial informations about hyponormal Toeplitz operators with trigonometric polynomial symbols. The goal of the present paper is to investigate the recent development on the study for hyponormal Toeplitz operators with trigonometric polynomial symbols and to present some new results on the rank of the self-commutator. Our approach emphasizes the use of the Carathéodory–Schur Interpolation Problem, and some proofs are done in a simpler way.

A bounded linear operator A on a complex Hilbert space $\mathscr{H}$ is called hyponormal if $[A^*, A] = A^*A - AA^* \geqslant 0$. Given $\phi \in L^\infty(\mathbb{T})$, the operator T_ϕ on the Hardy space $H^2(\mathbb{T})$ of the unit circle $\mathbb{T}$ defined by $T_\phi f = P(\phi \cdot f)$ (where $f \in H^2(\mathbb{T})$ and P denotes the orthogonal projection from $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$) is called the *Toeplitz operator with symbol* ϕ. The characterization of hyponormality in [3] requires one to solve a certain functional equation in the unit ball of $H^\infty(\mathbb{T})$. Suppose that $\phi \in L^\infty(\mathbb{T})$ is arbitrary and consider the following subset of the closed unit ball of $H^\infty(\mathbb{T})$:

$$\mathscr{E}(\phi) = \{k \in H^\infty(\mathbb{T}) : ||k||_\infty \leqslant 1 \text{ and } \phi - k\overline{\phi} \in H^\infty(\mathbb{T})\}. \tag{1}$$

Cowen's theorem states that T_ϕ is hyponormal if and only if $\mathscr{E}(\phi)$ is nonempty ([3], [19]). The hyponormality of Toeplitz operators has been studied by many authors (cf.

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Paper OaM-07-31

[2], [3], [4], [6], [7], [8], [12], [13], [14], [15], [16], [17], [19], [22] and etc.). K. Zhu [22] showed that the problem of finding a solution in $\mathscr{E}(\phi)$ is related to the classical interpolation problem so called Carathéodory–Schur Interpolation Problem and obtained an abstract characterization of those trigonometric polynomial symbols that correspond to hyponormal Toeplitz operators.

In Section 1 we investigate the recent development on the study for the hyponormality of *trigonometric Toeplitz operators*, i.e., Toeplitz operators with trigonometric polynomial symbols. In particular we focus on the relationship between hyponormality of trigonometric Toeplitz operators and Carathéodory–Schur Interpolation Problem. In Section 2 we derive several methods to compute the rank of the selfcommutator of hyponormal trigonometric Toeplitz operators. In Section 3, we consider the hyponormal extension problem of Toeplitz operators.

1. Basic properties

If ϕ is a trigonometric polynomial of the form $\phi(z) = \sum_{n=-m}^{N} a_n z^n$, where a_{-m} and a_N are nonzero, then the nonnegative integers N and m denote the analytic and co-analytic degrees of ϕ. If a function $k \in H^\infty$ satisfies $\phi - k\overline{\phi} \in H^\infty$, then we have

$$k \sum_{n=1}^{N} \overline{a_n} z^{-n} - \sum_{n=1}^{m} a_{-n} z^{-n} \in H^\infty . \tag{2}$$

If we write the Fourier coefficient $\widehat{k}(n) := c_n$, for $n = 0, 1, \ldots, N-1$, then by (2), $c_0, \ldots, c_{N-1}$ are determined uniquely from the coefficients of ϕ as follows: $c_0 = c_1 = \cdots = c_{N-m-1} = 0$ and

$$\begin{pmatrix} \overline{c_{N-m}} \\ \overline{c_{N-m+1}} \\ \vdots \\ \overline{c_{N-1}} \end{pmatrix} = \begin{pmatrix} a_{N-m+1} & a_{N-m+2} & \ldots & a_{N-1} & a_N \\ a_{N-m+2} & a_{N-m+3} & \ldots & a_N & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_N & 0 & \ldots & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} \overline{a_{-1}} \\ \overline{a_{-2}} \\ \vdots \\ \overline{a_{-m}} \end{pmatrix}. \tag{3}$$

The function $k_p(z) := \sum_{j=N-m}^{N-1} c_j z^j$ is the unique analytic polynomial of degree less than N satisfying $\phi - k\overline{\phi} \in H^\infty$. Thus the problem of finding a solution in $\mathscr{E}(\phi)$ is to find a function k in the closed unit ball of H^∞ interpolating k_p.

On the other hand, it was shown in [19] that if T_ϕ is a hyponormal operator such that its selfcommutator is of finite rank then $\mathscr{E}(\phi)$ contains a finite Blaschke product whose degree is exactly the rank of the selfcommutator $[T_\phi^*, T_\phi]$.

LEMMA 1.1. (Nakazi-Takahashi's Theorem) [19] *A Toeplitz operator T_ϕ is hyponormal and the rank of the selfcommutator $[T_\phi^*, T_\phi]$ is finite (e.g., ϕ is a trigonometric polynomial) if and only if there exists a finite Blaschke product $k \in \mathscr{E}(\phi)$ such that* $\deg(k) = \operatorname{rank}[T_\phi^*, T_\phi]$, *where* $\deg(k)$ *denotes the degree of k.*

If ϕ is a trigonometric polynomial then there are several conditions that ϕ must necessarily satisfy in order for T_ϕ to be a hyponormal operator.

LEMMA 1.2. (Conditions Necessary for Hyponormality) [8] *Suppose that* ϕ *is a trigonometric polynomial of the form* $\phi(z)=\sum_{n=-m}^{N} a_n z^n$, *where* a_{-m} *and* a_N *are nonzero. If* T_ϕ *is hyponormal then* $m \leqslant N$, $|a_{-m}| \leqslant |a_N|$, *and* $N-m \leqslant \operatorname{rank}[T_\phi^*, T_\phi] \leqslant N$.

Lemma 1.2 shows that the cases where $|a_{-m}|=|a_N|$ are, in some sense, extremal among all possibilities for hyponormality. In the below we treat such cases and the result will show, passing to the normality that one further feature, namely a symmetry property of the Fourier coefficients, is present. In general if $\phi(z)=\sum_{n=-m}^{N} a_n z^n$, then the hyponormality of T_ϕ is independent of the particular values of the Fourier coefficients $a_0, a_1, \cdots, a_{N-m}$ of ϕ.

We can have more:

LEMMA 1.3. [4, Lemma 1.5] *Suppose that* ϕ *is a trigonometric polynomial such that* $\phi := \overline{g}+f$, *where* f *and* g *are in* $H^\infty(\mathbb{T})$. *If* $\psi := \overline{g}+T_{\overline{z}^r} f$ $(r \leqslant N-m)$ *then* T_ϕ *is hyponormal if and only if* T_ψ *is.*

Lemma 1.3 shows that if $\phi(z)=\sum_{n=-m}^{N} a_n z^n$, then the analytic part of ϕ, $\sum_{n=0}^{N} a_n z^n$, can be "pulled back" to $\sum_{n=0}^{m} a_{N-m+n} z^n$ when studying the hyponormality of T_ϕ. For example, if $\phi(z)=a_{-1}z^{-1}+\sum_{n=0}^{N} a_n z^n$ and $\psi(z)=a_{-1}z^{-1}+a_N z$ then T_ϕ is hyponormal if and only if T_ψ is.

LEMMA 1.4. (Normality of T_ϕ) [7] *If* $\phi(z)=\sum_{n=-m}^{N} a_n z^n$ *then* T_ϕ *is normal if and only if* $m=N$, $|a_{-N}|=|a_N|$, *and*

$$\overline{a_N}\begin{pmatrix} a_{-1} \\ a_{-2} \\ \vdots \\ a_{-N} \end{pmatrix} = a_{-N}\begin{pmatrix} \overline{a_1} \\ \overline{a_2} \\ \vdots \\ \overline{a_N} \end{pmatrix}.$$

Proof. We here give a direct proof. By the Brown-Halmos theorem [1] T_ϕ is normal if and only if there are scalars $\alpha, \beta \in \mathbb{C}$ and a real-valued $\psi \in L^\infty$ such that $T_\phi = \alpha T_\psi + \beta I$. Hence T_ϕ is normal if and only if $m=N$ and

$$a_n = \begin{cases} \alpha\, \hat{\psi}(n) & \text{for } n=1,\cdots,N \\ \alpha\, \overline{\hat{\psi}(n)} & \text{for } n=-1,\cdots,-m, \end{cases}$$

or equivalently, $a_{-j}=e^{i\theta}\overline{a_j}$ for $j=1,\cdots,N$ and some fixed $\theta \in [0,2\pi)$. This gives the desired result. $\square$

In view of Lemma 1.2 the following is a characterization of hyponormality of trigonometric Toeplitz operators for an extremal case.

LEMMA 1.5. (Extremal Cases of ϕ) [7] *Suppose that* $\phi(z)=\sum_{n=-m}^{N}a_nz^n$, *where* $|a_{-m}|=|a_N|$. *Then* T_ϕ *is hyponormal if and only if the following equation holds:*

$$\overline{a_N}\begin{pmatrix}a_{-1}\\a_{-2}\\\vdots\\a_{-m}\end{pmatrix}=a_{-m}\begin{pmatrix}\overline{a_{N-m+1}}\\\overline{a_{N-m+2}}\\\vdots\\\overline{a_N}\end{pmatrix}.$$

Furthermore if T_ϕ *is hyponormal then the rank of* $[T_\phi^*,T_\phi]$ *is* $N-m$.

Although a proof was given in [7, Lemma 1.4] we give here a simple proof.

Proof. Suppose that T_ϕ is hyponormal and let $c_0,\cdots,c_{N-1}$ be the solution to the recurrence relation (3). In view of Lemma 1.1 we assume that $k(z)=\sum_{j=0}^{\infty}c_jz^j$ is a finite Blaschke product in $\mathscr{E}(\phi)$. Since $c_0=\cdots=c_{N-m-1}=0$, k is of the form

$$k(z)=e^{i\theta}z^{N-m}\prod_{j=1}^{r}\frac{z-\beta_j}{1-\overline{\beta_j}z}\qquad(r\leqslant m,\ 0<|\beta_j|<1).$$

But since $\frac{a_{-m}}{\overline{a_N}}=c_{N-m}=e^{i\theta}\prod_{j=1}^{r}(-\beta_j)$, it follows that $\prod_{j=1}^{r}|\beta_j|=|\frac{a_{-m}}{\overline{a_N}}|$. Since by our assumption $|a_{-m}|=|a_N|$, we can see that $k(z)=e^{i\theta}z^{N-m}$. By the Cowen's theorem, $a_{-j}=e^{i\theta}\overline{a_{N-m+j}}$ for $j=1,\cdots,m$. The converse follows at once from Theorems 1.3 and 1.4. The second assertion follows at once from Lemma 1.1 together with the fact that $k(z)=e^{i\theta}z^{N-m}$. $\square$

In the sequel, without loss of generality we may assume $m=N$ when we consider the hyponormality of T_ϕ. We now turn our attention to the relationship between the hyponormality of trigonometric Toeplitz operators and a finite interpolation problem.

From the preceding argument we can see that if $k(z)=\sum_{j=0}^{\infty}c_jz^j$ is a function in H^∞, then $\phi-k\overline{\phi}\in H^\infty$ if and only if $c_0,\cdots,c_{N-1}$ are given by (3). So by the Cowen's theorem, if $c_0,\cdots,c_{N-1}$ are given by (3) then the hyponormality of T_ϕ is equivalent to the existence of a function $k\in H^\infty$ satisfying

(i) $\widehat{k}(j)=c_j,\quad j=0,\cdots,N-1;$

(ii) $||k||_\infty\leqslant 1$.

This is exactly the classical interpolation theorem so called *Carathéodory–Schur Interpolation Problem* (CSIP). Thus the problem of hyponormality for T_ϕ reduces to CSIP. CSIP is analyzed by Schur numbers. We review here Schur's algorithm. Suppose that $k(z)=\sum_{j=0}^{\infty}c_jz^j$ is in the closed unit ball of H^∞. Let $k_0:=k$. Define by induction a sequence $\{k_n\}$ of functions in the closed unit ball of H^∞ as follows:

$$k_{n+1}(z)=\frac{k_n(z)-k_n(0)}{z\left(1-\overline{k_n(0)}k_n(z)\right)},\qquad |z|<1,\ n=0,1,2,\cdots.$$

Then $k_n(0)$ only depends on the coefficients $c_0, c_1, \cdots, c_n$. We write

$$k_n(0) = \Phi_n(c_0, \cdots, c_n) \qquad (n = 0, 1, 2, \cdots),$$

where Φ_n is a function of $n+1$ complex variables. We call the Φ_n's *Schur's functions* and the $|\Phi_n|$'s *Schur numbers*. Then CSIP is solvable if and only if $|\Phi_n(c_0, \cdots, c_n)| \leqslant 1$ for every $n = 0, 1, \cdots, N-1$ (cf. [20], [9]). Therefore if $\phi(z) = \sum_{n=-N}^{N} a_n z^n$ and if $c_0, \cdots, c_{N-1}$ are given by (3), then one can at once see that the following statements are equivalent (cf. [22]):

1. T_ϕ is a hyponormal operator.
2. $|\Phi_n(c_0, \cdots, c_n)| \leqslant 1$ for every $n = 0, 1, \cdots, N-1$.

By a straightforward calculation we can see that

$$\Phi_0(c_0) = c_0, \quad \Phi_1(c_0, c_1) = \frac{c_1}{1 - |c_0|^2}, \quad \text{and} \quad \Phi_2(c_0, c_1, c_2) = \frac{c_2(1 - |c_0|^2) + \overline{c_0} c_1^2}{(1 - |c_0|^2)^2 - |c_1|^2}.$$

Thus for example, if $\phi(z) = \sum_{n=-2}^{2} a_n z^n$ then T_ϕ is hyponormal if and only if $|c_1| \leqslant 1 - |c_0|^2$ or equivalently, $\left|\det\begin{pmatrix} \frac{a_{-1}}{\overline{a_1}} & \frac{a_{-2}}{\overline{a_2}} \end{pmatrix}\right| \leqslant |a_2|^2 - |a_{-2}|^2$ (cf. [6], [22]). However, with trigonometric polynomials of higher degree, the above criterion would be too complicated to be of much value because no closed-form for Schur's function Φ_n is known.

On the other hand, CSIP can be analyzed by a matricial argument (cf. [20]): CSIP is solvable if and only if the Toeplitz matrix

$$C := \begin{pmatrix} c_0 & 0 & 0 & \dots & 0 \\ c_1 & c_0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ c_{N-2} & & \ddots & \ddots & 0 \\ c_{N-1} & c_{N-2} & \dots & c_1 & c_0 \end{pmatrix}$$

is a contraction, i.e., $||C|| \leqslant 1$. Today this result is also called the Carathéodory-Fejér theorem. A proof of this result can be accomplished by means of the commutant lifting theorem (cf. [9], [11]). In particular, k is a solution of CSIP if and only if the Toeplitz operator T_k with symbol k is a contractive lifting of C which commutes with the unilateral shift on ℓ_2. We thus have that T_ϕ is a hyponormal operator if and only if the Toeplitz matrix C above is a contraction.

The hyponormality of trigonometric Toeplitz operators T_ϕ can be also determined by zeros of an analytic polynomial induced by ϕ. This was done in the cases where ϕ is a circulant polynomial [8] or where $z^N\phi$ satisfies the condition that the set $\{\zeta,\ 1/\overline{\zeta} :$ ζ and $1/\overline{\zeta}$ are zeros of $z^N\phi\}$ contains at least $(N+1)$ elements [14]. In [16], this was accomplished for the general polynomial symbols ϕ. The main idea runs as follows. Let f be an anlytic polynomial of the form $f(z) = \sum_{j=0}^{N} b_j z^j$ with $b_N = 1$. Then $g := \frac{f}{z^N \overline{f}} \in \mathscr{E}(\phi)$ if and only if (i) g satisfies the interpolation $\widetilde{g}(j) = c_j$ for $j = 0, \cdots, N-1$, where $\widetilde{g}(j)$ denotes the j-th Taylor coefficient for g and the c_j are given in (3); and (ii) $g \in H^\infty$. Then a straightforward calculation and a simplication shows that the first condition (i) is equivalent to the condition that if we let H denote the block Hankel matrix given by

$$H := \begin{pmatrix} 0 & \dots & \dots & 0 & A_0 \\ \vdots & & 0 & A_0 & A_1 \\ \vdots & . & . & . & \vdots \\ 0 & A_0 & \cdot & & A_{N-3} \\ A_0 & A_1 & \dots & A_{N-3} & A_{N-2} \end{pmatrix}, \quad \text{where } A_j := \begin{pmatrix} \operatorname{Re} c_j & \operatorname{Im} c_j \\ \operatorname{Im} c_j & -\operatorname{Re} c_j \end{pmatrix} \quad (j = 0, \cdots, N-2)$$

and let $V := (\operatorname{Re} c_1, \operatorname{Im} c_1, \operatorname{Re} c_2, \operatorname{Im} c_2, \cdots, \operatorname{Re} c_{N-1}, \operatorname{Im} c_{N-1}) \in \mathbb{R}^{2N-2}$, then the linear system

$$(I - H)X^T = V^T \tag{4}$$

(I is the identity matrix of degree $2N-2$ and the unknown is $X \in \mathbb{R}^{2N-2}$) is solvable. Also the second condition (ii) is equivalent to the condition that if f denote the analytic polynomial

$$f(z) := c_0 + \sum_{j=1}^{N-1} (x_j + i y_j) z^j + z^N, \tag{5}$$

where $X^T := (x_1, y_1, x_2, y_2, \cdots, x_{N-1}, y_{N-1})^T$ is a solution of the system (4) then for every zero ζ of f such that $|\zeta| > 1$, the number $1/\overline{\zeta}$ is a zero of f in the open unit disk $\mathbb{D}$ of multiplicity greater than or equal to the multiplicity of ζ. In fact the latter condition is equivalent to the condition that $\frac{f}{z^N \overline{f}}$ is a finite Blaschke product.

We can now summarize criteria for the hyponormality of trigonometric Toeplitz operators.

LEMMA 1.6. *Suppose $\phi(z) = \sum_{n=-N}^{N} a_n z^n$, where a_N is nonzero and that $c_0, \cdots, c_{N-1}$ are given by (3). Then the following statements are equivalent.*

1. *T_ϕ is a hyponormal operator.*

2. $|\Phi_n(c_0, \cdots, c_n)| \leqslant 1$ *for* $n = 0, 1, \cdots, N-1$.

3. $C = \begin{pmatrix} c_0 & 0 & 0 & \dots & 0 \\ c_1 & c_0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ c_{N-2} & & \ddots & \ddots & 0 \\ c_{N-1} & c_{N-2} & \dots & c_1 & c_0 \end{pmatrix}$ *is a contraction.*

4. $I - CC^* \geqslant 0$.

5. *The linear system* $(I - H)X^T = V^T$ *in* (4) *is solvable and if f is given by* (5) *then for every zero ζ of f such that $|\zeta| > 1$, the number $1/\overline{\zeta}$ is a zero of f in the open unit disk $\mathbb{D}$ of multiplicity greater than or equal to the multiplicity of ζ.*

6. *If f is given by* (5) *then* $\frac{f}{z^N\overline{f}}$ *is a finite Blaschke product such that* $\operatorname{rank}[T_\phi^*, T_\phi] = \deg\left(\frac{f}{z^N\overline{f}}\right)$.

It would be interesting to compare the third criterion and the fifth criterion. The former involves the norm of an $m\times m$ Toeplitz matrix C (and in turn, eigenvalues of C^*C). By comparison, the latter involves the zeros of an analytic polynomial of degree N induced by the values of entries of C. So the load of working with each criterion is on a par.

EXAMPLE 1.7. Consider the trigonometric polynomial

$$\phi(z) = -2z^{-4} + 9z^{-3} - 12z^{-2} + 4z^{-1} - 2z^2 + 9z^3 - 12z^4 + 4z^5.$$

Observe that $c_0 = 0$ and

$$\begin{pmatrix} \overline{c_1} \\ \overline{c_2} \\ \overline{c_3} \\ \overline{c_4} \end{pmatrix} = \begin{pmatrix} -2 & 9 & -12 & 4 \\ 9 & -12 & 4 & 0 \\ -12 & 4 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ -12 \\ 9 \\ -2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{3}{4} \\ \frac{3}{8} \\ \frac{3}{16} \end{pmatrix}.$$

First we use the criterion (5) of Lemma 1.6 to determine the hyponormality of T_ϕ. Observe

$$I_6 - H = \begin{pmatrix} 1 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & \frac{3}{2} & 0 & -\frac{3}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{3}{4} \\ \frac{1}{2} & 0 & -\frac{3}{4} & 0 & \frac{5}{8} & 0 \\ 0 & -\frac{1}{2} & 0 & \frac{3}{4} & 0 & \frac{11}{8} \end{pmatrix} \quad \text{and} \quad V^T = \begin{pmatrix} \frac{3}{4} \\ 0 \\ \frac{3}{8} \\ 0 \\ \frac{3}{16} \\ 0 \end{pmatrix}.$$

Since $\operatorname{rank}[I_6 - H] = 4 = \operatorname{rank}[I_6 - H : V^T]$, the system $(I_6 - H)X^T = V^T$ is solvable. If $X := (x_1, y_1, x_2, y_2, x_3, y_3)$ then a solution of this system is given by $x_1 = 1$, $x_2 = 0$, $x_3 = -\frac{1}{2}$, $y_1 = y_2 = y_3 = 0$. Thus the testing polynomial f is obtained by

$$f(z) = -\frac{1}{2} + z - \frac{1}{2}z^3 + z^4,$$

which has zeros at $z = \frac{1}{2}, -1, (-1)^{\frac{1}{3}}, -(-1)^{\frac{2}{3}}$. Therefore by the criterion (5), T_ϕ is hyponormal. Next we use the criterion (3) of Lemma 1.6. Write

$$C = \begin{pmatrix} -\frac{1}{2} & 0 & 0 & 0 \\ \frac{3}{4} & -\frac{1}{2} & 0 & 0 \\ \frac{3}{8} & \frac{3}{4} & -\frac{1}{2} & 0 \\ \frac{3}{16} & \frac{3}{8} & \frac{3}{4} & -\frac{1}{2} \end{pmatrix}.$$

To determine the hyponormality of T_ϕ we will check the contractivity of C. Recall that the norm of C is the largest singular value of C, i.e., $||C|| = \max\{\sqrt{\lambda} : \lambda \text{ is an eigenvalue of } C^*C\}$. A straightforward calculation shows that eigenvalues of C^*C are $\frac{1}{256}, 1$, and so $||C|| = 1$. Therefore by the criterion (3), T_ϕ is hyponormal.

EXAMPLE 1.8. Consider the trigonometric polynomial

$$\phi(z) = z^{-4} + z^{-3} + 2z^{-2} + z^2 + 2z^3 + 2z^4.$$

Observe

$$\begin{pmatrix} \overline{c_0} \\ \overline{c_1} \\ \overline{c_2} \\ \overline{c_3} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 & 2 \\ 1 & 2 & 2 & 0 \\ 2 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{3}{4} \\ -\frac{3}{4} \end{pmatrix}.$$

First we use the criterion (5) of Lemma 1.6. Observe

$$I_6 - H = \begin{pmatrix} 1 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{3}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{7}{4} \end{pmatrix} \quad \text{and} \quad V^T = \begin{pmatrix} 0 \\ 0 \\ \frac{3}{4} \\ 0 \\ -\frac{3}{4} \\ 0 \end{pmatrix}.$$

Then a straightforward calculation shows that rank $[I_6 - H] = 5 \neq 6 = \text{rank}\,[I_6 - H : V^T]$. Thus the system Thus the system (5) $(I_6 - H)X^T = V^T$ has no solution, and hence by the criterion (5), T_ϕ is not hyponormal. Next we use the criterion (3) of Lemma 1.6. Write

$$C = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ \frac{3}{4} & 0 & \frac{1}{2} & 0 \\ -\frac{3}{4} & \frac{3}{4} & 0 & \frac{1}{2} \end{pmatrix}.$$

Then a straightforward calculation shows that the largest singular value of C is approximately 1.39299, and so $||C|| \approx 1.39299$. Therefore by the criterion (3), T_ϕ is not hyponormal.

2. The set $\mathscr{E}(\phi)$ and rank of the selfcommutator $[T_\phi^*, T_\phi]$

If T_ϕ is a hyponormal operator then $\mathscr{E}(\phi)$ is nonempty. Further if $[T_\phi^*, T_\phi]$ is of finite rank then by the Nakazi-Takahashi theorem, $\mathscr{E}(\phi)$ contains a finite Blaschke product whose degree is equal to rank$[T_\phi^*, T_\phi]$. To see more informations on $\mathscr{E}(\phi)$, we review here the Carathéodory's theorem (cf. [10, Theorem I.2.1]) which states that for every function k in the closed unit ball of H^∞ there exists a sequence $\{B_n\}$ of finite Blaschke products that converges to $k(z)$ pointwise on $\mathbb{D}$. Its proof relies upon a construction of a sequence $\{B_n\}$ of finite Blaschke products satisfying that if $k(z) = \sum_{j=0}^\infty c_j z^j$ is in the closed unit ball of $H^\infty(\mathbb{T})$ then

$$\widehat{B_n}(j) = c_j \quad \text{for } j = 0, \cdots, n.$$

The construction runs as follows. Write Φ_n for the n-th Schur's function corresponding to the function k. Since $|\Phi_0| = |c_0| \leqslant 1$, we can take $B_0 := \frac{z+\Phi_0}{1+\overline{\Phi_0}z}$. If $|\Phi_0| = 1$ then

$B_0 = c_0$ is the Blaschke product such that $B_0 = k$. Write $B_0^{(0)} := B_0$. If $|\Phi_j| < 1$ for $j = 0, \cdots, n$, let

$$B_n^{(0)} := \frac{z + \Phi_n}{1 + \overline{\Phi_n} z}$$

and define by induction

$$B_n^{(j)} := \frac{z B_n^{(j-1)} + \Phi_{n-j}}{1 + \overline{\Phi_{n-j}} z B_n^{(j-1)}} \quad (j = 1, \cdots, n).$$

Set $B_n := B_n^{(n)}$. Then B_n satisfies the interpolation $\widehat{B_n}(j) = c_j$ for $j = 0, \cdots, n$. If $|\Phi_n| = 1$, then B_n is the finite Blaschke product such that $B_n = k$. This will be referred to the *Carathéodory construction*. In particular a careful analysis on the Carathéodory construction shows that if $|\Phi_n| < 1$ then $\deg(B_n) = n + 1$, and if instead $|\Phi_n| = 1$ then $\deg(B_n) = n$ (also see [16]).

We also recall the connection between Hankel and Toeplitz operators. For ϕ in $L^\infty(\mathbb{T})$, the *Hankel operator* $H_\phi : H^2 \to H^2$ is defined by $H_\phi f = J(I - P)(\phi f)$, where $J : (H^2)^\perp \to H^2$ is given by $J z^{-n} = z^{n-1}$ for $n \geqslant 1$. The following is a basic connection:

$$T_{\phi\psi} - T_\phi T_\psi = H_{\widetilde{\phi}}^* H_\psi \ \ (\phi, \psi \in L^\infty) \quad \text{and} \quad H_{\phi\psi} = T_{\widetilde{h}}^* H_\phi \ \ (h \in H^\infty),$$

where for $\zeta \in L^\infty(\mathbb{T})$, we define $\widetilde{\zeta} = \overline{\phi(\overline{z})}$. From this we can see that if $k \in \mathscr{E}(\phi)$ then

$$[T_\phi^*, T_\phi] = H_{\overline{\phi}}^* H_{\overline{\phi}} - H_\phi^* H_\phi = H_{\overline{\phi}}^* H_{\overline{\phi}} - H_{k\overline{\phi}}^* H_{k\overline{\phi}} = H_{\overline{\phi}}^* (1 - T_{\widetilde{k}} T_{\widetilde{k}}^*) H_{\overline{\phi}}, \tag{6}$$

which implies that $\ker H_{\overline{\phi}} \subseteq \ker [T_\phi^*, T_\phi]$.

We now have informations on $\mathscr{E}(\phi)$:

LEMMA 2.1. *Let $\phi \in L^\infty(\mathbb{T})$ be such that T_ϕ is a hyponormal operator. Then we have:*

1. *If $\phi(z) = \sum_{n=-N}^N a_n z^n$ ($a_N \neq 0$) is such that* $\operatorname{rank}[T_\phi^*, T_\phi] < N$ *then $\mathscr{E}(\phi)$ has exactly one element, which is a finite Blaschke product.*

2. *If $\phi(z) = \sum_{n=-N}^N a_n z^n$ ($a_N \neq 0$) is such that* $\operatorname{rank}[T_\phi^*, T_\phi] = N$ *then $\mathscr{E}(\phi)$ contains infinitely many elements which are finite Blaschke products.*

3. *If ϕ is not of bounded type ('bounded type' means quotient of two bounded analytic functions) then $\mathscr{E}(\phi)$ has exactly one element.*

Proof. The assertion (1) follows at once from [18, Corollary 5]. For the assertion (2), we suppose $\operatorname{rank}[T_\phi^*, T_\phi] = N$. Then there exists a Blaschke product $k \in \mathscr{E}(\phi)$ of degree N, namely $k = B_{N-1}$ in the Carathéodory construction. Therefore by the preceding considerations on the Carathéodory construction we have $|\Phi_{N-1}| < 1$: indeed if $|\Phi_{N-1}| = 1$, then there would exist a Blaschke product $b \in \mathscr{E}(\phi)$ of degree $N - 1$, which leads a contradiction by the uniqueness property (cf. [16, Lemma 1]) that if k and b are Blaschke product of, respectively, degrees m and r such that $m+r<2n$ and satisfy the finite interpolation $\widehat{k}(j)=\widehat{b}(j)$ for $j=0,\cdots,n-1$, then $k=b$. Therefore we can choose Φ_N so that $|\Phi_N|<1$, satisfying the interpolation $\widehat{B_N}(j)=c_j$ for $j=0,\cdots,N-1$. Then B_N is a Blaschke product in $\mathscr{E}(\phi)$ of degree $N+1$. Continuing this process one can get a sequence of Blaschke products B_n $(n=N-1,N,\cdots)$ in $\mathscr{E}(\phi)$ of degree n. For the assertion (3), we write $\phi=\overline{g}+f$ $(f,g\in H^\infty)$. Then by the Cowen's theorem we can easily see that $k\in\mathscr{E}(\phi)$ if and only if $||k||_\infty\leqslant 1$ and $H_{\overline{f}}k=\overline{z}\widetilde{g}$ (cf. [4], [5, Lemma 1]). But if ϕ is not of bounded type then $\ker H_{\overline{f}}=\ker H_{\overline{\phi}}=\{0\}$ (cf. [1, Lemma 3]), so that the solution k of the equation $H_{\overline{f}}k=\overline{z}\widetilde{g}$ is unique. This proves statement (3). $\square$

Note that if ϕ is of bounded type and is not a trigonometric polynomial then we have no informations on $\mathscr{E}(\phi)$. To see this consider the function

$$\phi(z)=z^{-1}+zb(z),\quad \text{where } b(z)=\frac{z-\frac{1}{2}}{1-\frac{1}{2}z}.$$

Then ϕ is of bounded type. We now claim that $\mathscr{E}(\phi)$ has exactly one element k, namely $k=b$. Indeed if $h\in\mathscr{E}(\phi)$ then $z^{-1}-hz^{-1}\overline{b}\in H^\infty$ and so $z^{-1}(1-h\overline{b})\in H^\infty$. Thus $h\overline{b}\in 1+zH^\infty$ and $||h\overline{b}||_\infty\leqslant 1$, which implies $h=b$. On the other hand, condisder the function

$$\psi(z)=\frac{1}{6}z^{-1}+\sum_{n=2}^{\infty}\frac{z^n}{2^{n-1}}.$$

If k and b are defined by

$$k(z)=\frac{1}{3}\frac{z-\frac{1}{2}}{1-\frac{1}{2}z}\quad\text{and}\quad b(z)=\frac{(z-\frac{1}{2})(z+\frac{1}{3})}{(1-\frac{1}{2}z)(1+\frac{1}{3}z)}$$

then a straightforward calculation shows that $k,b\in\mathscr{E}(\psi)$. Note that b is a Blaschke product, whereas k is not. Therefore $\mathscr{E}(\psi)$ contains at least two elements which includes a finite Blaschke product.

We now derive several formulae for the rank of the selfcommutator $[T_\phi^*,T_\phi]$ in the cases where T_ϕ is a hyponormal operator

THEOREM 2.2. *Suppose that $\phi(z)=\sum_{n=-m}^{N}a_nz^n$, where a_{-m} and a_N are nonzero, is such that T_ϕ is hyponormal. If $c_0,\cdots,c_{N-1}$ are given by (3) then the following statements are equivalent.*

1. $\operatorname{rank}[T_\phi^*,T_\phi]=r$.
2. *There exists a finite Blaschke product $k\in\mathscr{E}(\phi)$ of degree r.*
3. $|\Phi_r(c_0,\cdots,c_r)|=1$ *if* $r\leqslant N-1$; $|\Phi_{N-1}(c_0,\cdots,c_{N-1})|<1$ *if* $r=N$.

4. rank $(I-CC^*) = r$.

5. *If f is given by (5) with m in place of N and $\phi = \overline{g} + T_{\overline{z}^{N-m}}h$ in place of $\phi = \overline{g} + h$ ($g, h \in H^\infty$), then*
$$r = N - m + Z_{\mathbb{D}} - Z_{\mathbb{C}\setminus\overline{\mathbb{D}}},$$
where $Z_{\mathbb{D}}$ and $Z_{\mathbb{C}\setminus\overline{\mathbb{D}}}$ are the number of zeros of f in $\mathbb{D}$ and in $\mathbb{C}\setminus\overline{\mathbb{D}}$ counting multiplicity.

Proof. (1) $\Leftrightarrow$ (2): This is Lemma 1.1.

(2) $\Rightarrow$ (3): Suppose the function k is a finite Blaschke product in $\mathscr{E}(\phi)$ of degree r. If $r \leqslant N-1$, we assume to the contrary that $|\Phi_r| < 1$. Then by the Carathéodory construction there exists a Blaschke product $b \in \mathscr{E}(\phi)$ such that $\deg(b) = r+1$. Then by Lemma 2.1, we have $k = b$, a contradiction. If instead $r = N$ then we assume to the contrary that $|\Phi_{N-1}| = 1$. then again by the Carathéodory construction there exists a Blaschke product $b \in \mathscr{E}(\phi)$ of degree $N-1$. This leads a contradiction.

(3) $\Rightarrow$ (2): Immediate from the Carathéodory construction.

(2) $\Leftrightarrow$ (4): This follows from Lemma 2.1 together with an argument of S. Takahashi [21, Theorem], which states that if $I - CC^* \geqslant 0$ then there exists a finite Blaschke product whose degree is equal to the rank of $I - CC^*$.

(1) $\Leftrightarrow$ (5): Observe that if
$$\phi(z) := \sum_{n=-m}^{N} a_n z^n \quad \text{and} \quad \psi(z) := \sum_{n=-m}^{-1} a_n z^n + \sum_{n=0}^{m} a_{N-m+n} z^n,$$
then $\operatorname{rank}[T_\phi^*, T_\phi] = N - m + \operatorname{rank}[T_\psi^*, T_\psi]$ and that if f is given by (5) corresponding to $\psi(z)$ then
$$\operatorname{rank}[T_\psi^*, T_\psi] = \deg\left(\frac{f}{z^m \overline{f}}\right) = Z_{\mathbb{D}} - Z_{\mathbb{C}\setminus\mathbb{D}}.$$
Thus we can conclude that $\operatorname{rank}[T_\phi^*, T_\phi] = N - m + Z_{\mathbb{D}} - Z_{\mathbb{C}\setminus\overline{\mathbb{D}}}$. $\square$

3. A hyponormal extension problem

In this section we consider an extension problem. To do this we need:

LEMMA 3.1. *Suppose ϕ is a trigonometric polynomial such that $\phi = f + \overline{g}$, where f and g are analytic polynomials of degree N. Let*
$$g_0 := T_{\overline{z}^m} g, \quad f_0 := T_{\overline{z}^m} f, \quad \textit{and} \quad \psi := f_0 + \overline{g_0} \ (m < N).$$
If T_ϕ is hyponormal then T_ψ is hyponormal.

Proof. Suppose T_ϕ is hyponormal. By the Cowen's theorem there exists a function $k \in H^\infty$ such that $||k||_\infty \leqslant 1$ and $\overline{g} - k\overline{f} \in H^\infty$. Thus $z^m\overline{g} - kz^m\overline{f} \in H^\infty$, and hence $\overline{g_0} - k\overline{f_0} \in H^\infty$, which implies that T_ψ is hyponormal. $\square$

If ϕ is given as in Lemma 3.1 then we cannot, however, expect that $T_{\tilde{\phi}}$ is hyponormal, where $\tilde{\phi} = z^m f + \overline{z^m g}$. For example, if $\phi(z) = z^{-2} + z^{-1} + 4z + 2z^2$ and $\tilde{\phi} = z^{-3} + z^{-2} + 4z^2 + 2z^3$, then a straightforward calculation shows that T_ϕ is hyponormal, but $T_{\tilde{\phi}}$ is not.

In view of Lemma 3.1, if $\phi = f + \overline{g}$ $(f \in H^2,\ g \in zH^2)$ is a trigonometric polynomial then it seems to be natural that a *hyponormal extension* of T_ϕ is defined by a Toeplitz operator $T_{\tilde{\phi}}$ with the symbol $\tilde{\phi}$ of the form

$$\tilde{\phi} = \overline{z^m g} + \overline{q} + p + z^m f,$$

where p and q are analytic polynomials of degree $m-1$. Therefore the hyponormal extension problem is equivalent to the following completion problem: If $\phi = \overline{z^m g} + \overline{q} + p + z^m f$ (where f and g are analytic polynomials, and p and q are analytic polynomials of degree $m-1$), find necessary and sufficient conditions, in terms of the coefficients of p and q, for T_ϕ to be hyponormal when $T_{\overline{g}+f}$ is hyponormal. In general, for each p, there are many polynomials q for which T_ϕ is hyponormal when $T_{\overline{g}+f}$ is hyponormal.

We however have:

THEOREM 3.2. *Suppose that $\phi := \overline{z^m g} + \overline{q} + p + z^m f$, where f and g are analytic polynomials of degree N, and p and q are analytic polynomials of degree $m-1$. Let $\psi := \overline{g} + f$. If T_ψ is a hyponormal operator then for each polynomial p, there exists a polynomial q for which T_ϕ is hyponormal. In particular, if rank$[T_\psi^*, T_\psi] < N - m$, then q is unique.*

Proof. If T_ψ is hyponormal then by the Cowen's theorem there exists $k \in \mathscr{E}(\phi)$, i.e., $\overline{g} - k\overline{f} \in H^\infty$ and $||k||_\infty \leqslant 1$. Write $k(z) = \sum_{n=0}^{\infty} c_n z^n$. If $p = \sum_{n=1}^{m-1} a_n z^n$ is given and we write $p + z^m f = \sum_{n=1}^{m+N} a_n z^n$, define a_{-j} $(1 \leqslant j \leqslant m-1)$ by

$$\begin{pmatrix} \overline{a_{-1}} \\ \overline{a_{-2}} \\ \vdots \\ \overline{a_{-m+1}} \end{pmatrix} := \begin{pmatrix} \overline{a_1} & \dots & \dots & \overline{a_{m-1}} & \dots & \overline{a_{m+N}} \\ \vdots & & \cdot & & \cdot & 0 \\ \vdots & \cdot & & \cdot & \cdot & \vdots \\ \overline{a_{m-1}} & \dots & \overline{a_{m+N}} & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{m+N-1} \end{pmatrix}.$$

Thus if we define $q(z) := \sum_{n=1}^{\infty} a_{-n} z^{-n}$ then a straightforward calculation shows that $\overline{z^m g} + \overline{q} - k(\overline{p} + \overline{z^m f}) \in H^\infty$.

For the uniqueness, let rank$[T_\psi^*, T_\psi] < N - m$. Assume (q_1, p) and (q_2, p) $(q_1 \neq q_2)$ are pairs of analytic polynomials of degree $m-1$ such that the symbols $\phi_1 = \overline{z^m g} + \overline{q_1} + p + z^m f$ and $\phi_2 = \overline{z^m g} + \overline{q_2} + p + z^m f$ make T_{ϕ_1} and T_{ϕ_2} hyponormal. Assume $k_1 \in \mathscr{E}(\phi_1)$ and $k_2 \in \mathscr{E}(\phi_2)$. Then evidently, $k_1 \neq k_2$, and $\overline{z^m g} + \overline{q_i} - k_i(\overline{p} + \overline{z^m f}) \in H^\infty$ $(i = 1, 2)$ and so $\overline{g} + z^m \overline{q_i} - k_i(z^m \overline{p} + \overline{f}) \in H^\infty$, and hence $\overline{g} - k_i\overline{f} \in H^\infty$. Thus $k_i, k_2 \in \mathscr{E}(\psi)$, which contradicts the assumption that rank$[T_\psi^*, T_\psi] < N - m$, which by Lemma 2.1, implies that $\mathscr{E}(\psi)$ has exactly one element. $\square$

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