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DETERMINANT INEQUALITIES CONCERNING THE SOLUTION OF WAVE DIFFRACTION PROBLEMS WITH SEVERAL PARALLEL SOMMERFELD HALF PLANES

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Dedicated to the memory of Professor Erhard Meister

(*Communicated by I. M. Spitkovsky*)

Abstract. This paper deals with a system of parallel non-staggered half planes as scatterers for plane wave fields incident perpendicularly to the edges. In cases with first or second kind boundary conditions, the determinant of the positive definite real part of the corresponding L^2-lifted Wiener–Hopf matrix $H_N(\varepsilon)$, with $\varepsilon = \varepsilon_1, .., \varepsilon_{N-1}$ to lie in the unit circle, is shown to satisfy an inequality with lower and upper bounds for any positive spacings between the scatterers. The main result is based upon a sharpening of Hadamard's inequality. The derived relations can be used to prove a priori estimates in the construction of the inverses by operator Neumann series. The matrices which appear in the estimates are suitable to test the computation accuracy by machines via Corollary 3.1 and are related to infinite products of certain determinants, which could be of interest for number theory.

1. Introduction

In this article a set of complex–valued matrices (see Definition 2.1) is under consideration, which are Fourier symbols of pseudo differential operators for boundary value problems to linear wave equations. The matrices were derived in the paper of Meister, Rottbrand and Speck [15] in 1991, and have their origin back to the diploma thesis of the author. More precisely, they represent convolution kernels for quadratic systems of Wiener–Hopf boundary integral equations in the Lebesgue sense $\mathrm{L}^2(\mathbf{R})$. We make the real line a closed contour by going over infinity in the complex plane. Solutions of the boundary value problems are sought in H^s Sobolev energy norm spaces, that is $s = 1$ in particular. An exact analytical solution requires a suitable factorization of non-rational matrix functions with respect to the involved spaces. In the function theoretical sense this is solving discontinuous Riemann–Hilbert boundary value problems as matricial coupling problems for the left and right of certain contours. See the article of Meister [14] for instance. Introductory work in this field can be found in the books of Meister [13] and Muskhelishvili [18]. Clearly, to obtain an exact factorization often

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fails, and a numerical treatment is indicated instead. So, good error estimation formulas are needed when approximating inverse operators. Invertibility by a Neumann series as application of Banach's fixed point principle is assured for strongly elliptic operators, which is positive definiteness of the real part of the symbol matrix in this context. It is noted that invertibility and equivalently zero partial indices are a consequence of the strong ellipticity property. Fundamental theoretical work with applications in this direction is given by Meister and Speck [17], and the books of Speck [20], Litvinchuk and Spitkovsky [12], Clancey and Gohberg [6], Gohberg and Krupnik [9]. For a well working technique lower/upper bounds of determinants are of particular interest.

The matrices we are concerned with, have a special symmetric structure (Section 2). They are Toeplitz matrices and seem not investigated in detail in the existing literature. Our Wiener–Hopf Fourier symbol matrices stem from diffraction of plane waves by a set of parallel screens of vanishing thickness, say Sommerfeld half planes, with physically balancing boundary conditions of the first or second kind when approaching the banks.

The basic results presented in this work are not contained anywhere else to the author's best knowledge. Especially the determinant inequalities (Theorem 3.1 and 3.2, Example 3.1 and 3.2, where the program MatLab 5.3 was used). An alternative general proof of the determinant inequalities for $N=4$ starts with equation (18), and ends with (21), (22) in systematic manner representation with inner products. In Corollary 3.1 the lower and upper bounds in $\det[H_N(|\varepsilon|)] \leqslant \det[\Re H_N] \leqslant \det[H_N(\Re\varepsilon)] (\leqslant 1)$, both correspond to diagonal matrices with less amount of operations for the determinants (= products). The inequality on the right is sharper than the Hadamard inequality, from which $(\det[H_N(\Re\varepsilon)])^{1/2}$ would follow only.

Section 4 contains auxiliary formulas which serve to generate some (infinite) matrices and determinants, which can be of interest for testing accuracy of calculations by machines and for number theory. See Remark 4.2 and 4.3 for inverting matrices, Lemma 2.1 for infinite products, and the final part with the Riemann zeta function.

2. Sommerfeld parallel half planes problem matrices

DEFINITION 2.1. (SP-matrices.) Let $\xi \in \mathbf{R}$, ζ the extension of ξ to the complex plane, $N \in \mathbf{N}$, $h_m \in \mathbf{R}$, $m \in \{1,..,N-1\}$, $\kappa \in \mathbf{C}_+$, and $\sqrt{\xi^2-\kappa^2}$ be defined with branch cut half lines drawn from the branching points $\pm\kappa$ parallel to the imaginary axis to $\pm(\kappa + \mathrm{i}\infty)$ in order to have a positive real part in the strip $|\zeta| < |\kappa|$ containing the real axis, and set $\varepsilon_m := \exp(-[h_{m+1}-h_m]\sqrt{\xi^2-\kappa^2})$, where $h_{m+1} > h_m$ is assumed. We call the squared array

$$H_N := \begin{bmatrix} 1 & \varepsilon_1 & \varepsilon_1\varepsilon_2 & \varepsilon_1\varepsilon_2\varepsilon_3 & \varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4 & \varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4\varepsilon_5 & \cdots \\ \varepsilon_1 & 1 & \varepsilon_2 & \varepsilon_2\varepsilon_3 & \varepsilon_2\varepsilon_3\varepsilon_4 & \varepsilon_2\varepsilon_3\varepsilon_4\varepsilon_5 & \cdots \\ \varepsilon_1\varepsilon_2 & \varepsilon_2 & 1 & \varepsilon_3 & \varepsilon_3\varepsilon_4 & \varepsilon_3\varepsilon_4\varepsilon_5 & \cdots \\ \varepsilon_1\varepsilon_2\varepsilon_3 & \varepsilon_2\varepsilon_3 & \varepsilon_3 & 1 & \varepsilon_4 & \varepsilon_4\varepsilon_5 & \cdots \\ \varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4 & \varepsilon_2\varepsilon_3\varepsilon_4 & \varepsilon_3\varepsilon_4 & \varepsilon_4 & 1 & \varepsilon_5 & \cdots \\ \varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4\varepsilon_5 & \varepsilon_2\varepsilon_3\varepsilon_4\varepsilon_5 & \varepsilon_3\varepsilon_4\varepsilon_5 & \varepsilon_4\varepsilon_5 & \varepsilon_5 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{1}$$

to be a N-screen Sommerfeld parallel half planes problem matrix with respect to $\mathrm{L}^2(\mathbf{R})$. The components can be written as

$$H_{N;i,j}(\varepsilon_1,..,\varepsilon_{N-1}) := \prod_{m=\min(i,j)}^{\max(i,j)-1} \varepsilon_m. \tag{2}$$

It belongs to the class of Toeplitz matrices in case of equidistant screens with one $\varepsilon_m = \varepsilon$. Let $\delta_m = h_{m+1} - h_m$. It should be mentioned that (the L^∞) norms $|\varepsilon_m| \leqslant \exp(-\delta_m|\Im\kappa|) < 1$.

REMARK 2.1. (On the background.) This is a Fourier symbol matrix in ξ which belongs to a to L^2-lifted Wiener-Hopf system on the real line being equivalent to pure Dirichlet (Neumann) boundary value problems to the Helmholtz equation $(\Delta_2 + \kappa^2)u(x,y) = 0$ to hold in the exterior of N parallel half planes $x > 0$, $y = h_m$ to represent scatterers for given plane waves $exp(\mathrm{i}\kappa(x\cos\theta + y\sin\theta))$ with angle of incidence θ. For a formulation of such problems see the joint paper of Meister, Rottbrand and Speck [15]. For methods to obtain solutions by approximation techniques we refer to Speck [20] and the work of Brannan, Duan, Ervin and Razoumov [4],[5], where particular items can be found. For $N = 3$ equidistant half planes see the pioneering article of Jones [10], who obtained a function-theoretic Wiener-Hopf factorization. Meister and Rottbrand [16], [19] investigated elasto-dynamical scattering of several parallel crack half planes. Those works contain the N-screen Dirichlet (and Neumann) diffraction problem for the Helmholtz equation, and are the origin (see p. 102, p. 101) of the inequalities given in Corollary 3.1.

REMARK 2.2. In the general case the absolute value of the ε_m depends upon a polar angle ϕ: Letting $\sqrt{\xi^2 - \kappa^2} = r(\cos\phi + \mathrm{i}\sin\phi)$, while $r > 0$, and $\cos\phi > 0$, we face the products involved as

$$\Re[\varepsilon_m \cdot \ldots \cdot \varepsilon_p] = \exp(-\sum_{n=m}^{p} r\delta_n \cos\phi)\ \cos(\sum_{n=m}^{p} r\delta_n \sin\phi).$$

THEOREM 2.1. (Matrix of symmetric structure.) *Let* $N \in \mathbf{N}$, $\varepsilon_m \in \mathbf{C}$, $m \in \{1,..,N\}$, *and define the* $N \times N$ *matrix* H_N *through its components*

$$H_{N;i,j}(\varepsilon_1,..,\varepsilon_{N-1}) := \prod_{m=\min(i,j)}^{\max(i,j)-1} \varepsilon_m.$$

The following four statements (i)-(iv) hold:
(i) $\det[H_N] = \prod_{m=1}^{N-1}(1-\varepsilon_m^2)$.
(ii) While all $\varepsilon_m \neq \pm 1$, $m = 1,..,m = N-1$, *the inverse matrix* $G_N = H_N^{-1}$ *exists and*

is of symmetrical triple diagonal form. The elements on the diagonal read

$$G_{N;1,1} = \frac{1}{1-\varepsilon_1^2},$$
$$G_{N;j,j} = \frac{1-\varepsilon_{j-1}^2\varepsilon_j^2}{(1-\varepsilon_{j-1}^2)(1-\varepsilon_j^2)}, \quad j = 2,..,N-1,$$
$$G_{N;N,N} = \frac{1}{1-\varepsilon_{N-1}^2},$$

and the upper off diagonal elements

$$G_{N;j,j+1} = -\frac{\varepsilon_j}{1-\varepsilon_j^2}, \quad j = 1,..,N-1.$$

(iii) $\det[H_N(\ldots,-\varepsilon_m,\ldots)] = \det[H_N(\ldots,\varepsilon_m,\ldots)]$.
(iv) Let all ε_m *lie in unit circles* $|\varepsilon_m| < 1,\ m = 1,..,N-1$. *Then the real part* $\Re H_N$ *is positive definite.*

Proof. See the joint paper of Meister and Rottbrand [16]. □

LEMMA 2.1. (List of basic infinite product formulas.) *Let the numbers* $\lambda_n, \mu_n > 0,\ n \in \mathbf{N}$ *be zeroes of Bessel functions,* $J_0(\lambda_n) = 0 = J_1(\mu_n)$. *It holds that*

$$\frac{\pi}{4} = \prod_{n=1}^{\infty}\left(1-\left(\frac{1}{2n+1}\right)^2\right), \tag{3}$$

$$J_0(x) = \prod_{n=1}^{\infty}\left(1-\frac{x^2}{\lambda_n^2}\right), \tag{4}$$

$$\frac{J_1(x)}{x} = \prod_{n=1}^{\infty}\left(1-\frac{x^2}{\mu_n^2}\right), \tag{5}$$

$$\cos\left(\frac{\pi}{2}x\right) = \prod_{n=1}^{\infty}\left(1-\left(\frac{x}{2n-1}\right)^2\right), \tag{6}$$

$$\frac{\sin(\pi x)}{\pi x} = \prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2}\right). \tag{7}$$

Proof. The first, fourth and fifth relation follow directly by Lemma 4.2. The second and third equality can be found in Abramowitz and Stegun [1]. □

REMARK 2.3. (Infinite matrices.) Let $\kappa \in i\mathbf{R}$ in Definition 2.1. Then all $\varepsilon_m \in (-1,1) \subset \mathbf{R}$, and positive definiteness of H_N is directly seen by the determinant formula applied to the Hurwitz criterion. Lemma 2.1 can be used to generate sets of infinite matrices and formulas for their inverse matrices and determinants. For instance,

$$\det \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{15} & \frac{1}{105} & \frac{1}{945} & \frac{1}{10395} & \cdots \\ \frac{1}{3} & 1 & \frac{1}{5} & \frac{1}{35} & \frac{1}{315} & \frac{1}{3465} & \cdots \\ \frac{1}{15} & \frac{1}{5} & 1 & \frac{1}{7} & \frac{1}{63} & \frac{1}{693} & \cdots \\ \frac{1}{105} & \frac{1}{35} & \frac{1}{7} & 1 & \frac{1}{9} & \frac{1}{99} & \cdots \\ \frac{1}{945} & \frac{1}{315} & \frac{1}{63} & \frac{1}{9} & 1 & \frac{1}{11} & \cdots \\ \frac{1}{10395} & \frac{1}{3465} & \frac{1}{693} & \frac{1}{99} & \frac{1}{11} & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \frac{\pi}{4}. \tag{8}$$

Such matrices can be useful for testing accuracy of computer calculations, similar as Hilbert matrices are used.

REMARK 2.4. The first equality in Lemma 2.1 is known as Wallis' formula and used in equation (8).

3. The determinant inequalities

LEMMA 3.1. (A basic determinant inequality.) *Let the matrix Z=A+iB with A to be positive definite, and B real symmetric.*
It holds $|\det[Z]| \geqslant \det[A]$.

Proof. This can be found as an exercise in the book of Bellman [3], p. 126. □

REMARK 3.1. (On Lemma 3.1.) It is an extension of the statement for complex numbers, that $|\alpha + \mathrm{i}\beta| \geqslant \alpha$.

THEOREM 3.1. (Inequality for determinants.) *Under assumption of positive definite matrices* $\Re H_N$ *in accordance with Theorem 2.1 (iv) we have the lower/upper bounds*

$$\prod_{m=1}^{N-1}(1-|\varepsilon_m|^2) \leqslant \det[\Re H_N(\varepsilon_1,..,\varepsilon_{N-1})] \leqslant \prod_{m=1}^{N-1}|1-\varepsilon_m^2|. \tag{9}$$

Proof. The last inequality is immediate due to the previous Lemma 3.1. The left represents the smallest value of the term on the right. This will be demonstrated inductively. In the real case both terms coincide. Let $\varepsilon_m = |\varepsilon_m|(\cos\phi_m + \mathrm{i}\,\sin\phi_m)$ lie in the unit circle. We write the positive definite matrix $\Re H_N(\varepsilon_1,..,\varepsilon_{N-1})$ in the block partition

$$\Re H_N = \begin{pmatrix} \Re H_{N-1} & |\varepsilon_{N-1}|b \\ |\varepsilon_{N-1}|b^T & 1 \end{pmatrix} \tag{10}$$

where the last column vector of $\Re H_{N-1}$ is given by

$$a(\phi_1,..,\phi_{N-2}) = \begin{pmatrix} |\varepsilon_1|\cdot\cdot|\varepsilon_{N-2}|\cos(\phi_1+..+\phi_{N-2}) \\ |\varepsilon_2|\cdot\cdot|\varepsilon_{N-2}|\cos(\phi_2+..+\phi_{N-2}) \\ \vdots \\ |\varepsilon_{N-2}|\cos(\phi_{N-2}) \\ 1 \end{pmatrix},$$

and

$$b(\phi_1,..,\phi_{N-2},\phi_{N-1}) = \begin{pmatrix} |\varepsilon_1|\cdot\cdot|\varepsilon_{N-2}|\cos(\phi_1+..+\phi_{N-2}+\phi_{N-1}) \\ |\varepsilon_2|\cdot\cdot|\varepsilon_{N-2}|\cos(\phi_2+..+\phi_{N-2}+\phi_{N-1}) \\ \vdots \\ |\varepsilon_{N-2}|\cos(\phi_{N-2}+\phi_{N-1}) \\ \cos(\phi_{N-1}) \end{pmatrix}.$$

Be $\det[\Re H_{N-1}] \geqslant (1-|\varepsilon_1|^2)\cdot\ldots\cdot(1-|\varepsilon_{N-2}|^2) = \det[H_{N-1}(|\varepsilon_1|,..,|\varepsilon_{N-2}|)]$ fulfilled. That is $\phi_1 = 0,..,\phi_{N-2} = 0$ (modulo π due to symmetry! in the determinant formula for H_{N-1}), and $b(0,..,0,\phi_{N-1}) = a(0,..,0)\cdot\cos\phi_{N-1}$. It follows (note that an inverse matrix multiplied with the last column of the original matrix gives the unit vector $(0,..,0,1)^T$)

$$\begin{aligned} \det[\Re H_N] &\geqslant \det[H_{N-1}(|\varepsilon_1|,..,|\varepsilon_{N-2}|)](1-|\varepsilon_{N-1}|^2 \\ &\qquad (\cos\phi_{N-1})^2)\, a(0,..,0)^T[H_{N-1}(|\varepsilon_1|,..,|\varepsilon_{N-2}|)]^{-1}a(0,..,0)) \\ &= \det[H_{N-1}(|\varepsilon_1|,..,|\varepsilon_{N-2}|)](1-|\varepsilon_{N-1}|^2(\cos\phi_{N-1})^2) \\ &\geqslant (1-|\varepsilon_1|^2)\cdot\ldots\cdot(1-|\varepsilon_{N-1}|^2), \end{aligned}$$

and thus the left determinant inequality. □

Another proof of the left inequality is the following: Consider the hermitian $N\times N$ matrix $\widetilde{H}_N = A_N + \mathrm{i}\, C_N$ given through equation (26) with A_N to be positive definite (with symmetric square-root) being the real part of H_N also, and C_N to be real skew-symmetric. We split

$$\widetilde{H}_N = A_N^{1/2}(I_N + \mathrm{i}Y_N)A_N^{1/2}, \quad Y_N^T = -Y_N$$

real skew-symmetric with $Y_N = A_N^{-1/2}C_N A_N^{-1/2}$. We multiply $\widetilde{H}_N$ with its complex conjugate, that is, $(I_N+\mathrm{i}Y)(I_N-\mathrm{i}Y_N) = (I_N+\mathrm{i}Y_N)(I_N+\mathrm{i}Y_N^T) = I_N - Y_N Y_N^T + \mathrm{i}(Y_N+Y_N^T)$. The symmetric matrix $Y_N Y_N^T$ is at least positive semidefinite, and is subtracted from the unit matrix. Hence $\det[\widetilde{H}_N]\det[\overline{\widetilde{H}_N}] \leqslant \det[A_N]\det[A_N]$ together with the determinant formula (27) completes the proof, which includes the general situation in Remark 2.2.

DEFINITION 3.1. (Gramian determinant.) Determinants, being quadratic and symmetric with entries G_{ij} built up by inner products of vectors x_i, x_j in Euklidean N-space, $N \in \mathbf{N}$, are called Gramian determinants $G(1,..,N)$.

LEMMA 3.2. (Generalized Hadamard inequality.) *Let* $G(1,..,N)$, $N \in \mathbf{N}$, *be a positive Gramian determinant, and* $p \in \{1,..,N\}$. *It holds that*

$$G(1,..,N) \leqslant G(1,..,p)G(p+1,..,N).$$

Proof. See Gantmacher's book [8] on matrix theory. □

REMARK 3.2. (Special case of the generalized Hadamard inequality.) It follows that

$$G(1,..,N) \leqslant \prod_{j=1}^{N} G(j,j).$$

The volume of a parallel epiped is at most equal to the product of its edges. The equality holds only for orthogonal geometries. See Gantmacher [8], p. 264.

THEOREM 3.2. (Sharper inequality for determinants.) *It holds*

$$\det[\Re H_N(\varepsilon_1,..,\varepsilon_{N-1})] \leqslant \prod_{m=1}^{N-1}(1-(\Re\varepsilon_m)^2). \tag{11}$$

Proof. The cases $N = 1,2$ are obvious. Let $\mathrm{H}_N = \mathrm{A}_N + \mathrm{i}\mathrm{B}_N$, $N \geqslant 3$. We know that A_N is positive definite, and B_N real-symmetric, where we have due to the given structure that

$$\mathrm{H}_N = \begin{bmatrix} \mathrm{H}_{N-1} & \xi_{N-1} \\ \xi_{N-1}^T & 1 \end{bmatrix}, \quad \xi_{N-1} = \varepsilon_{N-1}\xi_{N-2}.$$

In the following $\Im$ shall denote the imaginary part. One obtains

$$\mathrm{a}_{N-1} := \Re\xi_{N-1} = \mathrm{a}_{N-2}\Re\varepsilon_{N-1} - \Im\varepsilon_{N-1}\Im\xi_{N-2}.$$

Hence we can express the positive definite matrix by $\Re\mathrm{H}_N =$

$$\begin{bmatrix} \begin{bmatrix} \mathrm{A}_{N-2} & \mathrm{a}_{N-2} \\ \mathrm{a}_{N-2}^T & 1 \end{bmatrix} & \Re\varepsilon_{N-1}\begin{bmatrix} \mathrm{a}_{N-2} \\ 1 \end{bmatrix} - \Im\varepsilon_{N-1}\begin{bmatrix} \Im\xi_{N-2} \\ 0 \end{bmatrix} \\ \Re\varepsilon_{N-1}\begin{bmatrix} \mathrm{a}_{N-2} \\ 1 \end{bmatrix}^T - \Im\varepsilon_{N-1}\begin{bmatrix} \Im\xi_{N-2} \\ 0 \end{bmatrix}^T & 1 \end{bmatrix}.$$

After some operations with rows and columns to obtain $Z_N = L_N \Re H_N L_N^T$, with I_{N-2} denoting a unit matrix in

$$L_N = \begin{pmatrix} I_{N-2} & & \\ & 1 & \\ & -\Re\,\varepsilon_{N-1} & 1 \end{pmatrix}, \tag{12}$$

and (as a convention) zeroes in the empty places, we see that the determinant is equal to the determinant of the positive definite matrix $Z = (L_N \Re H_N^{\frac{1}{2}})(L_N \Re H_N^{\frac{1}{2}})^T$, which reads

$$Z = \begin{bmatrix} \begin{bmatrix} \mathrm{A}_{N-2} & \mathrm{a}_{N-2} \\ \mathrm{a}_{N-2}^T & 1 \end{bmatrix} & -\Im\varepsilon_{N-1} \begin{bmatrix} \Im\xi_{N-2} \\ 0 \end{bmatrix} \\ -\Im\varepsilon_{N-1} \begin{bmatrix} \Im\xi_{N-2} \\ 0 \end{bmatrix}^T & 1-(\Re\varepsilon_{N-1})^2 \end{bmatrix}.$$

Now application of the generalized Hadamard inequality in Lemma 3.2 completes the proof. $\square$

COROLLARY 3.1. *We collect the results: Let $\varepsilon = \varepsilon_j$, $j = 1,2,..,N-1$ lie in the unit circle, and H_N be defined as in Theorem 2.1. The following inequalities hold,*

$$\det[H_N(|\varepsilon|)] = \prod_{j=1}^{N-1}\left(1-|\varepsilon_j|^2\right) \leqslant \det[\Re H_N] \leqslant \prod_{j=1}^{N-1}\left(1-(\Re\varepsilon_j)^2\right) = \det[H_N(\Re\varepsilon)]$$
$$\leqslant \prod_{j=1}^{N-1}\left|1-\varepsilon_j^2\right| = |\det[H_N(\varepsilon)]|.$$

EXAMPLE 3.1. (The case $N=3$.) We face a complex-valued matrix $\mathrm{Z} = \mathrm{A} + \mathrm{iB}$, $A := \Re \mathrm{Z}$, $B := \Im \mathrm{Z}$ of the form

$$\mathrm{Z} = \begin{pmatrix} 1 & \varepsilon_1 & \varepsilon_1\varepsilon_2 \\ \varepsilon_1 & 1 & \varepsilon_2 \\ \varepsilon_1\varepsilon_2 & \varepsilon_2 & 1 \end{pmatrix}, \quad |\varepsilon_j| < 1. \tag{13}$$

We write $\varepsilon_1 = a + \mathrm{i}\alpha$, $\varepsilon_2 = b + \mathrm{i}\beta$ to furnish the real part of Z as

$$\Re\mathrm{Z} = \begin{pmatrix} 1 & a & ab-\alpha\beta \\ a & 1 & b \\ ab-\alpha\beta & b & 1 \end{pmatrix}.$$

Direct calculation gives us that

$$(1-a^2)(1-b^2) \geqslant (1-a^2)(1-b^2) - \alpha^2\beta^2 = \det[\Re\mathrm{Z}], \tag{14}$$

and

$$\det[\Re \mathrm{Z}] \geqslant (1-a^2)(1-b^2)-\alpha^2\beta^2-\alpha^2(1-|\varepsilon_2|^2)-\beta^2(1-|\varepsilon_1|^2), \tag{15}$$

where the expression on the right coincides with $\prod_{j=1}^{2}(1-|\varepsilon_j|^2)$.

EXAMPLE 3.2. (On the case $N=4$.) Let us set $\mathrm{Z} := \mathrm{H}_N$,

$$\varepsilon_1 = \frac{1}{10}(8+1\mathrm{i}), \quad \varepsilon_2 = \frac{1}{10}(8-2\mathrm{i}), \quad \varepsilon_3 = \frac{1}{10}(5+4\mathrm{i}). \tag{16}$$

Then we have

$$\mathrm{Z}(\Re\varepsilon_1, \Re\varepsilon_2, \Re\varepsilon_3) = \begin{bmatrix} 1.00 & 0.80 & 0.64 & 0.32 \\ 0.80 & 1.00 & 0.80 & 0.40 \\ 0.64 & 0.80 & 1.00 & 0.50 \\ 0.32 & 0.40 & 0.50 & 1.00 \end{bmatrix},$$

$$\Re\mathrm{Z}(\varepsilon_1, \varepsilon_2, \varepsilon_3) = \begin{bmatrix} 1.000 & 0.800 & 0.660 & 0.362 \\ 0.800 & 1.000 & 0.800 & 0.480 \\ 0.660 & 0.800 & 1.000 & 0.500 \\ 0.362 & 0.480 & 0.500 & 1.000 \end{bmatrix},$$

and

$$\mathrm{Z}(|\varepsilon_1|, |\varepsilon_2|, |\varepsilon_3|) = \begin{bmatrix} 1 & \frac{\sqrt{65}}{10} & \frac{\sqrt{4420}}{100} & \frac{\sqrt{181220}}{1000} \\ \frac{\sqrt{65}}{10} & 1 & \frac{\sqrt{68}}{10} & \frac{\sqrt{2788}}{100} \\ \frac{\sqrt{4420}}{100} & \frac{\sqrt{68}}{10} & 1 & \frac{\sqrt{41}}{10} \\ \frac{\sqrt{181220}}{1000} & \frac{\sqrt{2788}}{100} & \frac{\sqrt{41}}{10} & 1 \end{bmatrix}.$$

The determinant inequalities (obtained with MatLab 5.3) then read (see Corollary 3.1)

$$0.066080... \leqslant \det[\Re\mathrm{Z}] = 0.094311... \leqslant 0.0972 \leqslant 0.20526... \tag{17}$$

In general we have a complex-valued matrix $\mathrm{Z} = \mathrm{A} + \mathrm{iB}$, $A := \Re\mathrm{Z}$, $B := \Im\mathrm{Z}$ of the form

$$\mathrm{Z} = \begin{pmatrix} 1 & \varepsilon_1 & \varepsilon_1\varepsilon_2 & \varepsilon_1\varepsilon_2\varepsilon_3 \\ \varepsilon_1 & 1 & \varepsilon_2 & \varepsilon_2\varepsilon_3 \\ \varepsilon_1\varepsilon_2 & \varepsilon_2 & 1 & \varepsilon_3 \\ \varepsilon_1\varepsilon_2\varepsilon_3 & \varepsilon_2\varepsilon_3 & \varepsilon_3 & 1 \end{pmatrix}, \quad |\varepsilon_j| < 1. \tag{18}$$

Let $\varepsilon_1 = a + \mathrm{i}\alpha$, $\varepsilon_2 = b + \mathrm{i}\beta$, $\varepsilon_3 = c + \mathrm{i}\gamma$ to represent the real part of Z as

$$\Re\mathrm{Z} = \begin{pmatrix} 1 & a & ab-\alpha\beta & abc-\alpha\beta c-a\beta\gamma-\alpha b\gamma \\ a & 1 & b & bc-\beta\gamma \\ ab-\alpha\beta & b & 1 & c \\ abc-\alpha\beta c-a\beta\gamma-\alpha b\gamma & bc-\beta\gamma & c & 1 \end{pmatrix}.$$

We obtain

$$\det[\Re Z] = (1-a^2)(1-b^2)(1-c^2) - \alpha^2 b^2 \gamma^2 (1-b^2-\beta^2) \\ -\beta^2\gamma^2(1-a^2-\alpha^2\beta^2) - \alpha^2\beta^2(1-c^2-\gamma^2 b^2), \tag{19}$$

and

$$\prod_{j=1}^{3}[1-|\varepsilon_j|^2] = (1-a^2)(1-b^2)(1-c^2) - \alpha^2(1-b^2)(1-c^2) \\ -\beta^2(1-a^2-\alpha^2)(1-c^2-\gamma^2) - \gamma^2(1-a^2-\alpha^2)(1-b^2). \tag{20}$$

It remains to show

$$\alpha^2(1-b^2)(1-c^2) + \beta^2(1-a^2-\alpha^2)(1-c^2-\gamma^2) + \gamma^2(1-a^2-\alpha^2)(1-b^2) \geqslant \\ \alpha^2 b^2\gamma^2(1-b^2-\beta^2) + \beta^2\gamma^2(1-a^2-\alpha^2\beta^2) + \alpha^2\beta^2(1-c^2-\gamma^2 b^2).$$

This is equivalent to

$$\alpha^2(1-b^2)(1-c^2-b^2\gamma^2) + \beta^2(1-a^2-\alpha^2)(1-c^2-\gamma^2) + \gamma^2(1-a^2-\alpha^2)(1-b^2) \geqslant \\ -\alpha^2 b^2\gamma^2\beta^2 + \beta^2\gamma^2(1-a^2-\alpha^2\beta^2) + \alpha^2\beta^2(1-c^2-\gamma^2 b^2).$$

Hence it should hold

$$\alpha^2(1-b^2-\beta^2)(1-c^2-b^2\gamma^2) + \beta^2(1-a^2-\alpha^2)(1-c^2-\gamma^2) \\ +\gamma^2(1-a^2-\alpha^2)(1-b^2) \geqslant \beta^2\gamma^2(1-a^2-\alpha^2(b^2+\beta^2)).$$

Noting that $a^2+\alpha^2<1$, $b^2+\beta^2<1$, thus $1-b^2>\beta^2$ on the left hand, $c^2+\gamma^2<1$, we arrive at

$$\alpha^2(1-b^2-\beta^2)(1-c^2-\gamma^2(b^2+\beta^2)) + \beta^2(1-a^2-\alpha^2)(1-c^2-\gamma^2) \geqslant 0,$$

which holds always true, indeed.

Now let us take the block partition (10) with vector d instead of b there, and note that $\det[\Re H_N] = \det[\Re H_{N-1}](1-|\varepsilon_{N-1}|^2 d^T[\Re H_{N-1}]^{-1}d)$. It should be mentioned that for $N=3$ the inner product

$$\frac{(ab-\alpha\beta, b)}{1-a^2}\begin{pmatrix} 1 & -a \\ -a & 1 \end{pmatrix}\begin{pmatrix} ab-\alpha\beta \\ b \end{pmatrix} = \frac{b^2(1-a^2)+\alpha^2\beta^2}{1-a^2} \leqslant b^2+\beta^2 = |\varepsilon_2|^2. \tag{21}$$

For $N=4$ the corresponding inner product reads ($det = (1-a^2)(1-b^2)-\alpha^2\beta^2>0$)

$$\begin{pmatrix} c(ab-\alpha\beta)-\gamma(a\beta+\alpha b) \\ cb-\gamma\beta \\ c \end{pmatrix}^T \begin{pmatrix} 1 & a & ab-\alpha\beta \\ a & 1 & b \\ ab-\alpha\beta & b & 1 \end{pmatrix}^{-1} \begin{pmatrix} c(ab-\alpha\beta)-\gamma(a\beta+\alpha b) \\ cb-\gamma\beta \\ c \end{pmatrix}$$

$$= \frac{c^2 det + \gamma^2(\beta^2+\alpha^2 b^2-\alpha^2 b^4-\beta^2 a^2-\beta^4\alpha^2-2\alpha^2 b^2\beta^2)}{det} \leqslant c^2+\gamma^2 = |\varepsilon_3|^2. \tag{22}$$

This can be seen after rewriting the γ^2 multiplier as

$$\begin{aligned} \beta^2(1-a^2)-\beta^2\alpha^2(b^2+\beta^2)+b^2\alpha^2(1-b^2-\beta^2) &\leqslant \\ \beta^2(1-a^2)-\beta^2\alpha^2(b^2+\beta^2)+(1-\beta^2)\alpha^2(1-b^2-\beta^2) &= \\ \beta^2(1-a^2)-\alpha^2\beta^2+\alpha^2(1-b^2-\beta^2) &\leqslant \\ (1-a^2)(\beta^2+1-b^2-\beta^2)-\alpha^2\beta^2 &= det, \end{aligned}$$

where for obtaining the first inequality, $b^2<1-\beta^2$, and for obtaining the second inequality, $\alpha^2<1-a^2$ are used. Hence we have verified that $\det[\Re H_4] \geqslant (1-|\varepsilon_1|^2)(1-|\varepsilon_2|^2)(1-|\varepsilon_3|^2)$.

We remark that in case of absolute values of $|\varepsilon|$ about $1-\delta$, $\delta>0$ small, computer calculation programs do not always tell the truth: For instance, the entries ε_j for H_N in Definition 2.1, calculated from $\xi=1$, $k=0.01+0.001\,\mathrm{i}$, $h_1=0.000001$, $h_2=0.00001$, $h_3=0.0001$, and $h_4=0.001$, produce wrong digits. The output obtained by Matlab 5.3 for Corollary 3.1 is:

$$\begin{aligned} &5.825312658629375e-012 \\ =\ &5.825312658625181e-012 \\ \leqslant\ &5.825312658630094e-012 \\ \leqslant\ &5.825312658625900e-012 \\ =\ &5.825312658630094e-012 \\ \leqslant\ &5.825312659499649e-012 \\ =\ &5.825312659503842e-012. \end{aligned}$$

The values $\xi=1$, $k=0.501+0.5001\,\mathrm{i}$, $h_1=0.2000001$, $h_2=0.400001$, $h_3=0.60001$, and $h_4=0.8001$ lead to the output

$$\begin{aligned} &0.03840941041282 \\ =\ &0.03840941041282 \\ \leqslant\ &0.03894630331134 \\ \leqslant\ &0.03894852511313 \\ =\ &0.03894852511313 \\ \leqslant\ &0.04163377676650 \\ =\ &0.04163377676650. \end{aligned}$$

The entries $\varepsilon_1=0.1+0.2i$, $\varepsilon_2=0.2-0.5i$, $\varepsilon_3=0.3+0.8i$, $\varepsilon_4=0.4-0.5i$, $\varepsilon_5=0.5+0.7i$ from the interior of the unit circle, give us

$$\begin{aligned} &0.02793644100000 \\ =\ &0.02793644100000 \\ \leqslant\ &0.26400081064711 \\ \leqslant\ &0.54486432000000 \end{aligned}$$

$$
\begin{aligned}
&= 0.54486432000000\\
&\leqslant 3.39134679807819\\
&= 3.39134679807819.
\end{aligned}
$$

The last example shows how Corollary 3.1 can be applied in order *to test errors* due to different amount of calculation operations needed for full matrices and diagonal matrices. The next section includes some more examples of matrices for testing accuracy in *inverting matrices* (note the use of Hilbert matrices), and for *infinite products* as special determinants (note the connection with Euler's identity and the Riemann zeta function).

4. Auxiliary formulas

LEMMA 4.1. (A multiplicative splitting.) *Let H_N be the matrix in Definition 2.1, and*

$$D_N := diag[1-\varepsilon_1^2, ..., 1-\varepsilon_{N-1}^2, 1], \tag{23}$$

$$U_N := \begin{bmatrix} 1 & -\varepsilon_1 & & \\ & 1 & \ddots & \\ & & 1 & -\varepsilon_{N-1} \\ & & & 1 \end{bmatrix}. \tag{24}$$

It holds that

$$U_N H_N U_N^T = D_N. \tag{25}$$

Proof. Direct calculation. We refer to Rottbrand [19]: p. 96. □

REMARK 4.1. This is not a diagonalization with eigenvalues.

We define the hermitian $N \times N$ matrix with $\varepsilon_1, \dots, \varepsilon_{N-1} \in \mathbf{C}$,

$$\widetilde{H}_N := \begin{bmatrix}
1 & \varepsilon_1 & \varepsilon_1\varepsilon_2 & \varepsilon_1\varepsilon_2\varepsilon_3 & \varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4 & \varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4\varepsilon_5 & \cdots \\
\bar{\varepsilon}_1 & 1 & \varepsilon_2 & \varepsilon_2\varepsilon_3 & \varepsilon_2\varepsilon_3\varepsilon_4 & \varepsilon_2\varepsilon_3\varepsilon_4\varepsilon_5 & \cdots \\
\bar{\varepsilon}_1\bar{\varepsilon}_2 & \bar{\varepsilon}_2 & 1 & \varepsilon_3 & \varepsilon_3\varepsilon_4 & \varepsilon_3\varepsilon_4\varepsilon_5 & \cdots \\
\bar{\varepsilon}_1\bar{\varepsilon}_2\bar{\varepsilon}_3 & \bar{\varepsilon}_2\bar{\varepsilon}_3 & \bar{\varepsilon}_3 & 1 & \varepsilon_4 & \varepsilon_4\varepsilon_5 & \cdots \\
\bar{\varepsilon}_1\bar{\varepsilon}_2\bar{\varepsilon}_3\bar{\varepsilon}_4 & \bar{\varepsilon}_2\bar{\varepsilon}_3\bar{\varepsilon}_4 & \bar{\varepsilon}_3\bar{\varepsilon}_4 & \bar{\varepsilon}_4 & 1 & \varepsilon_5 & \cdots \\
\bar{\varepsilon}_1\bar{\varepsilon}_2\bar{\varepsilon}_3\bar{\varepsilon}_4\bar{\varepsilon}_5 & \bar{\varepsilon}_2\bar{\varepsilon}_3\bar{\varepsilon}_4\bar{\varepsilon}_5 & \bar{\varepsilon}_3\bar{\varepsilon}_4\bar{\varepsilon}_5 & \bar{\varepsilon}_4\bar{\varepsilon}_5 & \bar{\varepsilon}_5 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}. \tag{26}$$

After operations with rows one finds the determinant formula

$$\det[\widetilde{H}_N] = \prod_{j=1}^{N-1} (1 - |\varepsilon_j|^2). \tag{27}$$

In the proof of Theorem 3.1 all $|\varepsilon_m| < 1$ is assumed. This is for positive definiteness of the real part of the matrix H_N (see Theorem 2.1), and thus of the real part of $\widetilde{H}_N$ too!

REMARK 4.2. (Inverse matrices $G_N = H_N^{-1}$ with rational entries.) Let a, b of integer multiplied with a fixed complex–valued parameter to build Pythagorean triples on the unit circle $u^2 + v^2 = 1$ given through $(u,v) = \frac{(a^2-b^2,\, 2ab)}{a^2+b^2}$, and set $\varepsilon_1 = \ldots = \varepsilon_{N-1} := u$ in Theorem 2.1 (ii), or equation (1), respectively. Hence the parameter cancels out. Then the inverse matrix reads as follows (triple diagonal form!):

$$G_N = \frac{1}{4a^2b^2} \times \begin{bmatrix} (a^2+b^2)^2 & -(a^2+b^2)(a^2-b^2) & & & \\ -(a^2+b^2)(a^2-b^2) & (a^2+b^2)^2+(a^2-b^2)^2 & -(a^2+b^2)(a^2-b^2) & & \\ & \ddots & \ddots & \ddots & \\ & & -(a^2+b^2)(a^2-b^2) & (a^2+b^2)^2+(a^2-b^2)^2 & -(a^2+b^2)(a^2-b^2) \\ & & & -(a^2+b^2)(a^2-b^2) & (a^2+b^2)^2 \end{bmatrix}.$$

The values $a = 5\lambda$, $b = 1\lambda$, $\lambda \in \mathbf{C}$, give us $\varepsilon_j = u = \frac{12}{13}$, $\det[H_N] = \left(\frac{25}{169}\right)^{N-1}$,

$$H_N^{-1} = \frac{1}{100} \begin{bmatrix} 676 & -624 & & & \\ -624 & 1252 & -624 & & \\ & \ddots & \ddots & \ddots & \\ & & -624 & 1252 & -624 \\ & & & -624 & 676 \end{bmatrix},$$

and, $a = 2\lambda$, $b = 1\lambda$, $\lambda \in \mathbf{C}$, give us $\varepsilon_j = u = \frac{3}{5}$, $\det[H_N] = \left(\frac{16}{25}\right)^{N-1}$,

$$H_N^{-1} = \frac{1}{16} \begin{bmatrix} 25 & -15 & & & \\ -15 & 34 & -15 & & \\ & \ddots & \ddots & \ddots & \\ & & -15 & 34 & -15 \\ & & & -15 & 25 \end{bmatrix},$$

which is equal to

$$\frac{1}{10000}\begin{bmatrix} 15625 & -9375 & & & \\ -9375 & 21250 & -9375 & & \\ & \ddots & \ddots & \ddots & \\ & & -9375 & 21250 & -9375 \\ & & & -9375 & 15625 \end{bmatrix}.$$

All the rational inverse matrices in Remark 4.2 are strictly diagonal dominant!

REMARK 4.3. (On Definition 2.1.) Let $a, b, \widetilde{a}, \widetilde{b}$ of integer, $\xi = \widetilde{a}^2 - \widetilde{b}^2$, $\kappa = \mathrm{i}\, 2\widetilde{a}\widetilde{b}$, and $h_{m+1} - h_m = \frac{1}{\widetilde{a}^2+\widetilde{b}^2} \ln\left(\frac{a^2+b^2}{a^2-b^2}\right)$ for $m = 1, \ldots, N-1$. Then each $\varepsilon_m = \frac{a^2-b^2}{a^2+b^2}$ is a rational number. Clearly, one may also choose $(\lambda a, \lambda b)$, $(\mu\widetilde{a}, \mu\widetilde{b})$, where $\lambda, \mu \in \mathbf{C}$.

4.1. Entries of infinite matrices in Remark 2.3

The aim of this section is to present some further interesting examples for other possible entries.

DEFINITION 4.1. (Gamma function.)

$$\Gamma(x) := \int_0^\infty \mathrm{e}^{-t} t^{x-1} \mathrm{d}t, \quad \Re x > 0. \tag{28}$$

REMARK 4.4. (Properties of Γ.)

$$\Gamma(x+1) = x\Gamma(x), \quad \Gamma(\frac{1}{2}) = \sqrt{\pi},$$
$$\Gamma(n+1) = \prod_{m=1}^{n} m =: n! \quad \text{(factorial } n\text{, when } n \in \mathbf{N}).$$

DEFINITION 4.2. (Bessel functions of the first kind.)

$$J_\nu(x) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\nu+n+1)} \left(\frac{x}{2}\right)^{\nu+2n}, \quad \nu \in \mathbf{R}. \tag{29}$$

For real zeroes of Bessel functions J_0 and J_1 see Abramowitz and Stegun [1], no 9.5.

LEMMA 4.2. (Infinite Products.) *Let* $k,n \in \mathbf{N}$, $k \in \{1,..,n\}$, $a_j, b_j > -1$ *satisfying* $\sum_{j=1}^{k}(a_j - b_j) = 0$, *and set* $P_n := \prod_{j=1}^{k} \frac{n+a_j}{n+b_j}$ *to define the infinite product* $P := \prod_{n\in\mathbf{N}} P_n$. *It holds*

$$P = \prod_{j=1}^{k} \frac{\Gamma(1+b_j)}{\Gamma(1+a_j)}.$$

Proof. This formula is due to Leonard Euler, confer Knuth [11], section 1.2.5, exercise 17 therein, where Stirling's formula is used and alternatively Fichtenholtz II [7], p. 808. □

EXAMPLE 4.1. (A concrete computation.) Let

$$P = \prod_{n=1}^{\infty}\left(1 + \frac{1}{n(n+2)}\right).$$

Thus we have $P_n = \frac{n+1}{n}\,\frac{n+1}{n+2}$, with $a_1 = a_2 = 1$, $b_1 = 0$, $b_2 = 2$, and obtain

$$P = \frac{\Gamma(1)}{\Gamma(2)}\frac{\Gamma(3)}{\Gamma(2)} = \frac{0!\;2!}{1!\;1!} = 2.$$

Another type of entries can be generated by the following function (Apostol [2]):

DEFINITION 4.3. (Riemann zeta function) Let $s = \sigma + it$ a fixed complex number with real part $\sigma > 1$. Then

$$\zeta(s) := \sum_{k=1}^{\infty} k^{-s}. \tag{30}$$

The series is known to be absolutely convergent. An analytic continuation (excepted the single pole $s = 1$) is given by

$$\zeta(s) := \frac{1}{1-2^{1-s}} \sum_{k=1}^{\infty} (-1)^{k-1} k^{-s}, \quad \sigma > 0, \tag{31}$$

with the sum defining the eta function $\eta(s)$. The Riemann zeta function stands in relation with prime numbers by Euler's identity:

$$\zeta(s) = \prod_{j=1}^{\infty} \frac{1}{1-p_j^{-s}} = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}, \quad \sigma > 1. \tag{32}$$

This means entries $\varepsilon_j = p_j^{-s/2}$ for $\det[H_\infty(\varepsilon)] = \frac{1}{\zeta(s)}$ with positive real part for all $\sigma > 1$. The entries $\varepsilon_j = (1 - \exp((-1)^j/j^s))^{\frac{1}{2}}$ lead to an infinite determinant taking the value $\exp(-\eta(s))$. It should be mentioned that the condition $|\varepsilon_j|^2 < 1$ leads to

$$\exp\left[\frac{(-1)^{j+1}\cos\ln(j^t)}{j^\sigma}\right]\cos\left(\frac{\sin\ln(j^t)}{j^\sigma}\right) > \frac{1}{2}, \tag{33}$$

with the expression on the left taking the maximal and minimal values $\exp(j^{-\sigma})$, and $\exp(-j^{-\sigma})$, respectively. For this one has to solve $\sin(t\ln j + \frac{(-1)^{j+1}}{j^\sigma}\sin(t\ln j)) = 0$, which gives us $t = \frac{k\pi}{\ln j}$, $k = 0, \pm 1, \pm 2, \ldots$ The last inequality obviously holds for $j = 1$, and j large enough such that $\sigma > -\frac{\ln\ln 2}{\ln j}$. Note that $\varepsilon_j = 0$ can not be satisfied for all j, because this leads to the condition $\cosh(\frac{\cos\ln(j^t)}{j^\sigma}) = \cos(\frac{\sin\ln(j^t)}{j^\sigma})$. Thus in order to have the infinite determinant to be equal one: that is $\eta(s) = 0$ (Riemann's hypothesis $\zeta(s) = 0$), there should appear factors with absolute values less than one and greater than one also. The question arises whether such a product can be constructed with almost all ε_j to lie in the unit circle. Note the last inequality in Corollary 3.1.

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REFERENCES

[1] M. ABRAMOWITZ AND I. STEGUN (Eds.), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical tables*, Reprint of the 1972 ed., John Wiley & Sons, New York, 1984.

[2] TOM M. APOSTOL, *Introduction to Analytic Number Theory*, Springer Science+Business Media LLC, New York, 1976.

[3] R. BELLMAN, *Introduction to Matrix Analysis*, McGraw-Hill, New York 1960.

[4] J. BRANNAN, J. DUAN, V. ERVIN AND L. RAZOUMOV, *Wiener-Hopf factorization of a multiple plate diffraction problem*, in: Bermúdez, Alfredo (ed.) et al., Mathematical and numerical aspects of wave propagation, 5th international conference, Santiago de Compostela, Spain, July 10–14, 2000, Philadelphia, PA: SIAM, Society for Industrial and Applied Mathematics (2000), 765–770.

[5] J. BRANNAN, J. DUAN, V. ERVIN AND L. RAZOUMOV, *A Wiener-Hopf Approximation Technique for a Multiple Plate Diffraction Problem*, Math. Meth. Appl. Sci. **27**, 1 (2004), 19–34.

[6] K. CLANCEY AND I. GOHBERG, *Factorization of Matrix Functions and Singular Integral Operators*, Birkhäuser, Basel, 1981.

[7] G. M. FICHTENHOLTZ, *Differential- und Integralrechnung II*, VEB Verlag der deutschen Wissenschaften, Berlin, 1979.

[8] R. F. GANTMACHER, *Matrizentheorie*, Springer, Berlin, 1986 (In Russian 1966).

[9] I. GOHBERG AND N. KRUPNIK, *Einführung in die Theorie der eindimensionalen singulären Integraloperatoren*, Birkhäuser, Basel 1979 (1973 in Russian).

[10] D. S. JONES, *Diffraction by three semi-infinite plates*, Proc. Royal Soc. London **A404** (1986), 299–321.

[11] DONALD E. KNUTH, *The Art of Computer Programming. Vol.1: Fundamental Algorithms*, Addison–Wesley, Reading MA, ^{3} 1997.

[12] G. S. LITVINCHUK AND I. M. SPITKOVSKY, *Factorization of Measurable Matrix Functions*, Birkhäuser, Basel, 1987.

[13] E. MEISTER, *Randwertaufgaben der Funktionentheorie*, Teubner, Stuttgart, 1983.

[14] E. MEISTER, *Some multiple-part Wiener-Hopf problems in mathematical physics*, St. Banach Center Publ. **15** (1985), 359–407.

[15] E. MEISTER, K. ROTTBRAND AND F.-O. SPECK, *Wiener–Hopf Equations for Waves Scattered by a System of Parallel Sommerfeld Half–Planes*, Math. Meth. Appl. Sci. **14** (1991), 525–552.

[16] E. MEISTER AND K. ROTTBRAND, *Elastodynamical Scattering by N Parallel Half–Planes in $\mathbf{R}^3$: II. Explicit Solutions for $N = 2$ by Explicit Symbol Factorization*, Integr. equ. oper. theory **29** (1997), 70–109.

[17] E. MEISTER AND F.-O. SPECK, *Some multi-dimensional Wiener-Hopf equations with applications*, Trends Appl. Pure Math. Mech. 2 Proc. Conf. in Kozubnik (Poland) 1977. Pitman, London 1979, 217–262.

[18] N. I. MUSKHELISHVILI, *Singular integral equations. Boundary problems of function theory and their applications to mathematical physics*, 2nd ed., Fizmatgiz, Moscow, 1962; English translation of 1st ed., Noordhoff, Groningen, 1953.

[19] K. ROTTBRAND, *Matrix-Wiener-Hopf-Faktorisierungsprobleme in der Elastodynamik bei mehreren parallelen Rissen*, Doctoral Thesis, Department of Mathematics, Technical University of Darmstadt, 1994.

[20] F.-O. SPECK, *General Wiener-Hopf factorization methods*, Pitman, London, 1985.

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