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A RESULT CONCERNING TWO–SIDED CENTRALIZERS ON ALGEBRAS WITH INVOLUTION

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Abstract. The purpose of this paper is to prove the following result. Let X be a complex Hilbert space, let $\mathscr{L}(X)$ be the algebra of all bounded linear operators on X and let $\mathscr{A}(X)\subset\mathscr{L}(X)$ be a standard operator algebra, which is closed under the adjoint operation. Let $T:\mathscr{A}(X)\to\mathscr{L}(X)$ be a linear mapping satisfying the relation $3T(AA^*A)=T(A)A^*A+AT(A^*)A+AA^*T(A)$ for all $A\in\mathscr{A}(X)$. In this case T is of the form $T(A)=\lambda A$ for all $A\in\mathscr{A}(X)$, where λ is some fixed complex number.

Throughout, R will represent an associative ring with center $Z(R)$. Given an integer $n\geqslant 2$, a ring R is said to be $n-$torsion free, if for $x\in R$, $nx=0$ implies $x=0$. An additive mapping $x\mapsto x^*$ on a ring R is called involution if $(xy)^*=y^*x^*$ and $x^{**}=x$ hold for all pairs $x,y\in R$. A ring equipped with an involution is called a ring with involution or $^*-$ring. Recall that a ring R is prime, if for $a,b\in R$, $aRb=(0)$ implies that either $a=0$ or $b=0$, and is semiprime in case $aRa=(0)$ implies $a=0$. We denoted by Q_r and C the Martindale right ring of quotients and the extended centroid of a semiprime ring R, respectively. For the explanation of Q_r and C we refer the reader to [2]. An additive mapping $T:R\to R$ is called a left centralizer in case $T(xy)=T(x)y$ holds for all pairs $x,y\in R$. In case R has the identity element, $T:R\to R$ is a left centralizer iff T is of the form $T(x)=ax$ for all $x\in R$, where a is some fixed element of R. For a semiprime ring R all left centralizers are of the form $T(x)=qx$ for all $x\in R$, where $q\in Q_r$ is some fixed element (see Chapter 2 in [2]). An additive mapping $T:R\to R$ is called a left Jordan centralizer in case $T(x^2)=T(x)x$ holds for all $x\in R$. The definition of right centralizer and right Jordan centralizer should be self-explanatory. We call $T:R\to R$ a two-sided centralizer in case T is both a left and a right centralizer. In case $T:R\to R$ is a two-sided centralizer, where R is a semiprime ring with extended centroid C, then T is of the form $T(x)=\lambda x$ for all $x\in R$, where $\lambda\in C$ is some fixed element (see Theorem 2.3.2 in [2]).

Zalar [21] has proved that any left (right) Jordan centralizer on a semiprime ring is a left (right) centralizer. Molnár [8] has proved that in case we have an additive mapping $T:A\to A$, where A is a semisimple H^*-algebra satisfying the relation $T(x^3)=T(x)x^2$ ($T(x^3)=x^2T(x)$) for all $x\in A$, then T is a left (right) centralizer. Let us recall that

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a semisimple H^*-algebra is a complex semisimple Banach$^*-$algebra, whose norm is a Hilbert space norm such that $(x,yz^*)=(xz,y)=(z,x^*y)$ is fulfilled for all $x,y,z\in A$. For basic facts concerning H^*-algebras we refer to [1]. Vukman [9] has proved that in case there exists an additive mapping $T:R\to R$, where R is a $2-$torsion free semiprime ring, satisfying the relation $2T(x^2)=T(x)x+xT(x)$ for all $x\in R$, then T is a two-sided centralizer. Kosi-Ulbl and Vukman [7] have proved the following result. Let A be a semisimple H^*-algebra and let $T:A\to A$ be an additive mapping such that $2T(x^{n+1})=T(x)x^n+x^nT(x)$ holds for all $x\in R$ and some fixed integer $n\geqslant 1$. In this case T is a two-sided centralizer. Recently, Benkovič, Eremita and Vukman [4] have considered the relation we have just mentioned above in prime rings with suitable characteristic restrictions. Vukman and Kosi-Ulbl [16] have proved that in case there exists an additive mapping $T:R\to R$, where R is a $2-$torsion free semiprime $^*-$ring, satisfying the relation $T(xx^*)=T(x)x^*$ $(T(xx^*)=xT(x^*))$ for all $x\in R$, then T is a left (right) centralizer. For results concerning centralizers on rings and algebras we refer to $[3,6-16,18-21]$, where further references can be found. Let X be a real or complex Banach space and let $\mathscr{L}(X)$ and $\mathscr{F}(X)$ denote the algebra of all bounded linear operators on X and the ideal of all finite rank operators in $\mathscr{L}(X)$, respectively. An algebra $\mathscr{A}(X)\subset\mathscr{L}(X)$ is said to be standard in case $\mathscr{F}(X)\subset\mathscr{A}(X)$. Let us point out that any standard operator algebra is prime, which is a consequence of a Hahn-Banach theorem. In case X is a real or complex Hilbert space, we denote by A^* the adjoint operator of $A\in\mathscr{L}(X)$. We denote by X^* the dual space of a real or complex Banach space X.

Vukman and Kosi-Ulbl [11] have proved the following result, which was motivated by the work of Brešar [5].

THEOREM 1. *Let R be a $2-$torsion free semiprime ring and let $T:R\to R$ be an additive mapping satisfying the relation*

$$3T(xyx)=T(x)yx+xT(y)x+xyT(x) \tag{1}$$

for all pairs $x,y\in R$. In this case T is of the form $T(x)=\lambda x$ for all $x\in R$, where λ is some fixed element from the extended centroid C.

Putting x for y in the relation (1), one obtains the relation

$$3T(x^3)=T(x)x^2+xT(x)x+x^2T(x),\ x\in R. \tag{2}$$

In case we have a $^*-$ring, we obtain, after putting x^* for y in the relation (1), the relation

$$3T(xx^*x)=T(x)x^*x+xT(x^*)x+xx^*T(x),\ x\in R. \tag{3}$$

The relation (2) is considered in [6] and [20] (actually, much more general situation is considered). It is our aim in this paper to consider the relation (3).

THEOREM 2. *Let X be a complex Hilbert space and let $\mathscr{A}(X)$ be a standard operator algebra, which is closed under the adjoint operation. Suppose $T:\mathscr{A}(X)\to\mathscr{L}(X)$ is a linear mapping satisfying the relation*

$$3T(AA^*A)=T(A)A^*A+AT(A^*)A+AA^*T(A) \tag{4}$$

for all $A \in \mathscr{A}(X)$. *In this case* T *is of the form* $T(A) = \lambda A$ *for all* $A \in \mathscr{A}(X)$, *where* λ *is a fixed complex number.*

Proof. Let us first consider the restriction of T on $\mathscr{F}(X)$. Let A be from $\mathscr{F}(X)$ (in this case we have $A^* \in \mathscr{F}(X)$). Let $P \in \mathscr{F}(X)$ be a self-adjoint projection with the property $AP = PA = A$ (we have also $A^*P = PA^* = A^*$). Putting P for A in (4) we obtain $3T(P) = T(P)P + PT(P)P + PT(P)$, which gives after some calculations

$$T(P) = T(P)P = PT(P) = PT(P)P. \tag{5}$$

Putting $A+P$ for A in the relation (4) we obtain

$$\begin{aligned} &3T(A^2 + AA^* + A^*A) + 6T(A) + 3T(A^*) \\ &= T(A)(A+A^*) + T(A)P + T(P)A^*A + T(P)(A+A^*) + AT(A^*)P \\ &\quad + PT(A^*)A + PT(A^*)P + AT(P)A + AT(P)P + PT(P)A \\ &\quad + (A+A^*)T(A) + PT(A) + AA^*T(P) + (A+A^*)T(P). \end{aligned}$$

Putting $-A$ for A in the above relation and comparing the relation so obtained with the above relation, we obtain

$$\begin{aligned} 3T(A^2 + AA^* + A^*A) = {} & T(A)(A+A^*) + T(P)A^*A + AT(A^*)P + PT(A^*)A \\ & + AT(P)A + (A+A^*)T(A) + AA^*T(P) \end{aligned} \tag{6}$$

and

$$\begin{aligned} 6T(A) + 3T(A^*) = {} & T(A)P + T(P)(A+A^*) + PT(A^*)P \\ & + AT(P)P + PT(P)A + PT(A) + (A+A^*)T(P). \end{aligned} \tag{7}$$

Putting iA for A in the relations (6) and (7) gives

$$\begin{aligned} 3T(A^2 - AA^* - A^*A) = {} & T(A)(A-A^*) - T(P)A^*A - AT(A^*)P - PT(A^*)A \\ & + AT(P)A + (A-A^*)T(A) - AA^*T(P) \end{aligned} \tag{8}$$

and

$$\begin{aligned} 6T(A) - 3T(A^*) = {} & T(A)P + T(P)(A-A^*) - PT(A^*)P + AT(P)P \\ & + PT(P)A + PT(A) + (A-A^*)T(P). \end{aligned} \tag{9}$$

Comparing (6) with (8) and (7) with (9) leads to

$$3T(A^2) = T(A)A + AT(P)A + AT(A) \tag{10}$$

and

$$6T(A) = T(A)P + T(P)A + AT(P)P + PT(P)A + PT(A) + AT(P).$$

After considering $PT(P)A = T(P)A$ and $AT(P)P = AT(P)$ from the relation (5), the above relation reduces to

$$6T(A) = T(A)P + PT(A) + 2T(P)A + 2AT(P). \tag{11}$$

Putting A^* for A in the relation (7) we obtain

$$\begin{aligned}6T(A^*)+3T(A) &= T(A^*)P+T(P)(A+A^*)+PT(A)P\\ &\quad +A^*T(P)P+PT(P)A^*+PT(A^*)+(A+A^*)T(P).\end{aligned}$$

Putting iA for A in the above relation and comparing the relation so obtained with the above relation, we obtain

$$3T(A) = T(P)A+PT(A)P+AT(P). \tag{12}$$

Multiplying the relation (12) by 2 and comparing the relation so obtained with (11) gives $T(A)P+PT(A)=2PT(A)P$, which after right multiplication by P gives $T(A)P=PT(A)P$ and $PT(A)=PT(A)P$ after left multiplication by P. Combining both identities, we get

$$T(A)P = PT(A) = PT(A)P. \tag{13}$$

Right multiplication by P in the relation (12) gives

$$3T(A)P = T(P)A+PT(A)P+AT(P)P.$$

After considering $PT(A)P=T(A)P$ from the relation (13) and $AT(P)P=AT(P)$ from the relation (5), the above relation reduces to

$$2T(A)P = T(P)A+AT(P).$$

According to the relation (13), we can write $2T(A)P=T(A)P+PT(A)$ in the above relation, which can now be written as

$$T(A)P+PT(A) = T(P)A+AT(P).$$

The above relation reduces the relation (11) to

$$2T(A) = T(P)A+AT(P). \tag{14}$$

From the above relation we can conclude that T maps $\mathscr{F}(X)$ into itself. Putting A^2 for A in the relation (14) gives

$$2T(A^2) = T(P)A^2+A^2T(P). \tag{15}$$

Right and left multiplication by A in the relation (14) gives, respectively,

$$T(P)A^2 = 2T(A)A-AT(P)A \tag{16}$$

and

$$A^2T(P) = 2AT(A)-AT(P)A. \tag{17}$$

Applying both (16) and (17) in the relation (15), we obtain

$$T(A^2) = T(A)A-AT(P)A+AT(A). \tag{18}$$

Adding the above relation to the relation (10) gives

$$2T(A^2) = T(A)A + AT(A). \tag{19}$$

We therefore have a linear mapping $T : \mathscr{F}(X) \to \mathscr{F}(X)$, satisfying the relation (19) for all $A \in \mathscr{F}(X)$. Since $\mathscr{F}(X)$ is prime, one can conclude, according to Theorem 1 in [9] that T is a two-sided centralizer on $\mathscr{F}(X)$. We intend to prove that there exists an operator $C \in \mathscr{L}(X)$, such that

$$T(A) = CA,\ A \in \mathscr{F}(X). \tag{20}$$

For any fixed $x \in X$ and $f \in X^*$ we denote by $x \otimes f$ an operator from $\mathscr{F}(X)$ defined by $(x \otimes f)y = f(y)x,\ y \in X$. For any $A \in \mathscr{L}(X)$ we have $A(x \otimes f) = (Ax) \otimes f$. Now let us choose such f and y that $f(y) = 1$ and define $Cx = T(x \otimes f)y$. Obviously, C is linear and applying the fact that T is a left centralizer on $\mathscr{F}(X)$, we obtain

$$(CA)x = C(Ax) = T((Ax) \otimes f)y = T(A(x \otimes f))y = T(A)(x \otimes f)y = T(A)x,$$

for any $x \in X$. We therefore have $T(A) = CA$ for any $A \in \mathscr{F}(X)$. As T is a right centralizer on $\mathscr{F}(X)$, we obtain $C(AB) = T(AB) = AT(B) = ACB$. We therefore have $[A,C]B = 0$ for any $A,B \in \mathscr{F}(X)$, whence it follows that $[A,C] = 0$ for any $A \in \mathscr{F}(X)$. Using closed graph theorem one can easily prove that C is continuous. Since C commutes with all operators from $\mathscr{F}(X)$, we can conclude that $Cx = \lambda x$ holds for any $x \in X$ and some $\lambda \in \mathbb{C}$, which gives together with the relation (20) that T is of the form

$$T(A) = \lambda A \tag{21}$$

for any $A \in \mathscr{F}(X)$ and some $\lambda \in \mathbb{C}$. It remains to prove that the relation (21) holds on $\mathscr{A}(X)$ as well. Let us introduce $T_1 : \mathscr{A}(X) \to \mathscr{L}(X)$ by $T_1(A) = \lambda A$ and consider $T_0 = T - T_1$. The mapping T_0 is, obviously, additive and satisfies the relation (4). Besides, T_0 vanishes on $\mathscr{F}(X)$. It is our aim to show that T_0 vanishes on $\mathscr{A}(X)$ as well. Let $A \in \mathscr{A}(X)$, let $P \in \mathscr{F}(X)$ be a one-dimensional self-adjoint projection and $S = A + PAP - (AP + PA)$. Such S can also be written in the form $S = (I-P)A(I-P)$, where I denotes the identity operator on X. Since $S - A \in \mathscr{F}(X)$, we have $T_0(S) = T_0(A)$. It is easy to see that $SP = PS = 0$. By the relation (4) we have

$$\begin{aligned}
&T_0(S)S^*S + ST_0(S^*)S + SS^*T_0(S)\\
&= 3T_0((S+P)(S+P)^*(S+P))\\
&= T_0(S+P)(S+P)^*(S+P) + (S+P)T_0((S+P)^*)(S+P)\\
&\quad +(S+P)(S+P)^*T_0(S+P)\\
&= T_0(S)S^*S + T_0(S)P + ST_0(S^*)S + ST_0(S^*)P + PT_0(S^*)S\\
&\quad +PT_0(S^*)P + SS^*T_0(S) + PT_0(S).
\end{aligned}$$

We therefore have

$$T_0(S)P + ST_0(S^*)P + PT_0(S^*)S + PT_0(S^*)P + PT_0(S) = 0.$$

Putting $-A$ for A in the above relation (in this case S becomes $-S$) and comparing the relation so obtained with the above relation, we obtain

$$T_0(S)P + PT_0(S^*)P + PT_0(S) = 0.$$

Putting iA for A in the above relation (in this case S^* becomes $-S^*$) and comparing the relation so obtained with the above relation, we obtain

$$T_0(S)P + PT_0(S) = 0.$$

Considering $T_0(S) = T_0(A)$ in the above relation, we obtain

$$T_0(A)P + PT_0(A) = 0. \tag{22}$$

Multiplication from both sides by P in the above relation leads to

$$PT_0(A)P = 0. \tag{23}$$

Right multiplication by P in the relation (22) and considering (23) gives

$$T_0(A)P = 0. \tag{24}$$

Since P is an arbitrary one-dimensional self-adjoint projection, it follows from (24) that $T_0(A) = 0$ for all $A \in \mathscr{A}(X)$, which completes the proof of the theorem. □

It should be mentioned that in the proof of Theorem 2 we used some ideas which are similar to those used by Molnár in [8] and by Vukman in [17].

We conclude with the following conjecture.

CONJECTURE 3. *Let R be a semiprime $^*-$ring with suitable torsion restrictions and let $T : R \to R$ be an additive mapping, satisfying the relation*

$$3T(xx^*x) = T(x)x^*x + xT(x^*)x + xx^*T(x)$$

for all $x \in R$. In this case T is a two-sided centralizer.

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