Operators and Matrices
Volume 7, Number 3 (2013), 587–602
doi:10.7153/oam-07-32

EIGENVALUE MULTIPLICITIES FOR SECOND ORDER ELLIPTIC OPERATORS ON NETWORKS

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(*Communicated by A. C. M. Ran*)

Abstract. We present some general bounds for the algebraic and geometric multiplicity of eigenvalues of second order elliptic operators on finite networks under continuity and weighted Kirchhoff flow conditions at the vertices. In particular the algebraic multiplicity of an eigenvalue is shown to be strictly bounded from above by the number of vertices if there are no eigenfunctions vanishing in all nodes, and to be bounded from above by the number of edges if there are such eigenfunctions.

1. Introduction

The present paper deals with the algebraic and geometric eigenvalue multiplicities of second order elliptic edge operators

$$L_j = a_j\partial_j^2 + b_j\partial_j + q_j$$

on a finite network with arbitrary edge lengths under continuity condition and general weighted Kirchhoff flow conditions

$$\sum_{j=1}^{N} d_{ij}c_{ij}\partial_j u_j(v_i) + \rho_i u(v_i) = 0$$

at all vertices v_i. Results for the geometric multiplicity have been obtained in [1]–[4], [13], [14]–[16], and [19] for finite networks and in [5]–[8] for the infinite case. The algebraic multiplicities of all eigenvalues of the canonical Laplacian under weighted homogeneous Kirchhoff laws have been determined in [9]. They play a key role in the determination of the asymptotic behavior of the eigenvalues in the general case. More general classes of linear vertex transition conditions as Kuchment conditions e.a. that lead to a variational setting and to self–adjoint operators have been treated by many authors, see e.g. [3, 11, 18] and the references therein.

It has been shown in [2]–[4] that in the case of *consistent* Kirchhoff conditions, the eigenvalue problem corresponds to a S–hermitian boundary eigenvalue problem

Mathematics subject classification (2010): 34B45, 34L20, 05C50, 35J25, 34L10.
Keywords and phrases: Elliptic operators on networks, eigenvalue multiplicities, adjacency and transition operators.

Paper OaM-07-32

that leads to a Hilbert space approach with real eigenvalues and coincidence of geometric and algebraic eigenvalue multiplicity. Thus, in the *inconsistent* case, i.e. when the conductivities c_{ij} in the Kirchhoff flow condition cannot be adapted to the principal part of the elliptic edge operators evaluated at the nodes, nonreal eigenvalues and multiplicity disparity can occur. We note in passing that by suitable tensor products of circuits of length 3 nonreal eigenvalues of arbitrarily high geometric and algebraic multiplicity can be found, see [7].

The present paper is organized as follows. After some graph theoretical preliminaries in Section 2, some basic upper bounds for the geometric eigenvalue multiplicity are presented for general elliptic edge operators of the form $L_j = a_j\partial_j^2 + b_j\partial_j + q_j$ in Section 3. The transition at the vertices is governed by a Kirchhoff flow condition (2) and by the continuity condition at ramification nodes (1). In Section 4 the adjacency calculus developed in [1, 4, 6] for weighted Laplacians is extended to general elliptic operators of second order. In particular, this calculus enables to deduce that the algebraic multiplicity of an eigenvalue is bounded from above either by the number of vertices minus 1 if there are no eigenfunctions vanishing in all nodes, or by the number of edges if there are such eigenfunctions, see Theorem 4.3. In Section 5 we recall that on trees, the algebraic and geometric multiplicities always coincide by showing that the operator becomes hermitian with respect to a suitable scalar product. Finally, some examples are presented in Section 6, in particular to illustrate the optimality of some of the established upper bounds.

2. Graphs and networks

For any graph $\Gamma = (V, E, \in)$, the vertex set is denoted by $V = V(\Gamma)$, the edge set by $E = E(\Gamma)$ and the incidence relation by $\in \subset V \times E$. The valency of each vertex v is denoted by $\gamma(v) = \mathrm{card}\{e \in E \,|\, v \in e\}$. Unless otherwise stated, all graphs considered in this paper are assumed to be nonempty, simple, connected and finite with

$$n = \#V, \qquad N = \#E.$$

The simplicity property means that Γ contains no loops, and at most one edge can join two vertices in Γ. By definition, a *circuit* is a connected and regular graph of valency 2. Number the vertices by $v_1, \ldots, v_n$, the respective valencies by $\gamma_1, \ldots, \gamma_n$, and the edges by $e_1, \ldots, e_N$. The *adjacency matrix* $\mathscr{A}(\Gamma) = (e_{ih})_{n\times n}$ of the graph is defined by

$$e_{ih} = \begin{cases} 1 & \text{if } v_i \text{ and } v_h \text{ are adjacent in } \Gamma \\ 0 & \text{else} \end{cases}$$

Note that $\mathscr{A}(\Gamma)$ is indecomposable iff Γ is connected. By simplicity, any two adjacent vertices v_i and v_h determine uniquely the edge e_s joining them, and we can set

$$s(i,h) = \begin{cases} s & \text{if } e_s \cap V = \{v_i, v_h\}, \\ 1 & \text{otherwise.} \end{cases}$$

For further graph theoretical terminology we refer to [20], and for the algebraic graph theory to [10] and [12].

Moreover, we consider each graph as a connected *topological graph* in $\mathbb{R}^m$, i.e. $V(\Gamma) \subset \mathbb{R}^m$ and the edge set consists of a collection of Jordan curves

$$E(\Gamma) = \{\pi_j : [0,\ell_j] \to \mathbb{R}^m \,|\, 1 \leqslant j \leqslant N\}$$

with the following properties: Each support $e_j := \pi_j([0,\ell_j])$ has its endpoints in the set $V(\Gamma)$, any two vertices in $V(\Gamma)$ can be connected by a path with arcs in $E(\Gamma)$, and any two edges $e_j \neq e_h$ satisfy $e_j \cap e_h \subset V(\Gamma)$ and $\#(e_j \cap e_h) \leqslant 1$. The arc length parameter of an edge e_j is denoted by x_j. Unless otherwise stated, we identify the graph $\Gamma = (V, E, \in)$ with its associated *network*

$$G = \bigcup_{j=1}^{N} \pi_j([0,\ell_j]),$$

especially each edge π_j with its support e_j. G is called a $\mathscr{C}^2$-network, if all $\pi_j \in \mathscr{C}^2([0,\ell_j],\mathbb{R}^m)$. Thus, endowed with the induced topology G is a connected and compact space in $\mathbb{R}^m$. We shall distinguish the *boundary vertices* $V_b = \{v_i \in V \,|\, \gamma_i = 1\}$ from the *ramification nodes* $V_r = \{v_i \in V \,|\, \gamma_i \geqslant 2\}$. The orientation of the graph Γ is given by the *incidence matrix* $\mathscr{D}(\Gamma) = (d_{ik})_{n \times N}$ with

$$d_{ij} = \begin{cases} 1 & \text{if } \pi_j(\ell_j) = v_i, \\ -1 & \text{if } \pi_j(0) = v_i, \\ 0 & \text{otherwise.} \end{cases}$$

For a function $u : G \to \mathbb{C}$ we set $u_j := u \circ \pi_j : [0,\ell_j] \to \mathbb{C}$ and use the abbreviations

$$u_j(v_i) := u_j(\pi_j^{-1}(v_i)), \quad \partial_j u_j(v_i) := \frac{\partial}{\partial x_j} u_j(x_j)\Big|_{\pi_j^{-1}(v_i)} \quad \text{etc.}$$

3. Vertex transition conditions and elliptic edge operators

As the basic geometric transition condition at ramification nodes we impose the *continuity condition*

$$\forall v_i \in V_r : \ e_j \cap e_s = \{v_i\} \implies u_j(v_i) = u_s(v_i), \tag{1}$$

that clearly is contained in the condition $u \in \mathscr{C}(G)$. Moreover, at all vertices we impose a weighted generalized Kirchhoff flow condition

$$\sum_{j=1}^{N} d_{ij} c_{ij} \partial_j u_j(v_i) + \rho_i u(v_i) = 0 \quad \text{for} \quad 1 \leqslant i \leqslant n \tag{2}$$

with weights $c_{ij} > 0$ and potential terms $\rho_i \in \mathbb{R}$. Note that this nonhomogeneous condition does not depend on the orientation. The validity of (2) in a function space will be indicated by the subscript GK.

On each edge we consider an elliptic differential operator of the form

$$L_j = a_j \partial_j^2 + b_j \partial_j + q_j \tag{3}$$

with continuous real coefficients a_j, b_j and q_j, where

$$a_j \geqslant \delta > 0 \quad \text{for all} \quad 1 \leqslant j \leqslant N \tag{4}$$

with some constant δ. Sometimes, it will be useful to consider on each edge the operator L_j in its formally self-adjoint form leading to the equivalent eigenvalue equation

$$\frac{1}{r_j} \partial_j \left(p_j \partial_j u_j \right) + q_j u_j = -\lambda u_j \tag{5}$$

on the same interval $[0, \ell_j]$ with

$$p_j(x_j) = \eta_j \exp\left(\int_0^{x_j} \frac{b_j(\xi_j)}{a_j(\xi_j)} d\xi_j \right), \quad r_j(x_j) = \frac{p_j}{a_j} \tag{6}$$

and with some parameter $\eta_j > 0$. Then *consistency* of the Kirchhoff conditions (2) means that, by a suitable parameter choice, each weight c_{ij} coincides with $p_j(v_i)$.

All together the L_j define the operator

$$L = \left(u \mapsto (L_j u_j)_{N \times 1} \right) : \mathscr{C}^2_{GK}(G) \to \prod_{j=1}^{N} \mathscr{C}[0, \ell_j] \tag{7}$$

on the $\mathscr{C}^2$–network G with the domain

$$\mathscr{C}^2_{GK}(G) = \{ u \in \mathscr{C}(G) | \forall j \in \{1, \ldots, N\} : u_j \in \mathscr{C}^2([0, \ell_j]), u \text{ satisfies } (2) \}.$$

Note that a corresponding weak setting leads to a sufficiently high regularity due to classical regularity results in one dimension. Thus, working in spaces of continuous functions does not constitute an essential restriction.

The main concern of our investigation are upper bounds for the algebraic multiplicity $m_a(\lambda)$ of the eigenvalues λ of $-L$ in $\mathscr{C}^2_{GK}(G)$. The eigenvalue problem in question reads

$$0 \neq u \in \mathscr{C}^2_{GK}(G) \quad \text{and} \quad L_j u_j = -\lambda u_j \quad \text{for} \quad 1 \leqslant j \leqslant N. \tag{8}$$

Recall that the *geometric multiplicity* $m_g(\lambda)$ of an eigenvalue $\lambda \in \mathbb{C}$ is defined by $m_g(\lambda) = \dim_{\mathbb{C}} \ker \left(L + \lambda I_{\mathscr{C}^2_{GK}(G)} \right)$, while its *algebraic multiplicity* $m_a(\lambda)$ is defined as

$$m_a(\lambda) = \dim_{\mathbb{C}} E^c(\lambda), \quad E^c(\lambda) := \ker \left(L + \lambda I_{\mathscr{C}^2_{GK}(G)} \right)^{\kappa}$$

with

$$\kappa = \kappa(\lambda) = \min\left\{k \in \mathbb{N} \,\middle|\, \ker\left(L + \lambda I_{\mathscr{C}^2_{GK}(G)}\right)^{k+1} = \ker\left(L + \lambda I_{\mathscr{C}^2_{GK}(G)}\right)^{k}\right\}.$$

The elements of $\ker\left(L + \lambda I_{\mathscr{C}^2_{GK}(G)}\right)^{k}$ are called *principal functions of order* k belonging to λ. Note that the kernel sequence becomes stationary since L has some compact resolvent, see e.g. [17]. Moreover,

$$L(E^c(\lambda)) \subset E^c(\lambda) \subset \mathscr{C}^2_{GK}(G). \tag{9}$$

In order to show the second inclusion, suppose that $u \in E^c(\lambda)$. If u is an eigenfunction, then $Lu \in \mathscr{C}^2_{GK}(G) \cap \ker\left(L + \lambda I_{\mathscr{C}^2_{GK}(G)}\right)$. By induction assume that $Lu, L^2u, \ldots, L^{k-1}u \in \mathscr{C}^2_{GK}(G) \cap E^c(\lambda)$. Then

$$0 = \left(L + \lambda I_{\mathscr{C}^2_{GK}(G)}\right)^{k} u = L^k u + \sum_{h=0}^{k-1} \binom{k}{h} \lambda^h L^{k-h} u,$$

which shows that $L^k u \in \mathscr{C}^2_{GK}(G) \cap E^c(\lambda)$ and (9).

Let $\Phi_j = \Phi_j(\cdot;\lambda) = \begin{pmatrix} \varphi_{j1} & \varphi_{j2} \\ \varphi'_{j1} & \varphi'_{j2} \end{pmatrix}$ denote the fundamental matrix associated to the first order system defined by the matrix $\begin{pmatrix} 0 & 1 \\ -\frac{q_j+\lambda}{a_j} & -\frac{b_j}{a_j} \end{pmatrix}$ on each k_j and satisfying $\Phi_j(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. By using the variation of constants formula, the solutions of the edge equation

$$L_j u_j + \lambda u_j = f_j \quad \text{with} \quad f_j \in \mathscr{C}[0, \ell_j] \tag{10}$$

are given by the formula

$$\begin{aligned} u_j(x_j) &= \varphi_{j1}(x_j)u_j(0) + \varphi_{j2}(x_j)\partial_j u_j(0) \\ &\quad + \int_0^{x_j} \frac{\varphi_{j2}(x_j)\varphi_{j1}(s) - \varphi_{j1}(x_j)\varphi_{j2}(s)}{\varphi_{j1}(s)\varphi'_{j2}(s) - \varphi'_{j1}(s)\varphi_{j2}(s)} \frac{f_j(s)}{a_j(s)}\, ds \\ &= \varphi_{j1}(x_j)u_j(0) + \varphi_{j2}(x_j)\partial_j u_j(0) \\ &\quad + \int_0^{x_j} \left(\varphi_{j2}(x_j)\varphi_{j1}(s) - \varphi_{j1}(x_j)\varphi_{j2}(s)\right) \frac{f_j(s)}{a_j(s)} \exp\left(\int_0^s \frac{b_j(\tau)}{a_j(\tau)} d\tau\right) ds. \end{aligned} \tag{11}$$

Clearly, prescribing $u_j(0)$ and $\partial_j u_j(0)$ determines uniquely the solution and leads to the following

LEMMA 3.1. *Suppose that* $\lambda \in \mathbb{C}$ *is not an eigenvalue of any* $-L_j$ *under* $0-$*Dirichlet boundary conditions on* $[0, \ell_j]$. *Then the dimension of the affine subspace* S *of* $\mathscr{C}^2(G)$ *defined by the functions* u *satisfying* (10) *on each edge for fixed* $f_j \in \mathscr{C}[0, \ell_j]$, *is given by the number of vertices* n. *In addition, those functions belonging to* S *that fulfill* (2) *form an affine subspace of dimension at most* $n-1$.

Proof. By construction all $\varphi_{j2}(0)=0$, thus, by hypothesis, each $\varphi_{j2}(\ell_j)\neq 0$, and each derivative $\partial_j u_j(0)$ is uniquely determined by $u_j(0), u_j(\ell_j), \varphi_{j1},\ \varphi_{j2}, f_j$ and by the coefficients of L_j. Under the continuity condition (1), the n values in the nodes determine uniquely the solution $u\in S$. As for Condition (2), choose some ramification node v_i of valency γ_i. Then the γ_i neighboring values are uniquely determined by the value in v_i and the γ_i derivatives in v_i. Among these, only γ_i-1 can be chosen freely under (2). □

As for the geometric multiplicity, we note first that $m_g(\lambda)\leqslant N$, since for an eigenfunction $u\in\mathscr{C}^2_{GK}(G)$, n among the N values of $u_1(0),\dots,u_N(0)$ determine all of them uniquely by (1). Moreover, at most N derivatives among the $2N$ ones can be chosen freely. The Kirchhoff condition (2) in turn implies that at each node at least one incident derivative is determined by the others and/or the value at the node. This reduces the maximal number of derivatives to choose freely to at most $N-n$.

Secondly, let us recall the optimal estimate for the geometric multiplicity given in [14]. For that purpose recall the construction of the parameter T of the graph Γ. If Γ has no bridges, then we put $T=2$. If Γ has bridges, then contract the connected components among the edges that are not bridges to single vertices and get a reduced tree. Then T denotes the number of boundary vertices of this tree.

THEOREM 3.2. ([14]) *The eigenvalues of* (8) *satisfy* $m_g(\lambda)\leqslant N-n+T$.

For the reader's convenience, a short proof will be given for trees in Lemma 5.2. In the case of nonreal eigenvalues this bound can be improved as follows.

THEOREM 3.3. *If λ is a nonreal eigenvalue of* (8)*, then*

$$m_g(\lambda)\leqslant N-n+1=\operatorname{corank}(\Gamma).$$

Proof. Suppose that there are $N-n+2$ or more linearly independent eigenfunctions. Then there is also an eigenfunction belonging to λ having zero derivatives at $m:=N-n+1$ arbitrary given points $p_1,\dots,p_m$ in the network G. Thus, as the dimension of the circuit space amounts to m, see [10], we can choose $p_1,\dots,p_m$ to be situated on suitable edge interiors of the circuits forming a basis of the circuit space of the graph such that, omitting these edges, the remaining graph is a forest. As a vanishing derivative at p_i corresponds to two Neumann boundary conditions at two boundary vertices identified with p_i, λ possesses an eigenfunction on a tree. But then λ must be real according to Theorem 5.1 below, which is impossible. □

On trees all eigenvalues are real under real coefficients, see Section 5. The smallest simple graph that displays nonreal eigenvalues for the canonical Laplacian is the circuit C_3 of equal lengths 1 with the Kirchhoff condition given by a row–stochastic matrix

$$\mathscr{C}=\begin{pmatrix}0 & c_{12} & c_{13}\\ c_{21} & 0 & c_{23}\\ c_{31} & c_{32} & 0\end{pmatrix}.$$

For $\det\mathscr{C} \leqslant \frac{1}{4}$, all eigenvalues are real, while for $\det\mathscr{C} > \frac{1}{4}$ nonreal eigenvalues occur, see [1]. Moreover, for $\det\mathscr{C} \neq \frac{1}{4}$ all algebraic multiplicities amount to 1. But for $\det\mathscr{C} = \frac{1}{4}$, the operator can be non symmetrizable. E.g. if $\mathscr{C}$ has the elements

$$\begin{pmatrix} 0 & \frac{1}{4} & \frac{3}{4} \\ \frac{2}{5} & 0 & \frac{3}{5} \\ \frac{1}{3} & \frac{2}{3} & 0 \end{pmatrix},$$

then $\mathscr{C}$ is not diagonalizable and has the simple eigenvalue 1 and the eigenvalue $-\frac{1}{2}$ with $m_g(-\frac{1}{2};\mathscr{C}) = 1$ and $m_a(-\frac{1}{2};\mathscr{C}) = 2$. The canonical Laplacian on C_3 has the eigenvalues satisfying $\cos\sqrt{\lambda} = -\frac{1}{2}$ with $m_g(\lambda) = 1$, $m_a(\lambda) = 2$ and $\ker(\Delta + \lambda I)^2 \cong \langle(-3,0,2)^t,(6,0,-4)^t\rangle_{\mathbb{R}}$.

4. The adjacency calculus

Following the transformations in [1, 4] the eigenvalue problem for L in question is equivalent to a matrix differential boundary eigenvalue problem incorporating the adjacency structure of the network. For that purpose we recall that the Hadamard product of matrices of the same size is defined as $(a_{ik})_{n\times n} \star (b_{ik})_{n\times n} = (a_{ik}b_{ik})_{n\times n}$. The vectors with constant entries equal to 1 are denoted by $\mathbf{e}$. Set $\rho = (\rho_i)_{n\times 1}$ and define the diagonal matrix having ρ as principal diagonal by

$$\mathrm{Diag}(\rho) = (\delta_{ik}\rho_i)_{n\times n}.$$

For a function $u : G \to \mathbb{C}$ denote its value distribution in the nodes by

$$\varphi = \mathbf{n}(u) = (u(v_i))_{n\times 1}. \tag{12}$$

For $x \in [0,1]$ define

$$\xi_{ih} = \ell_{s(i,h)}\left(\frac{1 + d_{is(i,h)}}{2} - x d_{is(i,h)}\right)$$

and the matrices

$$\mathbf{U}(x) = (u_{ih}(x))_{n\times n}, \qquad \mathbf{A}(x) = (a_{ih}(x))_{n\times n},$$
$$\mathbf{B}(x) = (b_{ih}(x))_{n\times n}, \qquad \mathbf{Q}(x) = (q_{ih}(x))_{n\times n},$$

the *length adjacency matrix* $\mathscr{L} = (\ell_{ih})_{n\times n}$, and the *adjacency conductivity matrix* $\mathscr{C} = (c_{ih})_{n\times n}$ by

$$u_{ih}(x) = e_{ih}u_{s(i,h)}(\xi_{ih}), \quad a_{ih}(x) = e_{ih}a_{s(i,h)}(\xi_{ih}), \quad b_{ih}(x) = e_{ih}b_{s(i,h)}(\xi_{ih}),$$
$$q_{ih}(x) = e_{ih}q_{s(i,h)}(\xi_{ih}), \quad \ell_{ih} = e_{ih}\ell_{s(i,h)}, \quad c_{ih} = e_{ih}c_{i\ s(i,h)},$$

respectively. Then the eigenvalue problem (8) reads:

$$u_{ih} \in C^2([0,1]) \text{ for all } i,h \in \mathbb{N} \tag{13}$$
$$e_{ih} = 0 \Rightarrow u_{ih} = 0 \text{ for all } i,h \in \mathbb{N} \tag{14}$$

$$\mathscr{L}^{(-2)} \star \mathbf{A}(x) \star \mathbf{U}'' + \mathscr{L}^{(-1)} \star \mathbf{B}(x) \star \mathbf{U}' + \mathbf{Q}(x) \star \mathbf{U} = -\lambda \mathbf{U} \text{ in } [0,1] \tag{15}$$

$$\mathbf{U}(0) = \varphi\, \mathbf{e}^* \star \mathscr{A} \quad \text{(continuity in } V_r(\Gamma)) \tag{16}$$

$$\mathbf{U}^*(x) = \mathbf{U}(1-x) \quad \text{for } x \in [0,1] \tag{17}$$

$$\left(\mathscr{C} \star \mathscr{L}^{(-1)} \star \mathbf{U}'(0)\right) \mathbf{e} + \operatorname{Diag}(\rho)\, \varphi = 0 \quad \text{(GK)} \tag{18}$$

Furthermore, introduce

$$\Phi := \mathbf{U}(0) = \varphi\, \mathbf{e}^* \star \mathscr{A}, \qquad \Psi := \mathbf{U}'(0)$$

and, using the fundamental solutions Φ_j on each edge, define

$$\Theta(x) = (\theta_{ih}(x))_{n\times n}, \quad \Sigma(x) = (\sigma_{ih}(x))_{n\times n}, \quad \mathbf{K}(x,s) = (\mathbf{k}_{ih}(x,s))_{n\times n}$$

by

$$\theta_{ih}(x) = e_{ih}\varphi_{s(i,h)1}(\xi_{ih}), \quad \sigma_{ih}(x) = e_{ih}\varphi_{s(i,h)2}(\xi_{ih})$$

and

$$\mathbf{k}_{ih}(x,s) = \frac{\sigma_{ih}(x)\theta_{ih}(s) - \theta_{ih}(x)\sigma_{ih}(s)}{a_{ih}(s)} \exp\left(\int_0^s \frac{b_{ih}(\tau)}{a_{ih}(\tau)} d\tau\right).$$

Using (11), the solution of (10) reads

$$\mathbf{U}(x) = \Phi \star \Theta(x) + \Psi \star \Sigma(x) + \int_0^x \mathbf{K}(x,s) \star F(s)\, ds \tag{19}$$

with $F(x) = \left(f_{s(i,h)}(\xi_{ih})\right)_{n\times n}$.

Before establishing a general upper bound for the algebraic multiplicity, we consider the case of the 0–Dirichlet condition at all vertices. For that purpose, a circuit ζ in Γ is said to be *compatible* with the operator L if ζ is the support of an eigenfunction of L belonging to $\mathscr{C}^2_{GK}(G) \cap \{u | \mathbf{n}(u) = 0\}$. Evidently, there is at most one independent eigenfunction vanishing at all nodes on the circuit ζ, since the eigenvalues under 0–Dirichlet condition on an interval are simple. E.g. for the canonical Laplacian (see 6.1) an odd circuit cannot be compatible for eigenvalues of the form $\cos\sqrt{\lambda} = -1$.

LEMMA 4.1. *If $\lambda \in \mathbb{C}$ is an eigenvalue of the problem*

$$0 \neq u \in \mathscr{C}^2(G) \cap \{u | \mathbf{n}(u) = 0\} \quad \textit{and} \quad L_j u_j = -\lambda u_j \quad \textit{for} \quad 1 \leqslant j \leqslant N, \tag{20}$$

then $\lambda \in \mathbb{R}$ and

$$m_a(\lambda) = m_g(\lambda) = N. \tag{21}$$

If, in addition, the Kirchhoff law (2) is imposed, then

$$N - n \leqslant m_a(\lambda) = m_g(\lambda) \leqslant \mathit{corank}(\Gamma) = N - n + 1. \tag{22}$$

Moreover, $m_a(\lambda) = m_g(\lambda) = N - n + 1$ holds if and only if the graph Γ contains only circuits that are compatible with L.

Proof. Using (5), the problem (20) corresponds to a selfadjoint one. This shows $\lambda \in \mathbb{R}$ and $m_a(\lambda) = m_g(\lambda)$ on Γ, as well as on any subgraph of Γ. Since the multiplicities on a single interval amount to 1, (21) is plain.

The Kirchhoff conditions (2) define n linear conditions whose rank amounts at least to $n-1 = \mathrm{rank}(\mathscr{D})$. Thus, at least $n-1$ of the N values are uniquely determined by the remaining ones. This shows the right inequality in (22).

As for the left inequality in (22), we reason by induction on the *Euler characteristic* $e := e(\Gamma) = N - n$. If $e = 0$, then the graph contains exactly one circuit and $m_g(\lambda) = 1$ or $m_g(\lambda) = 0$ according to whether the circuit is compatible or not. Next, suppose that $e > 0$. Let ζ be a circuit in Γ. For $k \in E(\zeta)$ define the subgraph Π_k by

$$E(\Pi_k) = E(\Gamma) \backslash \{k\} \quad \text{and} \quad V(\Pi_k) = V(\Gamma).$$

As Γ cannot be a circuit, and as no Π_k can be a tree, no edge of ζ can lie on all circuits of Γ. By induction

$$e(\Pi_k) = N - 1 - n \leqslant m_g(\lambda; L, \Pi_k).$$

If there is some eigenfunction $w \in \mathscr{C}^2_{GK}(G) \cap \{u | \mathbf{n}(u) = 0\}$ that does not vanish identically on some $k \in E(\zeta)$, then by simplicity of λ on k under 0–Dirichlet conditions,

$$E_\lambda(L;\Gamma) = \langle w \rangle \oplus E_\lambda(L;\Pi_k)$$

and

$$m_g(\lambda; L, \Gamma) = m_g(\lambda; L, \Pi_k) + 1 \geqslant N - 1 - n + 1 = N - n = e.$$

Thus we are led to the case that all eigenfunctions vanish on ζ, as well as on all other circuits of Γ, since ζ has been chosen arbitrarily. But this is impossible, since λ is supposed to be an eigenvalue of L in $\mathscr{C}^2_{GK}(G) \cap \{u | \mathbf{n}(u) = 0\}$, and since an eigenfunction has to vanish on edges incident to boundary vertices and cannot have forest–like support, but, must contain a circuit in its support.

As for the claimed equivalence, if all the circuits in Γ are compatible with L, then, in fact, the eigenspace is isomorphic to the circuit space of Γ, whose dimension amounts to $\dim\ker \mathscr{D} = N - n + 1$, see e.g. [10]. Conversely, we suppose that Γ has an incompatible circuit ζ. For $e = 0$, as above, $m_g(\lambda) = 0 = N - n \neq \mathrm{corank}(\Gamma)$. For $e > 0$, there must be some edge $k \notin E(\zeta)$ allowing a non vanishing restriction of some eigenfunction defined on the whole graph, since ζ is incompatible. Reasoning again by induction on e, we conclude as above that $m_g(\lambda; L, \Gamma) = 1 + m_g(\lambda; L, \Pi_k) = N - n$. $\square$

If the graph contains incompatible circuits, these can nevertheless be contained in the supports of some eigenfunction by means of *dumbbell*–like connected subgraphs δ in Γ. By definition, such a graph δ consists of two edge disjoint circuits ζ_1 and ζ_2 that are connected by some path π of length m that has exactly one vertex in common with each ζ_i. Note that $m = 0$ is admissible. It is easy to construct an eigenfunction $\mathscr{C}^2_{GK}(G) \cap \{u | \mathbf{n}(u) = 0\}$, whose support coincides with δ. Moreover, if ζ_1 and ζ_2 are incompatible, then $m_a(\lambda; L, \delta) = m_g(\lambda; L, \delta) = 1$.

The upper bound $\mathrm{corank}(\Gamma) = N - n + 1$ in (22) is optimal, since it is attained for any circuit and the canonical Laplacian $-\Delta$ (see Section 6.1) in the case $\cos\sqrt{\lambda} =$

1 bearing in mind that the eigenfunctions having non zero node distributions do not contribute here, see [1] and Theorems 6.1 and 6.2. The same holds for the lower bound $e = N - n$, take again the canonical Laplacian on a graph containing an odd circuit that cannot be compatible with $-\Delta$ for eigenvalues of the form $\cos\sqrt{\lambda} = -1$. Using Lemma 4.1, we conclude $m_g(\lambda) = N - n$. E.g. the graphs Y_1 and Y_2 in Fig. 1 have corank 2 and $m_g(\lambda) = 1$. The square in Y_1 is compatible with $-\Delta$ and the unique support of an eigenfunction, while the triangles are always incompatible for $\cos\sqrt{\lambda} = -1$. The dumbbell graph Y_2 and the graph Y_3 do not contain any compatible circuit. The support of an eigenfunction on Y_2 is necessarily the whole graph. The corank of Y_3 amounts to 3 while $m_g(\lambda) = 2$, and its eigenspace can only be generated by two eigenfunctions having dumbbell–like support.

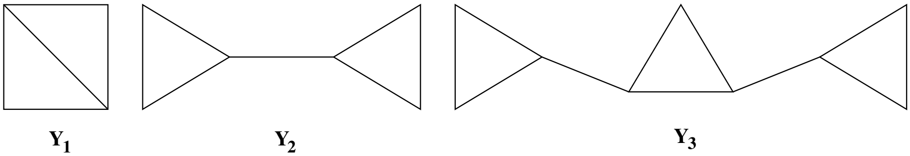

Figure 1: *Circuits that are incompatible with* $-\Delta$ *for* $\cos\sqrt{\lambda} = -1$.

LEMMA 4.2. *If* $\lambda \in \mathbb{C}$ *is an eigenvalue of Problem* (8) *and has no principal function vanishing in all nodes, then* $m_a(\lambda) \leqslant n - 1$.

Proof. By hypothesis, $\mathbf{n}$ defines an injective application from $E^c(\lambda)$ into $\mathbb{C}^n$. Thus, $m_a(\lambda) \leqslant n$. In order to refine this estimate, note that, again by hypothesis $\Sigma(1) \neq 0$, since otherwise, using $\Sigma(0) = 0$, $\lambda \in \mathbb{C}$ would be an eigenvalue under 0–Dirichlet boundary conditions on all the edges. By (19) a principal matrix solution satisfies

$$\mathbf{U}(1) = \Phi^t = \Phi \star \Theta(1) + \Psi \star \Sigma(1) + T, \tag{23}$$

where $T = \int_0^1 \mathbf{K}(1,s) \star F(s)\, ds$ stems from some iterated principal function matrix belonging to the characteristic space. Using Hadamard powers denoted by $\Sigma^{(k)} = \left(\sigma_{ih}^{(k)}\right)_{n\times n}$ for $k \in \mathbb{Z}$ and defined by

$$\sigma_{ih}^{(k)} = \begin{cases} \sigma_{ih}^k & \text{if } \sigma_{ih} \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

we get

$$\begin{aligned} \Psi &= \Sigma(1)^{(-1)} \star \left(\Phi^t - \Phi \star \Theta(1)\right) - \Sigma(1)^{(-1)} \star T \\ &= \Sigma(1)^{(-1)} \star \left(\mathbf{e}\,\varphi^t \star \mathscr{A} - \varphi\, \mathbf{e}^* \star \mathscr{A} \star \Theta(1)\right) - \Sigma(1)^{(-1)} \star T. \end{aligned}$$

Set

$$\mathbf{M} = \Sigma(1)^{(-1)} \star \mathscr{C} \star \mathscr{L}^{(-1)} - \mathrm{Diag}\left([\Sigma(1)^{(-1)} \star \mathscr{C} \star \mathscr{L}^{(-1)} \star \Theta(1)]\, \mathbf{e} + \rho\right).$$

Then (18) implies

$$\mathbf{M}\varphi = [\Sigma(1)^{(-1)} \star T]\,\mathbf{e},$$

and the eigenfunctions correspond exactly to the node vectors belonging to $\ker \mathbf{M}$. As the coefficients c_{ij} and ρ_i in the Kirchhoff law (2) at each node v_i can be multiplied by a positive constant α_i without altering the condition, a suitable choice of $\alpha_1, \dots, \alpha_n$ leads to a non zero trace of $\mathbf{M}$. This shows that $\mathbf{M}$ possesses eigenvalues different from 0 and that $m_a(0;\mathbf{M}) \leqslant n-1$. On the other hand, by hypothesis, independent principal functions belonging to λ lead to independent node distributions that are principal vectors for $\mathbf{M}$ belonging to 0. Thus $m_a(\lambda;L) \leqslant m_a(0;\mathbf{M}) \leqslant n-1$. $\square$

The upper bound in Lemma 4.2 is optimal as displayed by the first example in 6.2. Without the exclusion of 0–Dirichlet eigenvalues, the above estimate is false, see the second example in 6.2. Combining Lemmata 4.1 and 4.2 leads to the following

THEOREM 4.3. *The eigenvalues $\lambda \in \mathbb{C}$ of Problem (8) satisfy*

$$m_a(\lambda) \leqslant n-1,$$

if $\lambda \in \mathbb{C}$ has no eigenfunction vanishing in all nodes, and satisfy

$$m_a(\lambda) \leqslant N,$$

if $\lambda \in \mathbb{R}$ has some eigenfunction vanishing in all nodes.

Proof. The first assertion follows directly from Lemma 4.2. Decompose $E^c(\lambda) = E_0 \oplus \tilde{E}$ with $E_0 = \{u \in E^c(\lambda) | \mathbf{n}(u) = 0\}$. By Lemma 4.1, the dimension of E_0 is bounded from above by $N-n+1$, while by Lemma 4.2, the dimension of $\tilde{E}$ is bounded from above by $n-1$. $\square$

As already pointed out above, the first estimate is optimal. As for the second one, it is not clear in general whether it is optimal or not. The geometric multiplicity is always bounded by $N-n+T$, see Theorem 3.2, that reduces to the upper bound $N-n+2$ in the presence of eigenfunctions vanishing in all vertices, see [14, 15]. In particular, if the operator L is selfadjoint, the second bound N can never be attained. Thus, an example of optimality of the second bound would have to implicate the algebraic multiplicity of a real eigenvalue of a non selfadjoint operator L. For the canonical Laplacian (see Section 6.1), the upper bound N is never attained in the presence of eigenfunctions vanishing in all nodes, since λ could neither fulfill $\sin\sqrt{\lambda} = 0$, nor be a network immanent eigenvalue satisfying $\sin\sqrt{\lambda} \neq 0$ and $N = m_a(\cos\sqrt{\lambda}, \mathscr{Z}) \leqslant n-1$. In the latter case Γ would have to be a tree, that leads necessarily to $m_a(\lambda) \leqslant n-2$, since in that case the matrix $\mathscr{Z}$ has at least the eigenvalues $1 \neq \cos\sqrt{\lambda}$ and $-1 \neq \cos\sqrt{\lambda}$, see [1].

5. Trees

It has been shown in [9] that for operators of the form (3) on a tree, the algebraic eigenvalue multiplicity coincides with the geometric one. This result was applied in the proof of Theorem 3.3. For the reader's convenience we repeat the arguments here.

THEOREM 5.1. *Let T be a tree. Then, for each eigenvalue of Problem* (8)*, the algebraic and geometric multiplicities coincide. More precisely, the operator* $L = \left(u \mapsto (L_j u_j)_{N\times 1}\right)$ *is hermitian on* $\mathscr{C}^2_{GK}(T)$ *with respect to a suitable hermitian scalar product defined in* (24) *below. In particular, its eigenvalues are real.*

Proof. Without restriction, we can confine ourselves to the symmetric form (5) of the differential operators on the edges with parameters $\eta_1,\ldots,\eta_N$ to be specified later on. Next, orientate T such that some boundary vertex v_1 is a source, incident to e_1 and such that, at all other vertices, the indegree amounts to 1:

$$\gamma_i^+ := \operatorname{card}\{j\in\mathbb{N} | \, d_{ij} = 1\} = 1,$$
$$\gamma_i^- := \operatorname{card}\{j\in\mathbb{N} | \, d_{ij} = -1\} = \gamma_i - 1.$$

Put $\eta_1 = 1$ and, following the orientation, recursively at each node v_i with incoming edge e_m, set

$$p_j(0) = \eta_j = \frac{c_{ij}}{c_{im}} p_m(\ell_m) \qquad \text{if} \quad d_{ij} = -1, d_{im} = 1, v_i \in V.$$

Let m_i denote the edge index with $d_{im_i} = 1$. Then, introducing the scalar product on T

$$\langle u, w\rangle = \sum_{j=1}^{N} \int_0^{\ell_j} r_j u_j \overline{w_j}\, dx_j, \tag{24}$$

L is hermitian with respect to $\langle\cdot,\cdot\rangle$ on $\mathscr{C}^2_K(T;\mathbb{C})$, since the boundary terms stemming from integrations by parts match to 0:

$$\begin{aligned}
\sum_{j=1}^{N} [p_j \partial_j u_j \overline{w_j}]_0^{\ell_j} &= \sum_{i=1}^{n} \overline{w(v_i)} \left[p_{m_i}(\ell_{m_i}) \partial_{m_i} u_{m_i}(\ell_{m_i}) - \sum_{d_{ij}=-1} p_j(0)\partial_j u_j(0)\right] \\
&= \sum_{i=1}^{n} \overline{w(v_i)} \frac{p_{m_i}(\ell_{m_i})}{c_{im_i}} \left[c_{im_i} \partial_{m_i} u_{m_i}(\ell_{m_i}) - \sum_{d_{ij}=-1} c_{ij}\partial_j u_j(0)\right] \\
&= -\sum_{i=1}^{n} \overline{w(v_i)} \frac{p_{m_i}(\ell_{m_i})}{c_{im_i}} \rho_i u(v_i) = \sum_{j=1}^{N} [p_j u_j \partial_j \overline{w_j}]_0^{\ell_j}.
\end{aligned}$$

This shows that the eigenvalues are real, and, in turn, permits to follow a classical argument: For an eigenvalue λ of L on T and for $w \in \ker(L-\lambda I)^2$, it holds

$$0 = \left\langle (L-\lambda I)^2 w, w\right\rangle = \langle (L-\lambda I)w, (L-\lambda I)w\rangle,$$

which shows that w is an eigenfunction. This permits to conclude. □

Now we can present an easy proof of Theorem 3.2 for $m_a(\lambda)$ on trees.

LEMMA 5.2. *The eigenvalues* λ *of* (8) *on a tree* T *satisfy*

$$m_a(\lambda) = m_g(\lambda) \leqslant \#V_b - 1.$$

Proof. Choose a boundary vertex v_1 at which we prescribe the value of a presumed eigenfunction. This defines the value at the other node v_2 of the incident edge, as well as its derivative there. Thus, at v_2, we can prescribe exactly $\gamma_2 - 2$ derivatives imposing (2). Recursively, the number of free parameters to choose is bounded from above by

$$1 + \sum_{v_i \in V_r} (\gamma_i - 2) = 1 + 2N - 2n + \#V_b = \#V_b - 1. \quad \square$$

This upper bound is optimal, see 6.3. Moreover, if the tree is not just an interval, then the multiplicities are always bounded from above by $N-1 = n-2$. Combining (5.1) with the results from [1] or Theorems 6.1 and 6.2 below, we obtain

COROLLARY 5.3. *All principal functions of* $-\Delta_T^K$ *on a tree* T *are eigenfunctions. The node distributions of eigenfunctions of* $-\Delta_T^K$ *in* $\mathscr{C}_K^2(T)$ *either vanish and* $\sin\sqrt{\lambda} = 0$ *or are eigenvectors belonging to* $\cos\sqrt{\lambda}$ *of the matrix* $\mathscr{Z}$. *The multiplicities satisfy*

$$m_a(\lambda) = m_g(\lambda) = \begin{cases} 1 & \text{if } \sin\sqrt{\lambda} = 0, \\ m_g(\cos\sqrt{\lambda}, \mathscr{Z}) & \text{if } \sin\sqrt{\lambda} \neq 0. \end{cases}$$

6. Examples and remarks

6.1. The canonical Laplacian

For the *canonical* Laplacian Δ

$$\Delta = \Delta_G^K = \left(u \mapsto \left(\ell_j^2 \partial_j^2 u_j\right)_{N \times 1}\right) : \mathscr{C}_K^2(G) \to \prod_{j=1}^{N} \mathscr{C}[0, \ell_j]$$

under the weighted homogeneous Kirchhoff law (K)

$$\sum_{j=1}^{N} d_{ij} c_{ij} \ell_j^2 \partial_j u_j(v_i) = 0 \quad \text{for} \quad 1 \leqslant i \leqslant n \tag{25}$$

with weights $c_{ij} > 0$ the multiplicities can be determined with the aid of the row–stochastic *transition* matrix $\mathscr{Z} = \operatorname{Diag}\left((\mathscr{C} \star \mathscr{L})\,\mathbf{e}\right)^{-1} (\mathscr{C} \star \mathscr{L})$.

THEOREM 6.1. ([1, 4])

$$m_g(\lambda) = \begin{cases} 1 & \text{if } \lambda = 0, \\ m_g(\cos\sqrt{\lambda}, \mathscr{Z}) & \text{if } \sin\sqrt{\lambda} \neq 0, \\ N - n + 2 & \text{if } \cos\sqrt{\lambda} = 1, \\ N - n + 2 & \text{if } \cos\sqrt{\lambda} = -1, \Gamma \text{ bipartite}, \\ N - n & \text{if } \cos\sqrt{\lambda} = -1, \Gamma \text{ not bipartite}. \end{cases}$$

THEOREM 6.2. ([9])

$$m_a(\lambda) = \begin{cases} m_g(0) = 1 & \text{if } \lambda = 0, \\ m_a(\cos\sqrt{\lambda}, \mathscr{Z}) & \text{if } \sin\sqrt{\lambda} \neq 0, \\ m_g(\lambda) = N - n + 2 & \text{if } \cos\sqrt{\lambda} = 1, \\ m_g(\lambda) = N - n + 2 & \text{if } \cos\sqrt{\lambda} = -1, \Gamma \text{ bipartite}, \\ m_g(\lambda) = N - n & \text{if } \cos\sqrt{\lambda} = -1, \Gamma \text{ not bipartite}. \end{cases}$$

6.2. Optimal character of Lemma 4.2

For the canonical Laplacian on the complete graph K_n with equal edge lengths with $n \geqslant 2$ vertices under the classical Kirchhoff law

$$\sum_{j=1}^{N} d_{ij}\partial_j u_j(v_i) = 0 \quad \text{for} \quad 1 \leqslant i \leqslant n,$$

the eigenvalues satisfying $\cos\sqrt{\lambda} = \frac{-1}{n-1}$ have multiplicities $m_g(\lambda) = m_a(\lambda) = n-1$ and do not allow eigenfunctions vanishing in all nodes, see 6.1. This shows that the upper bound (4.2) is optimal in general. Without the exclusion of 0–Dirichlet eigenvalues, the estimate can be false. Take e.g. the canonical Laplacian on K_n as above with $n \geqslant 4$ that possesses the eigenvalues $\lambda > 0$ with $\cos\sqrt{\lambda} = 1$ and, according to Theorems 6.1 and 6.2, $m_a(\lambda) = m_g(\lambda) = \frac{n(n-1)}{2} - n + 2 > n - 1$.

6.3. Optimal character of Lemma 5.2

For the canonical Laplacian on a star graph under (25) the eigenvalues of the form $\sin\sqrt{\lambda} \neq 0$ always satisfy $m_a(\lambda) = m_g(\lambda) = N - 1 = \#V_b - 1$ for $\sin\sqrt{\lambda} \neq 0$, since the matrix $\mathscr{Z}$ has the form

$$\mathscr{Z} = \begin{pmatrix} 0 & z_{12} & \cdots & z_{1n} \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}$$

and the multiplicities $m_a(1) = m_a(-1) = 1$ and $m_g(0) = m_a(0) = n - 2$.

6.4. Another example with multiplicity disparity

Another example of multiplicity disparity can be constructed as follows. On $[0, 2\pi]$ consider the operator

$$Lu = u'' - (\sin^2 x)u' + (\sin x \cos x)u,$$

under periodic boundary conditions

$$u(0) = u(2\pi), \quad u'(0) = u'(2\pi).$$

This corresponds to the same operator on a loop, see [15]. Then $\lambda = 1$ is an eigenvalue of geometric multiplicity 1, while the algebraic one amounts to 2, since $(L+I)\sin x = 0$ and $(L+I)\cos x = \sin x$. Next, inserting at least two supplementary vertices on the loop and, thereby, creating a simple graph in the form of a circuit, we define the new edge operators by restriction of L and the new Kirchhoff laws by the $\mathscr{C}^1$–character. Then the multiplicities pertain since in fact, the eigensolutions are twice differentiable at the nodes.

Acknowledgement. Joachim von Below is grateful for invitations to the UPC Barcelona in 2009 and 2010. José A. Lubary is grateful for partial support by GenCat 2009SGR345 and MTM2008-06349-C03-01, Spain, and for the stay as invited professor to ULCO in Calais in 2009/2010. Both authors gratefully acknowledge valuable remarks by Professor Joan de Solà–Morales at Barcelona, by Baptiste Vasseur at Calais, and by the anonymous referee.

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(Received June 15, 2011)

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