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$\mathbb{R}$–ORBIT REFLEXIVE OPERATORS

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Abstract. We completely characterize orbit reflexivity and $\mathbb{R}$-orbit reflexivity for matrices in $\mathscr{M}_N(\mathbb{R})$. Unlike the complex case in which every matrix is orbit reflexive and $\mathbb{C}$-orbit reflexivity is characterized solely in terms of the Jordan form, the orbit reflexivity and $\mathbb{R}$-orbit reflexivity of a matrix in $\mathscr{M}_N(\mathbb{R})$ is described in terms of the linear dependence over $\mathbb{Q}$ of certain elements of $\mathbb{R}/\mathbb{Q}$. We also show that every $n \times n$ matrix over an uncountable field $\mathbb{F}$ is algebraically $\mathbb{F}$-orbit reflexive.

1. Introduction

The term *reflexive operator* was coined by P. R. Halmos [20], and studied by many authors, e.g., [1], [2], [3], [4], [5], [6], [7], [9], [12], [13], [16], [17], [18], [21], [22], [24], [25], [26], [30], [31], [33]. If $\mathscr{P}(T)$ denotes the set of all polynomials in the operator T, we say T is *reflexive* if S is in the strong operator topology closure $\mathscr{P}(T)^{-SOT}$ whenever S is an operator for which $Sx \in [\mathscr{P}(T)x]^-$ for every vector x. It was proved by J. Deddens and P. Fillmore [7] that an $n \times n$ complex matrix T is reflexive if and only if, for each eigenvalue λ of T, the two largest Jordan blocks corresponding to λ in the Jordan canonical form of T differ in size by at most 1. Later, D. Hadwin [12] characterized algebraic reflexivity (no closures) for an $n \times n$ matrix over an arbitrary field; in this setting the analog of the Jordan form contains blocks, which we will still call Jordan blocks, of the form

$$J_m(A) = \begin{pmatrix} A & I & 0 & \cdots & 0 \\ 0 & A & I & \ddots & \vdots \\ 0 & 0 & A & \ddots & 0 \\ \vdots & \vdots & \ddots & A & I \\ 0 & 0 & \cdots & 0 & A \end{pmatrix},$$

where A is the companion matrix of an irreducible factor of the minimal polynomial for T. When the irreducible factor has degree 1, the matrix A is 1×1 and an eigenvalue of T. Hadwin [12] proved that an $n \times n$ matrix T over a field $\mathbb{F}$ is (algebraically) reflexive if, for each eigenvalue of T, the two largest Jordan blocks differ in size by at most 1,

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and for an irreducible factor of the minimal polynomial of T that has degree greater than 1, the two largest Jordan blocks have the same size.

In [19] D. Hadwin, E. A. Nordgren, H. Radjavi and P. Rosenthal introduced the notion of an *orbit-reflexive operator*, where, in the definition of reflexivity, $\mathscr{P}(T)$ is replaced by

$$Orb(T) = \{T^n : n = 0, 1, 2, \ldots\}.$$

They proved that on a Hilbert space this class includes all normal operators, algebraic operators, compact operators, contractions and unilateral weighted shift operators. It was over twenty years before examples were constructed [10] and [29] (see also [8]) of operators that are not orbit reflexive. In [29] V. Müller and J. Vršovský proved that if $r(T) \neq 1$ ($r(T)$ denotes the spectral radius of T), then T is orbit reflexive. In [14] the authors proved that every polynomially bounded operator on a Hilbert space is orbit reflexive.

Recently, M. McHugh and the authors [15], [27] introduced the notion of $\mathbb{C}$-orbit reflexivity, where, in the definition of reflexivity, $\mathscr{P}(T)$ is replaced with

$$\mathbb{C}\text{-}orb(T) = \{\lambda T^n : \lambda \in \mathbb{C}, n \geqslant 0\},$$

and they proved that an $n \times n$ complex matrix T is $\mathbb{C}$-orbit reflexive if and only if it is nilpotent or, among all the Jordan blocks corresponding to all eigenvalues with modulus equal to the spectral radius $r(T)$ of T, the two largest blocks differ in size by at most 1. In [14] *null-orbit reflexivity* (where $Orb(T)$ is replaced with null-orb$(T) = Orb(T) \cup \{0\}$) was introduced, and it was shown that, while null-orbit reflexivity shares many nice properties with $\mathbb{C}$-orbit reflexivity, every $n \times n$ complex matrix is null-orbit reflexive.

In this paper we consider $\mathbb{R}$-orbit reflexivity. If T is an operator, we define

$$\mathbb{R}\text{-}orb(T) = \{\lambda T^n : \lambda \in \mathbb{R}, n \geqslant 0\},$$

and we say that T is $\mathbb{R}$*-orbit reflexive* if S is in the strong operator topology (SOT) closure of $\mathbb{R}$-orb(T) whenever S is an operator for which $Sx \in [\mathbb{R}\text{-}orb(T)x]^-$ for every vector x. In this paper we study $\mathbb{R}$-orbit reflexivity and orbit-reflexivity for a matrix in $\mathscr{M}_n(\mathbb{R})$. As mentioned above, in $\mathscr{M}_n(\mathbb{C})$ every matrix is orbit reflexive and $\mathbb{C}$-orbit reflexivity is characterized solely in terms of the Jordan form. Surprisingly, neither of these facts remain true for $\mathscr{M}_n(\mathbb{R})$; the characterizations involve a little number theory, i.e., linear dependence over $\mathbb{Q}$ of elements in $\mathbb{R}/\mathbb{Q}$.

2. Algebraic results

An irreducible factor $p(x)$ of a polynomial in $\mathbb{R}[x]$ has degree at most 2. If $p(x) \in \mathbb{R}[x]$ is monic and irreducible and $\deg p = 2$, then p has roots $\alpha \pm i\beta$ with $a, \beta \in \mathbb{R}$, $\beta \neq 0$, $p(x) = (x-\alpha)^2 + \beta^2$, and the corresponding companion matrix looks like $\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} = r \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, where

$$\alpha + i\beta = re^{i\theta}$$

with $r=\sqrt{\alpha^2+\beta^2}$ and $0\leqslant\theta<2\pi$. The matrix

$$R_\theta=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$$

acts on $\mathbb{R}^2$ as a counterclockwise rotation by the angle θ. More generally, if we identify $\mathbb{R}^2$ with $\mathbb{C}$, then $\begin{pmatrix}\alpha & -\beta\\ \beta & \alpha\end{pmatrix}$ acts as multiplication by $\alpha+i\beta$. An $m\times m$ Jordan block corresponding to $A=\begin{pmatrix}\alpha & -\beta\\ \beta & \alpha\end{pmatrix}$, is given by $J_m(A)$. However, $J_m(A)$ is similar to $rJ_m(R_\theta)$, and we will represent the Jordan blocks this way. A Jordan block J of T *splits,* or, *is splitting*, if the irreducible polynomial associated to it has degree 1, i.e., it corresponds to a real eigenvalue of T.

Since a real matrix may have empty spectrum, we let $\sigma_p(T)$ denote the *point spectrum* of T, the set of real eigenvalues of T. Note that $\sigma_p(T)=\varnothing$ is possible. We define the *spectral radius* to be

$$r(T)=\lim_{n\to\infty}\|T^n\|^{\frac{1}{n}},$$

which is the spectral radius of T considered as a matrix in $\mathscr{M}_n(\mathbb{C})$. Note that $r(J_m(R_\theta))$ $=1$ and $r\left(\begin{pmatrix}\alpha & -\beta\\ \beta & \alpha\end{pmatrix}\right)=\sqrt{\alpha^2+\beta^2}$.

If X is a vector space over a field $\mathbb{F}$, and T is a linear transformation on X, then $\mathscr{P}_{\mathbb{F}}(T)=\{p(T):p\in\mathbb{F}[t]\}$. A *linear manifold* M in X, is the translate of a linear subspace, i.e., nonempty subset M so that when $x\in M$, $M-x$ is a linear subspace.

We begin with a lemma on the cardinality of the field. In the case where the field is $\mathbb{R}$ or $\mathbb{C}$, the lemma is an immediate consequence of the Baire category theorem.

LEMMA 1. *If $\mathbb{F}$ is an uncountable field and n is a positive integer, then $\mathbb{F}^n$ is not a countable union of proper linear subspaces.*

Proof. Let $S=\left\{\left(1,x,x^2,\ldots,x^{n-1}\right):x\in\mathbb{F}\right\}$. Since any n distinct elements of S are linearly independent, the intersection of any proper linear subspace with S has cardinality at most $n-1$. However, S is uncountable, so S is not contained in a countable union of proper linear subspaces of $\mathbb{F}^n$. $\square$

THEOREM 2. *If $\mathbb{F}$ is an uncountable field, then every $T\in\mathscr{M}_N(\mathbb{F})$ is algebraically $\mathbb{F}$-orbit reflexive and algebraically orbit-reflexive.*

Proof. It is known from [16] that $\mathrm{AlgLat}_0(T)\cap\{T\}'=\mathscr{P}_{\mathbb{F}}(T)$, and that this algebra of operators has a separating vector e. We know from [15] that every nilpotent matrix is algebraically $\mathbb{F}$-orbit reflexive. Suppose A is an invertible $k\times k$ matrix and $S\in\mathbb{F}\text{-}\mathrm{OrbRef}_0(A)$. Then, for every $x\in\mathbb{F}^k$, there is a $\lambda\in\mathbb{F}$ and an $m\geqslant 0$ such that $Sx=\lambda A^m x$. Hence,

$$\mathbb{F}^k=\bigcup_{m=0}^{\infty}\ \bigcup_{\lambda\in\sigma_p(A^{-m}S)} Ker\left(A^{-m}S-\lambda\right),$$

which, by Lemma 1, implies there is an $m \geqslant 0$ and a $\lambda \in \mathbb{F}$ such that $S = \lambda A^m$. Hence A is algebraically $\mathbb{F}$-orbit reflexive. Since every $T \in \mathscr{M}_n(\mathbb{F})$ is the direct sum of a nilpotent matrix N and an invertible matrix A, it follows that every $S \in \mathbb{F}\text{-OrbRef}_0(T)$ is a direct sum of αN^s and βA^t for $\alpha, \beta \in \mathbb{F}$ and integers $s, t \geqslant 0$. It follows that $S \in \text{AlgLat}_0(T) \cap \{T\}'$; whence there is a polynomial $p \in \mathbb{F}[x]$ such that $S = p(T)$. However, there is a $\lambda \in \mathbb{F}$ and an $m \geqslant 0$ such that

$$p(T)e = Se = \lambda T^m e.$$

Since e is separating for $\mathscr{P}(T)$, we see that $S = p(T) = \lambda T^m$, which implies T is $\mathbb{F}$-orbit reflexive. The proof that T is algebraically orbit reflexive is very similar. $\square$

COROLLARY 3. *If $T \in \mathscr{M}_n(\mathbb{R})$ and $\{T^k : k \geqslant 0\}$ is finite, e.g., $T^N = I$ or $T^N = 0$ for some positive integer N, then* $\mathbb{R}\text{-OrbRef}(T) = \mathbb{R}\text{-OrbRef}_0(T) = \mathbb{R}\text{-Orb}(T)$ *and* $\text{OrbRef}(T) = \text{OrbRef}_0(T) = \text{Orb}(T)$.

Proof. Since $\{T^k : k \geqslant 0\}$ is finite, we know, for every vector x, that $\mathbb{R}\text{-Orb}(T)x$ and $\text{Orb}(T)x$ are closed, implying $\mathbb{R}\text{-OrbRef}(T) = \mathbb{R}\text{-OrbRef}_0(T)$ and $\text{OrbRef}(T) = \text{OrbRef}_0(T)$. $\square$

COROLLARY 4. *If $T \in \mathscr{M}_n(\mathbb{R})$, $T = A \oplus B$ with $A^N = I$ for some minimal $N \geqslant 1$ and $r(B) < 1$, then T is $\mathbb{R}$-orbit reflexive.*

Proof. Suppose $S \in \mathbb{R}\text{-OrbRef}(T)$. Then $S = S_1 \oplus S_2$ and, by Corollary 3, we know that $S_1 = \lambda A^s$ for some $\lambda \in \mathbb{R}$ and some $s \geqslant 0$. If $S_1 = 0$ it easily follows by considering $x \oplus y$ with $x \neq 0$ and y arbitrary, that $S_2 = 0$, which implies $S = 0$. Hence we can assume that $S_1 \neq 0$.

Note that

$$S_1^N = \lambda^N \left(A^N\right)^s = \lambda^N.$$

Let $E = \left\{e^{2\pi i k/n}\lambda : k = 1, \ldots, n\right\}$. Choose a separating unit vector x_0 for $\mathscr{P}_R(A)$. If $S_1 x_0 = \lambda_1 A^t x_0$, we have $S_1 = \lambda_1 A^t$, which implies $\lambda_1 \in E$. Suppose y is in the domain of B, then there is a sequence $\{k_m\}$ of positive integers and a sequence $\{\beta_m\}$ in $\mathbb{R}$ such that

$$\beta_m T^{k_m}(x_0 \oplus y) \to S_1 x_0 \oplus S_2 y.$$

We have $\beta_m A^{k_m} x_0 \to \lambda A^s x_0$, which implies $\{\beta_m\}$ is bounded. If $\{k_m\}$ is unbounded, then it has a subsequence diverging to ∞, which implies $S_2 y = 0$, since $\left\|B^k\right\| \to 0$ as $k \to \infty$. If $\{k_m\}$ is bounded, then it has a subsequence $\left\{\beta_{k_j}\right\}$ with a constant value t, and we get $\beta_{m_j} \to \lambda_1$ for some $\lambda_1 \in E$. Hence the domain of B is a countable union,

$$\ker S_2 \cup \bigcup_{k \in \mathbb{N}, \gamma \in E} \ker\left(S_2 - \gamma B^k\right).$$

It follows from Lemma 1 that $S_2 \in \mathscr{P}_{\mathbb{R}}(B)$. If we choose a vector y_0 that is separating for $\mathscr{P}_{\mathbb{R}}(B)$, we see from $S(x_0 \oplus y_0) \in [\mathbb{R}\text{-Orb(T)}(x_0 \oplus y_0)]^-$, that $S \in \mathbb{R}$-Orb(T). $\square$

3. Main results

A key ingredient in our proofs is the following well-known result from number theory. We sketch the elementary proof for completeness. For notation we let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle, $\mathbb{T}^k$ a direct product of k copies of $\mathbb{T}$, and $\mu_k = \mu \times \cdots \times \mu$ be Haar measure on $\mathbb{T}^k$, where μ is normalized arc length on $\mathbb{T}$. If $\lambda = (z_1, \cdots, z_k) \in \mathbb{T}^k$ and we define

$$\lambda^n = (z_1^n, \ldots, z_k^n)$$

for $n = 0, 1, 2, \ldots$.

LEMMA 5. *Suppose* $\theta_1, \ldots, \theta_k \in \mathbb{R}$, *and let* $\lambda = \left(e^{i\theta_1}, \ldots, e^{i\theta_k}\right)$. *The following are equivalent:*

1. $\left\{\lambda, \lambda^2, \ldots\right\}$ *is dense in* $\mathbb{T}^k$,
2. $\{1, \theta_1/2\pi, \ldots, \theta_k/2\pi\}$ *is linearly independent over* $\mathbb{Q}$,
3. *for every* $f \in C\left(\mathbb{T}^k\right)$ *we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(\lambda^n) = \int_{\mathbb{T}^k} f d\mu_k.$$

Proof. If $f(z_1, \ldots, z_k) = z_1^{m_1} \cdots z_k^{m_k}$ for integers $m_1, \ldots, m_k$, then statement (2) is equivalent to saying $f(\lambda) \neq 1$ whenever $(m_1, \ldots, m_k) \neq (0, \ldots, 0)$. For such a monomial f we know that $\int_{\mathbb{T}^k} f d\mu_k = 0$, and we know that $f(\lambda^n) = f(\lambda)^n$ for $n \geqslant 1$. Thus statement (2) implies that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(\lambda^n) = \lim_{N\to\infty} \frac{1}{N} \frac{1 - f(\lambda)^N}{1 - f(\lambda)} f(\lambda) \to 0 = \int_{\mathbb{T}^k} f d\mu_k.$$

It follows from the Stone-Weierstrass theorem that the span of the monomials is dense in $C\left(\mathbb{T}^k\right)$, so we see that $(2) \Longrightarrow (3)$. On the other hand (3) implies that, for every nonnegative continuous function f vanishing on $\left\{\lambda, \lambda^2, \ldots\right\}$ we must have $\int_{\mathbb{T}^k} f d\mu_k = 0$, which implies $f = 0$. If $x \in \mathbb{T}^k \backslash \left\{\lambda, \lambda^2, \ldots\right\}^-$, there is a nonnegative continuous function f vanishing on $\left\{\lambda, \lambda^2, \ldots\right\}$ with $f(x) \neq 0$. Hence $(3) \Longrightarrow (1)$. If f is a nonconstant monomial and $f(\lambda) = 1$, then the closure of $\left\{\lambda, \lambda^2, \ldots\right\}$ is contained in $f^{-1}(\{1\})$, which proves that $(1) \Longrightarrow (2)$. $\square$

The next two results show that in $\mathscr{M}_N(\mathbb{R})$ orbit reflexivity is not the same as in $\mathscr{M}_N(\mathbb{C})$.

LEMMA 6. *Suppose* $k \in \mathbb{N}$, $\theta_1, \ldots, \theta_k \in [0, 2\pi)$, *and* $T \in \mathscr{M}_N(\mathbb{R})$ *is a direct sum of* $R_{\theta_1} \oplus \cdots \oplus R_{\theta_k} \oplus B \oplus C$ *with* $B^2 = 1$ *and* $r(C) < 1$. *(The summands* B *and* C *might not be present.) The following are equivalent:*

1. *T is orbit reflexive*

2. *T is $\mathbb{R}$-orbit reflexive*

3. *There are nonzero integers $s_1,\ldots,s_k$ and an integer t such that*

$$\sum_{j=1}^{k} s_j\theta_j = 2\pi t.$$

4. *For every $j\in\{1,\ldots,k\}$, $\theta_j/2\pi \in sp_{\mathbb{Q}}(\{1\}\cup\{\theta_i/2\pi : 1\leqslant i\neq j\leqslant k\})$.*

Proof. The equivalence of (4) and (3) is easy.

$(1)\Longrightarrow(4)$ and $(2)\Longrightarrow(4)$. Assume (4) is false. We can assume that

$$\theta_1/2\pi \notin sp_{\mathbb{Q}}(\{1\}\cup\{\theta_i/2\pi : 2\leqslant i\leqslant k\}).$$

We can assume that $\{1,\theta_2/2\pi,\ldots,\theta s/2\pi\}$ is a basis for the linear span over $\mathbb{Q}$ of $\{1\}\cup\{\theta_i/2\pi : 2\leqslant i\leqslant k\}$, which makes $\theta_1/2\pi,\theta_2/2\pi,\ldots,\theta_s/2\pi$ irrational, and makes $\{1,\theta_1/2\pi,\ldots,\theta s/2\pi\}$ linearly independent over $\mathbb{Q}$. Since each $\theta_j/2\pi$, $s<j\leqslant k$ is a rational linear combination of $1,\theta_2/2\pi,\ldots,\theta s/2\pi$, there is a positive integer d such that, for $s<j\leqslant k$, each $d\theta_j/2\pi$ is an integral linear combination of $1,\theta_2/2\pi,\ldots,\theta s/2\pi$. Suppose $\alpha\in[0,2\pi)$. Since $\{1,\theta_1/4\pi d,\ldots,\theta s/4\pi d\}$ is linearly independent over $\mathbb{Q}$, it follows from Lemma 5 that there is a sequence $\{m_n\}$ of positive integers such that $m_n\to\infty$,

$$R_{\theta_1}^{m_n} = R_{m_n\theta_1} \to R_{\alpha/2d},$$

$$R_{\theta_j}^{m_n} = R_{m_n\theta_j} \to I$$

for $2\leqslant j\leqslant s$. This implies that $R_{\theta_1}^{2dm_n} = R_{2dm_n\theta_1}\to R_\alpha$ and $R_{\theta j}^{2dm_n} = R_{2dm_n\theta_j}\to I$ for $2\leqslant j\leqslant s$. If $s<j\leqslant k$, there are integers $t_2,\ldots,t_s$ and t such that $d\theta_j = t2\pi + \sum_{i=2}^{s} t_i\theta_i$, which implies

$$R_{\theta j}^{2dm_n} = I^{2tm_n}\prod_{i=2}^{s}\left(R_{m_n\theta_i}\right)^{2t_i} \to I.$$

Moreover,

$$(B\oplus C)^{2dm_n} = B^{2dm_n}\oplus C^{2dm_n} \to I\oplus 0 = P.$$

Let $F=\begin{pmatrix}0&1\\1&0\end{pmatrix}$, and define $S = F\oplus I\oplus\cdots\oplus I\oplus P$. It follows from the fact that, for every $x\in\mathbb{R}^2$ there is an $\alpha\in[0,2\pi)$ such that $Fx = R_\alpha x$, that $S\in\mathrm{OrbRef}(T)\subseteq\mathbb{R}$-$\mathrm{OrbRef}(T)$. Since $FR_{\theta_1}\neq R_{\theta_1}F$ (because $\sin\theta_1\neq 0$), it follows that $ST\neq TS$, and we see that both (1) and (2) are false.

$(3)\Longrightarrow(2)$. Suppose (3) is true. If $k=1$, then $\theta_1/2\pi\in\mathbb{Q}$, and $R_{\theta_1}^N = I$ for some positive integer N, which, by Corollary 4, implies T is $\mathbb{R}$-orbit reflexive. Hence we can assume $k\geqslant 2$, which, by (3), implies $\theta_1/2\pi\notin\mathbb{Q}$. Suppose $S\in\mathbb{R}$-$\mathrm{OrbRef}(T)$. Since

$\mathbb{R}$-OrbRef(T) is contained in AlgLat(T), we can write $S = S_1 \oplus \cdots \oplus S_k \oplus\ D \oplus E$. Suppose $x \neq 0$ is in the domain of S_1. We consider two cases:

Case 1. $S_1 x = 0$. If y is any vector orthogonal to the domain of S_1, there is a sequence $\{m_n\}$ of nonnegative integers and a sequence $\{\lambda_n\}$ in $\mathbb{R}$ such that $S(x \oplus y) = \lim \lambda_n T^{m_n}(x \oplus y)$. Thus $|\lambda_n|\,\|x\| \to \|S_1 x\| = 0$, which implies $\lambda_n \to 0$, and since $\{\|T^n\|\}$ is bounded, we see that $S(x \oplus y) = 0$. Thus $0 = S_2 = \cdots = S_k$ and $D = 0, E = 0$. Since $k \geqslant 2$, and arguing as above (when we showed $S_1 = 0 \Longrightarrow S_2 = 0$), we know $S_1 = 0$, and thus $S = 0$.

Case 2. $S_1 x \neq 0$. Let $x_1 = x$, and choose x_j in the domain of S_j for $2 \leqslant j \leqslant k$ with each $\|x_j\| = \|x\|$, and let $u = x \oplus x_2 \oplus \cdots \oplus x_k \oplus 0 \oplus 0$. Since $R_{\theta_1} \oplus \cdots \oplus R_{\theta_k}$ is an isometry and $S \in \mathbb{R}$-OrbRef(T), it follows that there is a sequence $\{m_n\}$ of nonnegative integers and a sequence $\{\lambda_n\}$ in $\mathbb{R}$ such that $0 \neq Su = \lim_{n\to\infty} \lambda_n T^{m_n} u$. Hence, $\{\lambda_n\}$ is bounded, so we can assume that $\lambda_n \to \lambda$ for some nonzero $\lambda \in \mathbb{R}$, and we can assume that $T^{m_n} \to R_{\alpha_1} \oplus \cdots \oplus R_{\alpha_k} \oplus F \oplus G$ with $0 \leqslant \alpha_1, \ldots, \alpha_k < 2\pi$. We know that $|\lambda| = \|S_1 x\| \neq 0$, and, for $1 \leqslant j \leqslant k$, $S_j x_j = \|S_1 x\| R_{\alpha_j} x_j$ if $\lambda > 0$ and $S_j x_j = \|S_1 x\| R_{\alpha_j + \pi} x_j$ if $\lambda < 0$. Moreover, since $R_{\theta_j}^{m_n} \to R_{\alpha_j}$ for $1 \leqslant j \leqslant k$, we have, from (3), that $\sum_{j=1}^{k} s_j \alpha_j \in 2\pi\mathbb{Z}$, and thus $\sum_{j=1}^{k} s_j(\alpha_j + \pi) \in \pi\mathbb{Z}$. Suppose now we replace x_1 with another vector y in the domain of S_1 with $\|y\| = \|x_1\|$, we get real numbers $\beta_1, \ldots, \beta_k$ such that $S_1 y = \|S_1 y\| R_{\beta_1} y$ and $S_j x_j = \|S_1 y\| R_{\beta_j} x_j = \|S_1 y\| R_{\alpha_j} x_j$ for $2 \leqslant j \leqslant k$, and such that $\sum_{j=1}^{k} s_j \beta_j \in \pi\mathbb{Z}$. However, for $2 \leqslant j \leqslant k$, we must have $\beta_j - \alpha_j \in \pi\mathbb{Z}$. Hence, $s_1\beta_1 - s_1\alpha_1 \in \pi\mathbb{Z}$. Hence the domain of S_1 is the union

$$\bigcup_{n\in\mathbb{Z}} \ker\left(S_1 - \|S_1 x\| R_{\alpha_1 + n\pi/s_1}\right),$$

which, by Lemma 1, implies that there is a $\gamma_1 \in [0, 2\pi) \cap \left(\alpha_1 + \frac{\pi}{s_1}\mathbb{Z} + 2\pi\mathbb{Z}\right)$ such that $S_1 = \|S_1 x\| R_{\gamma_1}$. Similarly, we get, for $2 \leqslant j \leqslant k$, that $S_j = \|S_1 x\| R_{\gamma_j}$ for some $\gamma_j \in [0, 2\pi)$.

Applying the same reasoning we see that $D = \|S_1 x\| B$ or $D = -\|S_1 x\| B$. Also, for every f in the domain of C we get $Ef \in \mathbb{R}$-Orb$(C)f$, so, by Theorem 2, $E \in \mathbb{R}$-Orb(C). We therefore have $S_j \in \mathscr{P}_{\mathbb{R}}\left(R_{\theta_j}\right)$ for $1 \leqslant j \leqslant k$, $D \in \mathscr{P}_{\mathbb{R}}(B)$, and $E \in \mathscr{P}_{\mathbb{R}}(C)$. If we choose separating vectors v_j for each $\mathscr{P}_{\mathbb{R}}\left(R_{\theta_j}\right)$ $(1 \leqslant j \leqslant k)$ and w_1 for $\mathscr{P}_{\mathbb{R}}(B)$ and w_2 for $\mathscr{P}_{\mathbb{R}}(C)$, and we let $\eta = v_1 \oplus \cdots \oplus v_k \oplus w_1 \oplus w_2$, then there is a sequence $\{q_n\}$ of nonnegative integers and a sequence $\{t_n\}$ in $\mathbb{R}$ such that

$$t_n T^{q_n} \eta \to S\eta,$$

and it follows that

$$t_n T^{q_n} \to S.$$

Thus $S \in \mathbb{R}$-Orb$(T)^{-SOT}$.

$(2) \Longrightarrow (1)$. Suppose (2) is true, let e be a separating vector for $\mathscr{P}_{\mathbb{R}}(T)$, and suppose $S \in \mathrm{OrbRef}(T) \subseteq \mathbb{R}\text{-}\mathrm{OrbRef}(T) = \mathbb{R}\text{-}\mathrm{Orb}(T) \subseteq \mathscr{P}_{\mathbb{R}}(T)$ (by (2)). Since there is a sequence $\{m_n\}$ of nonnegative integers such that $T^{m_n}e \to Se$, it follows that $T^{m_n} \to S$. Hence (1) is proved. $\square$

THEOREM 7. *A matrix $T \in \mathscr{M}_N(\mathbb{R})$ fails to be orbit reflexive if and only if it is similar to a matrix of the form in Lemma 6 that is not orbit reflexive.*

Proof. We know from [14, Lemma 17] that if one of the sets $\left\{x \in \mathbb{R}^N : T^k x \to 0\right\}$ or $\left\{x \in \mathbb{R}^N : \left\|T^k x\right\| \to \infty\right\}$ is not a countable union of nowhere dense subsets of $\mathbb{R}^N$, then T is orbit reflexive. Thus if $r(T) < 1$, then T is orbit reflexive. If $r(T) > 1$, then the Jordan form shows that $\left\{x \in \mathbb{R}^N : \left\|T^k x\right\| \to \infty\right\}$ has nonempty interior, which implies T is orbit reflexive. Hence we are left with the case where $r(T) = 1$. Moreover, if the Jordan form of T has an $m \times m$ block of the form $\begin{pmatrix} A & I_2 & \cdots & 0 \\ 0 & A & \ddots & \vdots \\ \vdots & 0 & \ddots & I_2 \\ 0 & \cdots & 0 & A \end{pmatrix}$ with $A = \pm I$ or $A = R_\theta$, then for any vector $x \in \mathbb{R}^N$ whose m^{th}-coordinate relative to this summand is nonzero, we have $\left\|T^k x\right\| \to \infty$; whence T is orbit reflexive. Thus the Jordan form of a matrix that is not orbit reflexive must be as the matrix in Lemma 6. $\square$

If X is a Banach space over $\mathbb{R}$, and $T \in B(\mathbb{R})$ is algebraic, i.e., there is a nonzero polynomial $p \in \mathbb{R}[x]$ such that $p(T) = 0$, then, as a linear transformation, T has a Jordan form with finitely many distinct blocks, but possibly with some of the blocks having infinite multiplicity.

COROLLARY 8. *Suppose X is a Banach space over $\mathbb{R}$ and $T \in B(X)$ is algebraic. Then T fails to be orbit-reflexive if and only if $r(T) = 1$, and the Jordan form for T has one block R_{θ_1} of multiplicity 1, other blocks of the form $R_{\theta_2}, \ldots, R_{\theta_k}$ with $\theta_1/2\pi \notin sp_{\mathbb{Q}}\{1, \theta_2, \ldots, \theta_k\}$, the remaining blocks of the form $\pm I$ or blocks with spectral radius less than 1.*

Proof. Suppose T has the indicated form. Then there is an invertible operator $D \in B(X)$ such that $D^{-1}TD = R_{\theta_1} \oplus A \oplus B$ with $r(A) = 1$ and $r(B) < 1$. Let $S = F \oplus 1 \oplus 0$. Suppose $x \in X$. Choose a finite-dimensional invariant subspace M for T of the form $M = M_1 \oplus M_2 \oplus M_3$, with M_1 equal to the domain of S_1 such that $x \in M$. It follows from the assumptions on T and the proof of Theorem 7 that $S|M \in \mathrm{OrbRef}(T|M)$. In particular, Sx is in the closure $\mathrm{Orb}(T)x$. Thus $S \in \mathrm{OrbRef}(T)$, but $ST \neq TS$, so T is not orbit reflexive.

On the other hand, if T does not have the described form, then, given $S \in \mathrm{OrbRef}(T)$, vectors $x_1, \ldots, x_n$ and $\varepsilon > 0$, there is a finite-dimensional invariant subspace E of X containing $x_1, \ldots, x_n$ such that $T|E$ is orbit reflexive because of the conditions in Theorem 7. Hence, since $S|E \in \mathrm{OrbRef}(T|E)$, there is an integer $m \geqslant 0$ such that

$$\left\|Sx_j - T^m x_j\right\| < \varepsilon$$

for $1 \leqslant j \leqslant n$. Thus S is in the strong operator closure of $\mathrm{Orb}(T)$. Thus T is orbit reflexive. $\square$

THEOREM 9. *A matrix $T \in \mathscr{M}_N(\mathbb{R})$ fails to be $\mathbb{R}$-orbit reflexive if and only if $r(T) \neq 0$ with the largest size of a Jordan block with spectral radius $r(T)$ being m, and either*

1. *every Jordan block of T with spectral radius $r(T)$ splits over $\mathbb{R}$, and the largest two such blocks differ in size by more than 1, or*
2. *there exist $k \in \mathbb{N}$, $\theta_1, \dots, \theta_k \in [0, 2\pi)$ such that the direct sum of the non-splitting $m \times m$ Jordan blocks of $T/r(T)$ that have spectral radius 1 is similar to*

$$J_m\left(R_{\theta_1}\right) \oplus \cdots \oplus J_m\left(R_{\theta_k}\right)$$

with $\theta_1/2\pi \notin sp_{\mathbb{Q}}\{1, \theta_2/2\pi, \dots, \theta_k/2\pi\}$.

Proof. We know that if $r(T) = 0$, then T is nilpotent, which, by Corollary 3, implies T is $\mathbb{R}$-orbit reflexive. Hence we can assume that $r(T) > 0$. Replacing T by $T/r(T)$, we can, and do, assume $r(T) = 1$.

In the case where every Jordan block of T with spectral radius $r(T)$ splits, the proof that T is not $\mathbb{R}$-orbit reflexive is equivalent to the condition in (1) is exactly the same at the proof of Theorem 7 in [15].

Next suppose T satisfies (2). Then, as in the proof of $(1) \Longrightarrow (4)$ in Lemma 6, given $\alpha \in [0, 2\pi)$, we can choose a sequence $\{s_d\}$ of positive integers converging to ∞ such that $s_d - m + 1$ is even for each $d \geqslant 1$ and such that $R_{\theta_1}^{s_d - m + 1} \to R_\alpha$ and $R_{\theta_j}^{s_d - m + 1} \to I$ for $2 \leqslant j \leqslant k$. It follows that

$$\frac{1}{\binom{s_d}{m-1}} J_m^{s_d}\left(R_{\theta_1}\right) \to \begin{pmatrix} 0 & \cdots & 0 & R_\alpha \\ 0 & 0 & \cdots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

and for any of the other splitting or non-splitting $m \times m$ Jordan block J with $r(J) = 1$, we have

$$\frac{1}{\binom{s_d}{m-1}} J^{s_d} \to \begin{pmatrix} 0 & \cdots & 0 & I \\ 0 & 0 & \cdots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

For any block J with $r(J) < 1$ or with size smaller than $m \times m$, we have

$$\frac{1}{\binom{s_d}{m-1}} J^{s_d} \to 0.$$

Arguing as in the proof of $(1) \Longrightarrow (4)$ in Lemma 6, we see that, if F is the flip matrix, and S is the matrix that is $\begin{pmatrix} 0 \cdots & 0 & F \\ 0 \ 0 & \cdots & \vdots \\ \vdots \ \vdots & \ddots & 0 \\ 0 \ 0 & \cdots & 0 \end{pmatrix}$ on the domain of $J_m\left(R_{\theta_1}\right)$, $\begin{pmatrix} 0 \cdots & 0 & I \\ 0 \ 0 & \cdots & \vdots \\ \vdots \ \vdots & \ddots & 0 \\ 0 \ 0 & \cdots & 0 \end{pmatrix}$ on the domains of each of the remaining $m \times m$ blocks J with $r(J) = 1$, and 0 on the domains of the remaining blocks, then $S \in \mathbb{R}\text{-OrbRef}(T)$, but $ST \neq TS$. Hence T is not $\mathbb{R}$-orbit reflexive.

We need to show that if (2) holds with the condition on θ_1 replaced with condition (3) in Lemma 6, then T must be $\mathbb{R}$-orbit reflexive. If $m = 1$, then T has the form as in Lemma 6, so we can assume that $m > 1$. Suppose $S \in \mathbb{R}\text{-OrbRef}(T)$ and $0 \neq \begin{pmatrix} 0 \\ 0 \\ \vdots \\ x \end{pmatrix} = X$ is in the domain of $J_m\left(R_{\theta_1}\right)$. We consider three cases.

Case 1. $S_1(X) = S(X) = 0$, where S_1 is the restriction of S to the domain of $J_m\left(R_{\theta_1}\right)$. Suppose Y is orthogonal to the domain of $J_m\left(R_{\theta_1}\right)$, and using the fact that there is a sequence $\{m_n\}$ of nonnegative integers and a sequence $\{\lambda_n\}$ in $\mathbb{R}$ such that

$$S(X+Y) = \lim_{n\to\infty} \lambda_n T^{m_n}(X+Y),$$

which means that

$$0 = S(X) = \lim_{n\to\infty} \lambda_n T^{m_n}(X),$$

and

$$S(Y) = \lim_{n\to\infty} \lambda_n T^{m_n}(Y).$$

However, the former implies

$$\lim_{n\to\infty} |\lambda_n| \binom{m_n}{m-1} = 0,$$

which implies $S(Y) = 0$. If $k \geqslant 2$, then $S_2 = 0$, where S_2 is the restriction of S to the domain of $J_m\left(R_{\theta_2}\right)$, so the preceding arguments imply that $S_1 = 0$; whence, $S = 0$.

We therefore suppose $k = 1$, and it follows from (3) that $\theta_1/2\pi \in \mathbb{Q}$, i.e., $\theta_1 = 2\pi p/q$ with $1 \leqslant p < q$ relatively prime integers. We can identify $\mathbb{R}^2$ with $\mathbb{C}$, and we can write $x = re^{\alpha}$ with $r > 0$. Since $S(X) = 0$, we have $S\left(\frac{1}{r}X\right) = 0$, so we can assume $x = e^{i\alpha}$. Then $\left\{\lambda R_{\theta_1}^s x : \lambda \in \mathbb{R}, 1 \leqslant s \leqslant q\right\}$ is the set of all complex numbers whose argument belongs to $\{\alpha + jp2\pi/q : 1 \leqslant j \leqslant q\} + \pi\mathbb{Z}$. Choose numbers β and γ with $\alpha < \beta < \gamma < \alpha + \pi/8$ such that

$$[\{\gamma + jp2\pi/q : 1 \leqslant j \leqslant q\} + \pi\mathbb{Z}] \cap [\{\beta + jp2\pi/q : 1 \leqslant j \leqslant q\} + \pi\mathbb{Z}] = \varnothing.$$

Since the argument of $e^{i\alpha}+te^{i\gamma}$ ranges over (α,γ) as t ranges over $(0,\infty)$, we can chose $t>0$ so that the argument of $e^{i\alpha}+te^{i\gamma}$ is β. Now let $W=\begin{pmatrix}0\\0\\\vdots\\te^{i\gamma}\end{pmatrix}$ in the domain of $J_m\left(R_{\theta_1}\right)$. Then $S(X+W)=SX+SW=SW$. However, the nonzero coordinates of any vector in the closure of $\mathbb{R}$-Orb$(T)(X+W)$ are all complex numbers with arguments in $\{\gamma+jp2\pi/q:1\leqslant j\leqslant q\}+\pi\mathbb{Z}$ and the nonzero coordinates of any vector in the closure of $\mathbb{R}$-Orb$(T)(X+W)$ are all complex numbers with arguments in $\{\beta+jp2\pi/q:1\leqslant j\leqslant q\}+\pi\mathbb{Z}$ Hence $S_1\begin{pmatrix}0\\0\\\vdots\\y\end{pmatrix}=0$ for every choice of y. We can apply similar arguments to each of the other coordinates to get $S_1=0$, which implies $S=0$.

Case 2. $S(X)=S_{1^{\text{‘}}}(X)=\lambda_0T^{n_0}(X)\neq 0$. Note that if $\lambda T^s(X)=\begin{pmatrix}x_1\\x_2\\\vdots\\x_m\end{pmatrix}\neq 0$, then

$$\frac{\|x_{m-1}\|}{\|x_m\|}=s,\ \text{and}\ R_{\theta_1}^{-s}x_m=\lambda x.$$

This means that if $\{m_n\}$ is a sequence of nonnegative integers and $\{\lambda_n\}$ is a sequence in $\mathbb{R}$, and $T^{m_n}(X)\to S(X)$, then, eventually $m_n=n_0$ and $\lambda_n\to\lambda_0$. It follows that $S=\lambda_0T^{n_0}$ on the orthogonal complement of the domain of S_1. If $k\geqslant 2$, we can argue (using S_2) that $S=\lambda_0T^{n_0}$. If $k=1$, we can use M_1,M_2,M_3 as in Case 1 to show that $S=\lambda_0T^{n_0}$.

Case 3. $S(X)=S_1(X)=\begin{pmatrix}x_1\\x_2\\\vdots\\x_m\end{pmatrix}\neq 0$, but $x_m=0$. If $\{s_n\}$ is a sequence of nonnegative integers and $\{\lambda_n\}$ is a sequence in $\mathbb{R}$ and $\lambda_nT^{s_n}(X)\to S(X)$, we must have $\lambda_n\to 0$, and thus $s_n\to\infty$, and $\left\{|\lambda_n|\binom{s_n}{m-1}\right\}$ bounded. Thus $S_1(X)=\begin{pmatrix}x_1\\0\\\vdots\\0\end{pmatrix}$. It follows that if J is an $m\times m$ Jordan block of T with $r(J)=1$ and whose domain is orthogonal to the domain of S_1, then the restriction of S to the domain of J is a matrix whose only nonzero entry is in the first row and m^{th} column. The restriction of S to the domain of a block J with $r(J)<1$ or whose size is smaller than $m\times m$ must be 0. If $k\geqslant 2$, the S_1 also has an operator matrix whose only nonzero entry is in the first row and m^{th} column. If $k=1$, then $\theta_1/2\pi$ is rational, and we can argue with M_1, M_2, M_3 as in Case 1 to see that S_1 has a matrix whose only nonzero entry is in the first row and m^{th} column. If the $m\times m$ Jordan blocks of T are $J_m\left(R_{\theta_1}\right)\oplus\cdots\oplus J_m\left(R_{\theta_k}\right)\oplus J_m\left(I_a\right)\oplus J_m\left(-I_b\right)$ (I_a is an $a\times a$ identity matrix), then the corresponding decomposition of S is a direct sum of $\begin{pmatrix} 0 & \cdots & 0 & A_j \\ 0 & 0 & \cdots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix}$, $1\leqslant j\leqslant k+2$. It is easily seen that $A_1\oplus\cdots\oplus A_{k+2}$ is in $\mathbb{R}$-OrbRef$\left(R_{\theta_1}\oplus\cdots\oplus R_{\theta_k}\oplus I_a\oplus -I_b\right)$. Since $\theta_1,\ldots,\theta_k$ satisfy condition (3) in Lemma 6, it follows from Lemma 6 that $R_{\theta_1}\oplus\cdots\oplus R_{\theta_k}\oplus I_a\oplus -I_b$ is $\mathbb{R}$-orbit reflexive, so there is a sequence $\{s_n\}$ with $s_n\to\infty$ and a sequence $\{\lambda_n\}$ in $\mathbb{R}$ such that $\lambda_n\left(R_{\theta_1}\oplus\cdots\oplus R_{\theta_k}\oplus I_a\oplus -I_b\right)^{s_n-m+1}\to A_1\oplus\cdots\oplus A_{k+2}$. Hence

$$\lambda_n T^{s_n}\to S.$$

Hence T is $\mathbb{R}$-orbit reflexive. $\square$

REMARK 10. Using the ideas of the proof of Corollary 8 it is possible to characterize $\mathbb{R}$-orbit reflexivity for an algebraic operator on a Banach space in terms of its algebraic Jordan form.

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