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A POLYNOMIAL IDENTITY AND ITS APPLICATION TO INVERSE SPECTRAL PROBLEMS IN STIELTJES STRINGS

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Abstract. The equation $\Phi = P_1Q_2 + P_2Q_1$ is studied where Φ, Q_1, Q_2 are known real polynomials while P_1 and P_2 are unknown polynomials. Condition are obtained for the solution (P_1,P_2) to exist and to be such that $P_1^{-1}Q_1$ and $P_2^{-1}Q_2$ are Stieltjes functions. This result is used to prove the existence of a tree with two complementary subtrees of Stieltjes strings such that the spectrum of the Neumann boundary value problem on the tree is exactly the set of zeros of Φ and the spectra of Dirichlet problems on the subtrees are the sets of zeros of Q_1 and Q_2. This result is generalized to the equation $\Phi = \sum_{i=1}^{q}\left[P_i \prod_{j\neq i} Q_j\right]$, which is then applied to solve the inverse several spectra problem for trees of Stieltjes strings.

1. Introduction

The problem of interlacing sequences appears in rather different areas of finite and infinite dimensional analysis connected with inversion procedure. Strict interlacing of the eigenvalues of two boundary value problems is involved in the necessary and sufficient conditions of existence of the solution for the inverse Sturm-Liouville problem by two spectra [16], [17], [18]. Also we meet strictly interlacing sequences in finite dimensional case solving inverse problem for the so-called Stieltjes string [9] (see also [13]). In linear algebra it is known that the so-called tree-patterned matrix can be found for a pair of strictly interlacing sequences such that one of the sequences is the spectrum of the matrix while the other is the spectrum of its principal submatrix [7], [19].

The direct problem for a Stieltjes string was first studied in the monograph [9]. The problem arises from mechanical systems and is interesting in that the solution of the inverse spectral problem can be expressed in terms of a continued fraction. This simple finite-dimensional model was used in [14] and [8] to describe certain effects in a train vibrations. It should be mentioned that the same equations appear in the so-called Cauer method in the synthesis of electrical circuits [4]. A nice historical excursus into the applications can be found in the review article [5] where experiments are also described.

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Then the so-called inverse three spectral problem was studied in [3], with the tools of Nevanlinna functions developed in [12] (see also [1]). The three spectral problems [24], [10], [11], [6], [3] lead to non-strictly interlacing sequences of Dirichlet and Neumann boundary value problems.

The spectral problem on a star graph of Stieltjes strings was solved in [2] (see [20] for the case of much more general strings and [21], [22] for the Sturm-Liouville case).

The inverse problem on a star graph can be considered as a particular case of an inverse problem on a tree where the spectra of the Dirichlet and Neumann boundary value problems are also non-strictly interlaced (see [23] where the relations between the Neumann and Dirichlet characteristic functions of a tree and its complementary subtrees established in [15] were used).

In this paper, we are interested in the following *several spectra problem*. Let T be a metric tree with q complementary subtrees T_i $(i=1,\cdots,q)$. That is $\cup_{i=1}^{q} T_i = T$, and $T_i \cap T_j = \{\mathbf{v}\}$, where $\mathbf{v}$ is the root of T. Now given $q+1$ sequences of positive numbers $\{\lambda_k\}_{k=1}^{n}$ and $\{\nu_k^{(i)}\}_{k=1}^{n_i}$ $(i=1,\cdots,q)$ such that $n=\sum_1^q n_i$, we want to find distribution of point masses on the edges of T such that the spectra of the corresponding Dirichlet problems on T and T_i are exactly $\{\lambda_k\}_{k=1}^{n}$ and $\{\nu_k^{(i)}\}_{k=1}^{n_i}$ respectively.

2. A polynomial identity

Consider the identity

$$\Phi(z) = P_1(z)Q_2(z) + P_2(z)Q_1(z) \tag{1}$$

where Φ is a polynomial of degree n, P_1, Q_1 are polynomials of degree n_1, and P_2, Q_2 are polynomials of degree n_2. Suppose that Φ, Q_1 and Q_2 are known such that $\Phi(0), Q_1(0), Q_2(0) \neq 0$. Our aim is to find polynomials P_1 and P_2 such that $P_1(0) = C_1^{(1)}$, a given constant. The method of reconstruction is as follows. Let the set of zeros of Φ be denoted by $\{\lambda_k\}_{k=1}^{n}$, the sets of zeros of Q_1 and Q_2 be denoted by $\{\nu_k^{(1)}\}_{k=1}^{n_1}$ and $\{\nu_k^{(2)}\}_{k=1}^{n_2}$ respectively. Then (1) implies

$$P_1(\nu_k^{(1)}) = \frac{\Phi(\nu_k^{(1)})}{Q_2(\nu_k^{(1)})}, \qquad P_2(\nu_k^{(2)}) = \frac{\Phi(\nu_k^{(2)})}{Q_1(\nu_k^{(2)})}$$

Also we let $P_1(0) = C_1^{(1)}$. Similarly we want

$$C_1^{(2)} = \frac{\Phi(0) - C_1^{(1)} Q_2(0)}{Q_1(0)}.$$

Thus, by Lagrange interpolation, for $i=1,2$,

$$P_i(z) = \sum_{k=1}^{n_i} \left[\frac{\Phi(\nu_k^{(i)})}{\prod_{j\neq i} Q_j(\nu_k^{(i)})} \frac{z}{\nu_k^{(i)}} \prod_{j=1, j\neq k}^{n_i} \frac{z-\nu_j^{(i)}}{\nu_k^{(i)} - \nu_j^{(i)}} \right] + C_1^{(i)} \prod_{j=1}^{n_i} \frac{z-\nu_j^{(i)}}{-\nu_j^{(i)}}. \tag{2}$$

This procedure works if all the $\nu_k^{(1)}$'s and $\nu_k^{(2)}$'s are different, and nonzero. In this case, the solution of our problem exists and is unique.

However, in general, the situation is more complicated. To deal with it, we need the notion of a Nevanlinna function. It is also called R-function or Herglotz function. Its definition also varies. In this paper, we use the definition below.

DEFINITION. A function $f(z)$ is said to be a Nevanlinna function if

(i) it is analytic in the half-planes $\mathrm{Im} z > 0$ and $\mathrm{Im} z < 0$;

(ii) $f(\bar{z}) = \overline{f(z)}$, when $\mathrm{Im} z \neq 0$;

(iii) $\mathrm{Im} z \mathrm{Im} f(z) \geqslant 0$, when $\mathrm{Im} z \neq 0$.

DEFINITION.

(a) A Nevanlinna function $f(z)$ analytic on $\mathbf{C} \setminus [0, \infty)$ is said to be an S-function if $f(z) \geqslant 0$ when z is real and $z < 0$;

(b) A meromorphic S-function is said to be a S_0-function if 0 is not a pole.

In this paper, we study about polynomials. If f is a rational function with a positive leading coefficient, then it is an S-function if and only if its zeros $\{z_i\}_{i=1}^n$ and poles $\{w_i\}_{i=1}^m$ are all simple and strictly interlaced in the following way:

$$n = m,\ w_1 < z_1 < \cdots < w_n < z_n; \qquad \text{or} \qquad n = m+1,\ z_1 < w_1 < \cdots < w_n < z_{n+1}$$

We remark that in [1, Appendix II], there is a concise and interesting discussion on the properties of Nevanlinna functions.

LEMMA 2.1. *Suppose that f and g are Nevanlinna functions, then $f+g$ and $-\frac{1}{f}$ are also Nevanlinna functions.*

In general, it is possible that the points $\{\lambda_k\}$, $\{\nu_k^{(1)}\}$ and $\{\nu_k^{(2)}\}$ may overlap. It is also desirable to have each $\frac{Q_i(z)}{P_i(z)}$ to be an S_0-function.

THEOREM 2.2. *Let the sets of distinct positive numbers $\{\lambda_k\}_{k=1}^n$, $\{\nu_k^{(1)}\}_{k=1}^{n_1}$ and $\{\nu_k^{(2)}\}_{k=1}^{n_2}$ be given ($n = n_1 + n_2$) together with the positive numbers C, C_0^i and $C_1^{(1)}$. For $i = 1, 2$ denote by*

$$\Phi(z) = C \prod_{k=1}^{n} \left(1 - \frac{z}{\lambda_k}\right), \qquad Q_i(z) = C_0^{(i)} \prod_{k=1}^{n_i} \left(1 - \frac{z}{\nu_k^{(i)}}\right). \tag{3}$$

Also let $\{\zeta_k\}_{k=1}^n = \{\nu_k^{(1)}\}_{k=1}^{n_1} \cup \{\nu_k^{(2)}\}_{k=1}^{n_2}$ satisfying

(i) $0 < \lambda_1 \leqslant \zeta_1 \leqslant \cdots \leqslant \lambda_n \leqslant \zeta_n$*;*

(ii) $\lambda_k = \zeta_{k-1}$ *if and only if* $\lambda_k = \zeta_k$*;*

(iii) Let $C_1^{(1)} > 0$ *satisfy* $C > C_1^{(1)} C_0^{(2)}$.

Then the equation (1) possesses a solution $(P_1(z), P_2(z))$ *which are polynomials of degree* n_1 *and* n_2*, such that* $P_1(0) = C_1^{(1)}$*,* $P_2(0) = \frac{C - C_1^{(1)}}{C_0^{(1)}}$ *and each* $\frac{Q_i(z)}{P_i(z)}$ *is an* S_0*-function. In addition, if* $\{\nu_k^{(1)}\}_{k=1}^{n_1} \cap \{\nu_k^{(2)}\}_{k=1}^{n_2} = \emptyset$*, then this solution is unique.*

Proof. The case $\{\nu_k^{(1)}\}_{k=1}^{n_1} \cap \{\nu_k^{(2)}\}_{k=1}^{n_2} = \emptyset$ was discussed above. By Lagrange interpolation, each P_i has $n_i + 1$ nodes of interpolation and so its degree is n_i. Thus (P_1, P_2) is uniquely determined. It remains to show that $\frac{Q_1(z)Q_2(z)}{\Phi(z)}$ is an S_0-function.

Notice that in this case, due to (ii), the interlacing between $\{\lambda_k\}_{k=1}^{n}$ and $\{\zeta_k\}_{k=1}^{n}$ is strict in (i). Let $\zeta_k = \nu_p^{(1)}$ and $\zeta_{k+s} = \nu_{p+1}^{(1)}$. Then

$$\Phi(\nu_p^{(1)})(-1)^k = \Phi(\zeta_k)(-1)^k > 0 \quad \text{and} \quad \Phi(\nu_{p+1}^{(1)})(-1)^{k+s} = \Phi(\zeta_{k+s})(-1)^{k+s} > 0.$$

Also $Q_2(\nu_p^{(1)})(-1)^{p-k} > 0$ and $Q_2(\nu_{k+1}^{(1)})(-1)^{p+1-k-s} > 0$. Therefore,

$$\frac{\Phi(\nu_p^{(1)})}{Q_2(\nu_p^{(1)})}(-1)^p > 0, \qquad \frac{\Phi(\nu_{p+1}^{(1)})}{Q_2(\nu_{p+1}^{(1)})}(-1)^{p+1} > 0,$$

for $p = 1, \ldots, n_1 - 1$. That means, $P_1(\nu_p^{(1)})(-1)^p > 0$ for $p = 1, \ldots, n_1$. Since $P_1(0) = C_1^{(1)} > 0$ and $\deg P_1 = n_1$, P_1 has exactly one zero $\mu_{p+1}^{(1)}$ between $\nu_p^{(1)}$ and $\nu_{p+1}^{(1)}$. That is,

$$0 < \mu_1^{(1)} < \nu_1^{(1)} < \cdots < \mu_{n_1}^{(1)} < \nu_{n_1}^{(1)}.$$

Thus $\frac{Q_1}{P_1}$ is an S_0-function. Similarly one can show that the zeros of P_2 and Q_2 interlace strictly, and so $\frac{Q_2}{P_2}$ is also an S_0-function.

Next we consider the case when $\nu_{k_j}^{(1)} = \nu_{p_j}^{(2)}$ for $j = 1, \ldots, r$. Here we choose arbitrary real numbers $C_{k_1}^{(1)}, \ldots, C_{k_r}^{(1)}$ such that

$$(-1)^{k_j} C_{k_j}^{(1)} > 0 \qquad \text{and} \qquad |C_{k_j}^{(1)}| < |\Phi'(\nu_{k_j}^{(1)})|\,|Q_2'(\nu_{k_j}^{(1)})|^{-1}$$

Letting these $C_{k_j}^{(1)}$ be the values of $P_1(\nu_{k_j}^{(1)})$, we have

$$P_1(z) = \sum_{k=1, k\neq k_j}^{n_1} \left[\frac{\Phi(\nu_k^{(1)})}{Q_2(\nu_k^{(1)})} \frac{z}{\nu_k^{(1)}} \prod_{j\neq k} \frac{z - \nu_j^{(1)}}{\nu_k^{(1)} - \nu_j^{(1)}} \right] + C_1^{(1)} \prod_{j=1}^{n_1} \frac{z - \nu_j^{(1)}}{-\nu_j^{(1)}}$$
$$+ \sum_{j=1}^{r} C_{k_j}^{(1)} \frac{z}{\nu_{k_j}^{(1)}} \prod_{s\neq k_j} \frac{z - \nu_s^{(1)}}{\nu_{k_j}^{(1)} - \nu_s^{(1)}}.$$

Now (ii) implies $\Phi(\nu_{k_j}^{(1)}) = 0$ and $\Phi'(\nu_{k_j}^{(1)}) \neq 0$. Thus we may define

$$C_{p_j}^{(2)} = \frac{\Phi'(\nu_{k_j}^{(1)}) - Q_2'(\nu_{k_j}^{(1)}) C_{k_j}^{(1)}}{Q_1'(\nu_{k_j}^{(1)})}. \tag{4}$$

Hence we can construct P_2 as

$$P_2(z) = \sum_{k=1, k\neq k_j}^{n_2} \left[\frac{\Phi(v_k^{(2)})}{Q_1(v_k^{(2)})} \frac{z}{v_k^{(2)}} \prod_{j\neq k} \frac{z-v_j^{(2)}}{v_k^{(2)}-v_j^{(2)}} \right] + \frac{C - C_1^{(1)} C_0^{(2)}}{C_0^{(1)}} \prod_{j=1}^{n_2} \frac{z-v_j^{(2)}}{-v_j^{(2)}}$$
$$+ \sum_{j=1}^{r} C_{p_j}^{(2)} \frac{z}{v_{p_j}^{(2)}} \prod_{s\neq p_j} \frac{z-v_s^{(2)}}{v_{p_j}^{(2)}-v_s^{(2)}}.$$

To prove that (P_1,P_2) given above is a solution of (1), we let

$$\Omega(z) = \Phi(z) - P_1(z)Q_2(z) - P_2(z)Q_1(z).$$

With the definition of these polynomials at 0, it is easy to verify that $\Omega(0) = 0$. Then since $Q_1(v_k^{(1)}) = 0$, we have $\Omega(v_k^{(1)}) = 0$ for each $k \neq k_j$. Then at k_j's, $\Phi(v_{k_j}^{(1)}) = Q_1(v_{k_j}^{(1)}) = Q_2(v_{k_j}^{(1)}) = 0$. Thus Ω vanishes there too. Similarly, $\Omega(v_k^{(2)}) = 0$ for each $k = 1,\dots,n_2$. By (P_1,P_2) constructed above and (4), we have $\Omega'(v_{k_j}^{(1)}) = 0$. So the polynomial $\Omega(z)$ of degree $n_1+n_2 = n$ has at least $n+1$ zeros, counting multiplicities. Therefore $\Omega \equiv 0$.

Finally, it is trivial to see that $\frac{Q_1}{P_1}$ is an S_0-function. Then we observe that

$$(-1)^{k_j} Q_1'(v_{k_j}^{(1)}) > 0, \qquad (-1)^{p_j} Q_2'(v_{k_j}^{(1)}) > 0, \qquad (-1)^{k_j+p_j} \Phi'(v_{k_j}^{(1)}) > 0.$$

Thus, $(-1)^{p_j} C_{p_j}^{(2)} > 0$. This implies $(-1)^{p_j} P_2(v_{k_j}^{(1)}) > 0$. Therefore, the zeros of P_2 and Q_2 interlace strictly and so $\frac{Q_2}{P_2}$ is an S_0-function. $\square$

Next, we consider the more general polynomial equation

$$\Phi(z) = \sum_{i=1}^{q} \left[P_i(z) \prod_{j\neq i} Q_j(z) \right]. \tag{5}$$

where $\Phi(z)$ is a polynomial of degree n, and P_i, Q_i are polynomials of degree n_i $(i = 1,\dots,q)$, with $n = \sum n_i$. We also let Φ and each Q_i be as given in (3), with real positive zeros $\{\lambda_k\}_{k=1}^{n}$ and $\{v_k^{(i)}\}_{k=1}^{n_i}$ respectively, and $C, C_0^{(i)} > 0$.

THEOREM 2.3. *Define $\{\eta_k\}_{k=1}^{n} = \cup_{i=1}^{q} \{v_k^{(i)}\}_{k=1}^{n_i}$. Suppose that the conditions (i) and (ii) in Theorem 2.2 and (iii') below hold.*

(iii') Let $C_1^{(j)}$ $(j = 1,\dots,q-1)$ such that

$$C - \sum_{j=1}^{q-1} \left[C_1^{(j)} \prod_{i\neq j, i=1}^{q} C_0^{(i)} \right] > 0. \tag{6}$$

Then the identity (5) possesses a solution $(P_1,\dots,P_q)$, where each P_i are degree n_i polynomials, and $P_i(0)=C_i^{(1)}$ for $i=1,\dots,q-1$. Furthermore, each $\frac{Q_i}{P_i}$ is an S_0-function. If in addition, $\cap_{i=1}^q\{\nu_k^{(i)}\}_{k=1}^{n_i}=\emptyset$, then the solution is unique.

Proof. When $\cap_{i=1}^q\{\nu_k^{(i)}\}_{k=1}^{n_i}=\emptyset$, then we let

$$C_1^{(q)}:=\frac{C-\sum_{i=1}^{q-1}C_1^{(i)}\prod_{j\neq i}C_0^{(j)}}{\prod_{i=1}^{q-1}C_1^{(i)}}>0.$$

Noting that from (5),

$$P_i(\nu_k^{(i)})=\frac{\Phi(\nu_k^{(i)})}{\prod_{j\neq i}Q_j(\nu_k^{(i)})}.$$

Thus we may define the polynomial P_i as in (2) uniquely by Lagrange interpolation. Furthermore, the interlace is strict. Let $\eta_k=\nu_p^{(1)}$ $(k\geqslant p)$ and $\eta_{k+s}=\nu_{p+1}^{(1)}$. Then $\Phi(\nu_p^{(1)})(-1)^k=\Phi(\eta_k)(-1)^k>0$, and $\Phi(\nu_{p+1}^{(1)})(-1)^k=\Phi(\eta_{k+s})(-1)^k>0$. Also, $\prod_{j\neq 1}Q_j(\nu_p^{(1)})(-1)^{k-p}>0$ and $\prod_{j\neq 1}Q_j(\nu_{p+1}^{(1)})(-1)^{(s+k)-(p+1)}>0$. Hence

$$P_1(\nu_p^{(1)})(-1)^p=\frac{\Phi(\nu_p^{(1)})}{\prod_{j\neq 1}Q_j(\nu_p^{(1)})}(-1)^p>0.$$

Since $P_1(0),\ Q_1(0)>0$ and $deg(P_1)=deg(Q_1)$, this means that the zeros of P_1 and Q_1 are strictly interlacing. Therefore $\frac{Q_1}{P_1}\in S_0$. The proof for other i's is similar.

When the zeros of Q_i's overlap, the situation is more complicated. We give some typical cases.

Case 1. If $\nu_{k_j}^{(1)}=\nu_{p_j}^{(2)}$, $j=1,\cdots,r$, while $\cap_{i\neq 1}\{\nu_k^{(i)}\}_{k=1}^{n_i}=\cap_{i\neq 2}\{\nu_k^{(i)}\}_{k=1}^{n_i}=\emptyset$. Choose $C_{k_j}^{(1)}$, $j=1,\cdots,r$ to satisfy $(-1)^{k_j}C_{k_j}^{(1)}>0$, $P_1(\nu_{k_j}^{(1)})=C_{k_j}^{(1)}$, and

$$|C_{k_j}^{(1)}|<|\Phi'(\nu_{k_j}^{(1)})|(|Q_2'(\nu_{k_j}^{(1)})\prod_{l\neq 1}Q_l(\nu_{k_j}^{(1)})|)^{-1}.$$

Also we let

$$P_2(\nu_{p_j}^{(2)})=C_{p_j}^{(2)}:=\frac{\Phi'(\nu_{k_j}^{(1)})-C_{k_j}^{(1)}Q_2'(\nu_{k_j}^{(1)})\prod_{l\geqslant 3}Q_l(\nu_{k_j}^{(1)})}{Q_1'(\nu_{k_j}^{(1)})\prod_{l\geqslant 3}Q_l(\nu_{k_j}^{(1)})}.$$

Construct the Lagrange interpolating polynomials for P_i as follows:

$$\begin{aligned}P_1(z)\ =\ &\sum_{k=1,k\neq k_j}^{n_1}\left[\frac{\Phi(\nu_k^{(1)})}{\prod_{l\neq 1}Q_l(\nu_k^{(1)})}\frac{z}{\nu_k^{(1)}}\prod_{j\neq k}\frac{z-\nu_j^{(1)}}{\nu_k^{(1)}-\nu_j^{(1)}}\right]\\&+\sum_{j=1}^{r}C_{k_j}^{(1)}\frac{z}{\nu_{k_j}^{(1)}}\prod_{s\neq k_j}\frac{z-\nu_s^{(1)}}{\nu_{k_j}^{(1)}-\nu_s^{(1)}}+C_1^{(1)}\prod_{j=1}^{n_1}\frac{z-\nu_j^{(1)}}{-\nu_j^{(1)}},\end{aligned}$$

$$P_2(z) = \sum_{k=1,k\neq p_j}^{n_2} \frac{\Phi(\nu_k^{(2)})}{\prod_{l\neq 2} Q_l(\nu_k^{(2)})} \frac{z}{\nu_k^{(2)}} \prod_{j\neq k} \frac{z-\nu_j^{(2)}}{\nu_k^{(2)}-\nu_j^{(2)}}$$
$$+\sum_{j=1}^{r} C_{p_j}^{(2)} \frac{z}{\nu_{p_j}^{(2)}} \prod_{s\neq p_j} \frac{z-\nu_s^{(2)}}{\nu_{p_j}^{(2)}-\nu_s^{(2)}} + C_1^{(2)} \prod_{j=1}^{n_2} \frac{z-\nu_j^{(2)}}{-\nu_j^{(2)}},$$

The other polynomials P_i's can be defined as in (2). To prove $(P_1,\dots,P_q)$ is a solution of (3) consider the polynomial

$$\Omega(z) = \Phi(z) - \sum_{i=1}^{q} \left[P_i(z) \prod_{j\neq i} Q_j(z) \right].$$

It is not difficult to see that $\Omega(0) = \Omega(\nu_{k_i}^{(i)}) = 0,\ k_i = 1,2,3,\cdots,n_i$ with $i = 1,2,3,\cdots,q$, and $\Omega'(\nu_{k_j}^{(1)}) = 0,\ j = 1,2,\cdots,r$. This implies that $\Omega \equiv 0$ in this case.

Then we show $\frac{Q_i}{P_i} \in S_0,\ i = 1,2,\cdots,q.\ \ i = 1$ is ok by the assumption of $C_{k_j}^{(1)}$, we have that $(-1)^{k_j} P_1(\nu_{k_j}^{(1)}) = (-1)^{k_j} C_{k_j}^{(1)} > 0$. By interlacing of zeros, we have that $(-1)^{p_j} C_{p_j}^{(2)} = (-1)^{p_j} P_2(\nu_{p_j}^{(2)}) > 0$. And $(-1)^k P_i(\nu_k^{(i)}) > 0$ for all the other i's.

Case 2. $\nu_{k_j}^{(1)} = \nu_{p_j}^{(2)} = \nu_{g_j}^{(3)} := \mu_j,\ j = 1,2,3,\cdots,r$, and no other common zeros between any two zero sets of Q_i.

Choose $P_1(\nu_{k_j}^{(1)}) = C_{k_j}^{(1)}$ and $C_{p_j}^{(2)} = P_2(\nu_{p_j}^{(2)})$ such that $(-1)^{k_j} C_{k_j}^{(1)} > 0$, $(-1)^{p_j} C_{p_j}^{(2)} > 0$, and at $z = \mu_j$,

$$2|C_{k_j}^{(1)} Q_2' Q_3' \prod_{l>3} Q_l| + 2|C_{p_j}^{(2)} Q_1') Q_3' \prod_{l>3} Q_l| < |\Phi''|.$$

Define

$$C_{g_j}^{(3)} = \left. \frac{\Phi'' - 2C_{k_j}^{(1)} Q_2' Q_3' \prod_{l>3} Q_l + 2C_{p_j}^{(2)} Q_1' Q_3' \prod_{l>3} Q_l}{2Q_1' Q_2' \prod_{l>3} Q_l} \right|_{z=\mu_j}. \tag{7}$$

Hence $(P_1,\dots,P_q)$ can be determined. Furthermore $\Omega \equiv 0$.

Then we show that $\frac{Q_i}{P_i} \in S_0,\ i = 1,2,3$. By the choices of $C_{k_j}^{(1)}$ and $C_{p_j}^{(2)},\ i = 1,2$ are ok. From (7), we have $(-1)^{g_j} C_{g_j}^{(3)} = (-1)^{g_j} P_3(\nu_{g_j}^{(3)}) > 0$. So $i = 3$ is also ok. The rest is trivial.

Case 3. $\nu_{k_j}^{(1)} = \nu_{p_j}^{(2)},\ j = 1,2,3,\cdots r$, and $\nu_{h_j}^{(1)} = \nu_{l_j}^{(3)},\ j = 1,2,\cdots m$, where $k_j \neq h_j$.

Choose $C_{k_j}^{(1)}$ and $C_{h_j}^{(1)}$ to satisfy $(-1)^{k_j} C_{k_j}^{(1)} > 0,\ (-1)^{h_j} C_{h_j}^{(1)} > 0$, and

$$|C_{k_j}^{(1)}| < |\Phi'(\nu_{k_j}^{(1)})|(|Q_2'(\nu_{k_j}^{(1)}) \prod_{l>2} Q_l(\nu_{k_j}^{(1)})|)^{-1}$$
$$|C_{h_j}^{(1)}| < |\Phi'(\nu_{h_j}^{(1)})|(|Q_3'(\nu_{h_j}^{(1)}) \prod_{l\neq 1,3} Q_l(\nu_{h_j}^{(1)})|)^{-1}.$$

Define

$$C^{(2)}_{p_j} = \frac{\Phi'(v^{(1)}_{k_j}) - C^{(1)}_{k_j} Q'_2(v^{(1)}_{k_j}) \prod_{l>2} Q_l(v^{(1)}_{k_j})}{Q'_1(v^{(1)}_{k_j}) \prod_{l>2} Q_l(v^{(1)}_{k_j})}$$

$$C^{(3)}_{l_j} = \frac{\Phi'(v^{(1)}_{h_j}) - C^{(1)}_{h_j} Q'_3(v^{(1)}_{h_j}) \prod_{l\neq 1,3} Q_2(v^{(1)}_{h_j})}{Q'_1(v^{(1)}_{h_j}) \prod_{l\neq 1,3} Q_2(v^{(1)}_{h_j})}.$$

Then $(P_1,\ldots,P_q)$ can be determined. The rest is similar $\square$

3. An existence problem

Let T be a metric tree rooted at $\mathbf{v}$, having $q = d(\mathbf{v})$ complementary subtrees T_i $(i = 1,\ldots,q)$. Thus $\cup_{i=1}^{q} T_i = T$, and $T_i \cap T_j = \{\mathbf{v}\}$. For each i, let T_i have γ_i edges, and each edge $e_{i,j}$ has length $L_{i,j}$ $(j = 1,\ldots,\gamma_i)$.

It is assumed that the tree T of Stieltjes strings is stretched and vibrates in the direction orthogonal to the equilibrium position of the strings. The transverse displacement of the mass $m^{i,j}_k$ is denoted by $w^{(i,j)}_k(t)$. Let $\mathbf{v}$ be the root of T and all the edges $e_{i,j}$ are directed towards $\mathbf{v}$, i.e., the local coordinates of its endpoints are 0 and $L_{i,j}$ associated with vertices v_1 and v_2 respectively. We say $e_{i,j}$ is outgoing from v_1 and ingoing to v_2, while the displacement at v_1 and v_2 associated with $e_{i,j}$ is denoted by $w^{(i,j)}_0$ and $w^{(i,j)}_{\tau_{i,j}}$ respectively. Using such notation vibrations of the graph can be described by the system of equations

$$\frac{w^{(i,j)}_k(t) - w^{(i,j)}_{k+1}(t)}{l^{(i,j)}_k} + \frac{w^{(i,j)}_k(t) - w^{(i,j)}_{k-1}(t)}{l^{(i,j)}_{k-1}} + m^{(i,j)}_k \frac{\partial^2 w^{(i,j)}_k}{\partial t^2}(t) = 0$$

$(k = 1,2,\ldots,\tau_{i,j};\ j = 1,2,...,\gamma_i)$. For each interior vertex v with ingoing edges $e_{i,j}$'s and outgoing edge $e_{i,r}$ we impose the continuity conditions $w^{(i,r)}_0(t) = w^{(i,j)}_{\tau_{i,j}+1}(t)$. Balance of forces at v implies

$$\frac{w^{(i,r)}_1(t) - w^{(i,r)}_0(t)}{l^{(i,r)}_0} = \sum_j \frac{w^{(i,j)}_1(t) - w^{(i,j)}_0(t)}{l^{(i,j)}_{\tau_{i,j}}}$$

For an edge $e_{i,j}$ incident with a pendant vertex, we impose Dirichlet boundary condition $w^{i,j}_0(t) = 0$. The continuity conditions at the root $\mathbf{v}$ are $w^{(i,j)}_{\tau_{i,j}+1}(t) = w^{(i,r)}_{\tau_{i,r}+1}(t)$ for all pairs of edges incident with $\mathbf{v}$. We need to impose one more condition at the root. We consider two cases: Dirichlet case with $w^{(i,j)}_{\tau_{i,j}+1}(t) = 0$; and Neumann case $\sum\limits_j \frac{w^{(i,j)}_{\tau_{i,j}+1}(t) - w^{(i,j)}_{\tau_{i,j}}(t)}{l^{(i,j)}_{\tau_{i,j}}} = 0$.

Substituting $w_k^{(i,j)}(t) = e^{i\rho t} u_k^{(i,j)}$ into the above equations, we obtain the Dirichlet problem described below. For each edge:

$$\frac{u_k^{(i,j)} - u_{k+1}^{(i,j)}}{l_k^{(i,j)}} + \frac{u_k^{(i,j)} - u_{k-1}^{(i,j)}}{l_{k-1}^{(i,j)}} - m_k^{(i,j)} \lambda u_k^{(i,j)} = 0,\ \lambda = \rho^2. \tag{8}$$

$(k = 1,2,\ldots,\tau_{i,j},\ j = 1,2,...,\gamma_i)$. For each interior vertex with incoming edges e_j and outgoing edge e_r we have

$$u_0^{(i,r)} = u_{\tau_{i,j}+1}^{(i,j)}. \tag{9}$$

$$\frac{u_1^{(i,r)} - u_0^{(i,r)}}{l_0^{(i,r)}} = \sum_j \frac{u_{\tau_{i,j}+1}^{(i,j)} - u_{\tau_{i,j}}^{(i,j)}}{l_{\tau_{i,j}}^{(i,j)}}. \tag{10}$$

For each edge $e_{i,j}$ incident with a pendant vertex,

$$u_0^{(i,j)} = 0. \tag{11}$$

At the root $\mathbf{v}$:

$$u_{\tau_{i,j}+1}^{(i,j)} = 0 \tag{12}$$

for all of edges incident with $\mathbf{v}$.

The conditions

$$u_{\tau_{i,k}+1}^{(i,k)} = u_{\tau_{i,j}+1}^{(i,j)} \tag{13}$$

for all pair of edges $e_{i,k}$ and $e_{i,j}$ incident with the root together with

$$\sum_j \frac{u_{\tau_{i,j}+1}^{(i,j)} - u_{\tau_{i,j}}^{(i,j)}}{l_{\tau_{i,j}}^{(i,j)}} = 0 \tag{14}$$

we call Neumann conditions at the root. If the root is a pendant vertex than (13), (14) are equivalent to the usual Neumann condition. In what follows problem (8)–(12) is called Dirichlet problem at $\mathbf{v}$ for the tree T and problem (8)–(11), (13), (14) is called Neumann problem at $\mathbf{v}$.

We let $z = \lambda^2$ and $R_k^{(i,j)}(z)$ be polynomials (see [9]) which satisfy the initial conditions $R_0^{(i,j)}(z) = 1$, $R_{-1}^{(i,j)}(z) = \frac{1}{l_j^0}$ such that $u_k^{(i,j)} = R_{2k-2}^{(i,j)}(z)$ on the edge $e_{i,j}$ is a solution to (8), while

$$R_{2k-1}^{(j)}(z) = \frac{R_{2k}^{(i,j)}(z) - R_{2k-2}^{(i,j)}(z)}{l_k^{(i,j)}},$$

Then these polynomials satisfy the relations [9]:

$$\begin{cases} R_{2k-1}^{(i,j)}(z) = -z m_k^{(i,j)} R_{2k-2}^{(i,j)}(z) + R_{2k-3}^{(i,j)}(z), \\ R_{2k}^{(i,j)}(z) = l_k^{(i,j)} R_{2k-1}^{(i,j)}(z) + R_{2k-2}^{(i,j)}(z) \end{cases}. \tag{15}$$

Associated with the root $\mathbf{v}$ for T, we define ϕ_N to the characteristic function with continuity and Kirchhoff conditions (or Neumann condition) at $\mathbf{v}$, and ϕ_D to be the characteristic function with Dirichlet condition at $\mathbf{v}$. We call ϕ_N Neumann characteristic function and ϕ_D Dirichlet characteristic function. Similarly we let $\phi_N^{(i)}$ and $\phi_D^{(i)}$ the Neumann and Dirichlet characteristic function for the subtree T_i respectively. As proved in [23, Corollary 2.2], we have

THEOREM 3.1.

$$\phi_N = \sum_{i=1}^{q}\left(\phi_N^{(i)}\prod_{j\neq i}\phi_D^{(j)}\right), \qquad \phi_D = \prod_{i=1}^{q}\phi_D^{(i)}.$$

It is well known that on the interval, the Weyl-Titchmarsh m-function, which is a ratio of characteristic functions for different boundary conditions, uniquely determines the potential function. Furthermore this m-function is equivalent to the spectral function, and is a Nevanlinna function. So, as an analog, for any tree T of Stieltjes strings, we define its *M-function at* $\mathbf{v}$ to be $\frac{\phi_D}{\phi_N}$.

THEOREM 3.2. *The M-function at $\mathbf{v}$, $\frac{\phi_D}{\phi_N}$, for the tree T is a S_0 function.*

The above theorem was proved in [23, Theorem 2.8]. We shall give a simpler proof in the appendix. Now we are going to state the main theorem of this paper. We define

Problem I: Neumann (continuity and Kirchhoff) conditions at $\mathbf{v}$.

Problem I_i: Dirichlet conditions at the root $\mathbf{v}$ of subtrees T_i $(i=1,\ldots,q)$

THEOREM 3.3. *Suppose $q+1$ sequence $\Lambda=\{\lambda_k\}_{k=1}^{n}$ and $V^{(i)}=\{\nu_k^{(i)}\}_{k=1}^{n_i}$ disjoint positive numbers be given such that $n=\sum_{i=1}^{q}n_i$. Let $\{\zeta_k\}_{k=1}^{n}=\cup_{i=1}^{q}V^{(i)}$ such that conditions (i) and (ii) in Theorem 2.2 are satisfied. Let also a tree T be given together with its complementary subtrees T_J $(j=1,2,\ldots q)$ and the lengths $L_{i,j}$ of the edges.*

Then there exist sequences of real numbers $\mathscr{M}_{i,j,k}=\{m_k^{(i,j)}:\ 1\leqslant k\leqslant\tau_{i,j},\ 1\leqslant j\leqslant\gamma_i,\ 1\leqslant i\leqslant q\}$, and $\mathscr{L}_{i,j,k}=\{l_k^{(i,j)}:\ 0\leqslant k\leqslant\tau_{i,j},\ 1\leqslant j\leqslant\gamma_i,\ 1\leqslant i\leqslant q\}\}$, such that $\sum_{j=1}^{\gamma_i}\tau_{i,j}=n_i$, $\sum_{k=0}^{\tau_{i,j}}l_k^{(i,j)}=L_{i,j}$, and all $m_k^{(i,j)}$'s and $l_k^{(i,j)}$'s are positive while $l_0^{(i,j)}$'s are nonnegative. Furthermore, with the tree of Stieltjes string thus formed, the spectra of Problem I and Problems I_i are exactly $\{\lambda_k\}_{k=1}^{n}$ and $\{\nu_k^{(i)}\}_{k=1}^{n_i}$.

REMARK. Hence we use $2n$ eigenvalues plus $\sum_{i=1}^{q}\gamma_i$ constants to recover totally n masses and $n+\sum_{i=1}^{q}\gamma_i$ lengths.

Before the proof is given, we need a few symbols to express some operations with continued fractions. It is customary to use $[a_1,a_2,\ldots,a_j]$ to denote a continued fraction

$$a_1+\cfrac{1}{a_2+\frac{1}{a_3+\cdots+\frac{1}{a_j}}}.$$

Hence if $A = [a_1, a_2, \ldots, a_j]$, then

$$A^{-1} = \frac{1}{A} = [0, a_1, a_2, \ldots, a_j].$$

Let $B = [b_1, b_2, \ldots, b_k]$ be another continued fraction. We define a composition of A and B to be

$$[A;B] := [a_1, \ldots, a_j, b_1, \ldots, b_k].$$

For simplicity, we also let $[A;B;C] = [A;[B;C]]$, and so on.

In [9, 2], it was shown that for a q-star graph with root at the interior vertex $\mathbf{v}$, M-function $G_i(z)$ at $\mathbf{v}$ of each edge e_i is given by

$$G_i(z) = \frac{R^{(i)}_{2n_i}(z)}{R^{(i)}_{2n_i-1}(z)} = [l_{n_i}, -m_{n_i}z, l_{n_i-1}, -m_{n_i-1}z, \ldots, -m_1 z, l_0],$$

while the M-function $G(z)$ of the q-star graph at $\mathbf{v}$, by Theorem 3.3, is

$$G(z) = (\sum_{i=1}^{q} G_i(z)^{-1})^{-1}.$$

Note that $G_i(0) = L_i$, length of the ith edge e_i. Hence $G(0) = (\sum_{i=1}^{q} L_i^{-1})^{-1}$. Inductively, for any tree T, the value of the M-function at $z = 0$ can be given as a rational function of its edgelengths, or more precisely, as a continued fraction of its edgelengths. Assuming each F_i is the M-function for Dirichlet problem for the ith edge, The M-function F associated with continuity and Kirchhoff conditions at $\mathbf{v}$ is given by

$$F(z) = \left\{F_1^{-1} + F_2^{-1} + [F_3;F_4;F_5]^{-1}\right\}^{-1}.$$

Proof of Theorem 3.3. Let $F_i(z) = \frac{\phi^{(i)}_D(z)}{\phi^{(i)}_N(z)}$ be the M-function of the subtree T_i at $\mathbf{v}$. By the recursive formulas given in Theorem 3.1, the M-function $F(z)$ of T at $\mathbf{v}$ is given by

$$F(z) = (\sum_{i=1}^{q} F_i(z)^{-1})^{-1}. \tag{16}$$

We define

$$Q_i(z) = C_0^{(i)} \prod_{k=1}^{n_i} \left(1 - \frac{z}{\nu_k^{(i)}}\right).$$

We also let

$$\Phi(z) = C \prod_{k=1}^{n} \left(1 - \frac{z}{\lambda_k}\right).$$

Here $C_0^{(i)} = F_i(0)$ and $C = F(0)$. Then the equation (3) is formed, while the conditions (i), (ii) and (iii') in Theorem 2.3 are satisfied. Hence we may apply Theorem 2.3 to solve (3) for the polynomials $P_i(z)$ $(i = 1, \ldots, q)$ and each quotient $\frac{Q_i}{P_i}$ is a S_0 function.

Now as the polynomial Q_i is a scalar multiple of $\phi_D^{(i)}$, by uniqueness, we conclude that P_i is also a scalar multiple of $\phi_N^{(i)}$, and $F_i = \frac{Q_i}{P_i}$ is the M-function of the subtree T_i at $\mathbf{v}$. Therefore $F(z)$ can be recovered by (16). Furthermore, the Dirichlet spectrum and Neumann spectrum of T_i at $\mathbf{v}$ are now known. Thus by [23, Theorem 3.1], there exists sets $\mathscr{M}_{i,j}$ and $\mathscr{L}_{i,j}$ representing the point masses $\{m_k^{(i,j)}\}$ and $\{l_k^{(i,j)}\}$ respectively. $\square$

REMARK. In general, the solutions might not be unique. If Γ and V are not strictly interlaced, there might be multiple solutions of $(P_1,\ldots,P_q)$. For each solution, $\frac{Q_i}{P_i}$ is an S_0 function, plus the zeros of P_i and Q_i are different and strictly interlaced. Hence by [23, Theorem 3.1], zeros of Q_i and P_i represent the Dirichlet and Neumann spectra of the subtree T_i at the point $\mathbf{v}$, and so there exists masses $\mathscr{M}_{i,j}$ and lengths $\mathscr{L}_{i,j}$ $(j=1,\ldots,\gamma_i)$ on the subtree T_i of Stieltjes string. Even when Γ and V are strictly interlaced, there is only one solution $(P_1,\ldots,P_q)$. However the resulting M-function $F_i = \frac{Q_i}{P_i}$ might be associated with different point mass distribution, as the subtree T_i might be too complicated. In case Γ and V are strictly interlaced, and T is a star graph, there exists a unique solution.

4. Appendix

Here we give a simple proof of Theorem 3.2:

First consider the case of an interval. Let $f = \frac{\phi_D}{\phi_N}$ be the M-function for an interval. Let ϕ_D and ϕ_N be both polynomials of degree n. With notations similar to those in (15), we have $f(z) = \dfrac{R_{2n}(z)}{R_{2n-1}(z)}$, where $R_0(z) = 1$ and $R_{-1}(z) = 1/l_0$. Also we have the system of difference equations

$$R_{2k+1}(z) = R_{2k-1}(z) - zm_kR_{2k-2}(z) \tag{17}$$
$$R_{2k}(z) = l_kR_{2k-1}(z) + R_{2k-2}(z) \tag{18}$$

Obviously $f(z) = [l_n, -m_kz, l_{n-1}, \cdots, -m_1z, l_0]$, and so $f(\overline{z}) = \overline{f(z)}$ and $f(0) = l_n > 0$. For $z \leqslant 0$, $f(z) > 0$. So for this case, it remains to show $\mathrm{Im}z\mathrm{Im}f(z) \geqslant 0$ whenever $Imz \neq 0$. From (17), we have

$$(R_{2k+1} - R_{2k-1})\overline{R_{2k}} = -zm_{k+1}|R_{2k}|^2.$$

Hence

$$\mathrm{Im}\left(R_{2k+1}\overline{R_{2k}} - R_{2k-1}\overline{R_{2k-2}}\right) = -(\mathrm{Im}z)m_{k+1}|R_{2k}|^2. \tag{19}$$

Also from (18),

$$R_{2k-1}(\overline{R_{2k}} - \overline{R_{2k}}) = l_k|R_{2k-1}|^2 \in \mathbf{R}.$$

Hence $\mathrm{Im}(R_{2k}\overline{R_{2k}}) = \mathrm{Im}(R_{2k-1}\overline{R_{2k-2}}$. Thus (19) becomes

$$\mathrm{Im}\left(R_{2k-1}\overline{R_{2k}} - R_{2k-1}\overline{R_{2k-2}}\right) = -(\mathrm{Im}z)m_{k+1}|R_{2k}|^2.$$

Now sum from $k=0$ to $k=n-1$ to obtain

$$\operatorname{Im}\left(R_{2n-1}\overline{R_{2n}} - R_{-1}\overline{R_{-2}}\right) = -(\operatorname{Im}z)\sum_{k=0}^{n-1} m_{k+1}|R_{2k}|^2.$$

That is

$$\operatorname{Im}\left(\frac{R_{2n-1}}{R_{2n}}\right) = \frac{-\operatorname{Im}z}{|R_{2n}|^2}\sum_{k=0}^{n-1} m_{k+1}|R_{2k}|^2.$$

Thus $\operatorname{Im}f = \operatorname{Im}(\frac{R_{2n}}{R_{2n-1}})$ has the same sign as $\operatorname{Im}z$.

In general, consider a tree T which is the union of complementary subtrees T_i ($i=1,\ldots,q$) connected at a vertex $\mathbf{v}$. By Theorem 3.1,

$$\frac{\phi_N}{\phi_D} = \sum_{i=1}^{q} \frac{\phi_N^{(i)}}{\phi_D^{(i)}}.$$

Hence if each M-function $f_i = \frac{\phi_D^{(i)}}{\phi_N^{(i)}}$ of T_i at $\mathbf{v}$ is an S_0 function, then $\frac{\phi_D}{\phi_N}$ is also an S_0 function. The proof is complete. $\square$

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