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FURTHER RESULTS ON GENERALIZED BOTT–DUFFIN INVERSES

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Abstract. Let A be a bounded linear operator, $P_{\mathscr{M}}$ be an orthogonal projection with range $\mathscr{M}$ and $P_{\mathscr{M},\mathscr{N}}$ be an idempotent with range $\mathscr{M}$ and kernel $\mathscr{N}$. This paper presents some novel relations between Bott-Duffin inverse $A^+_{\mathscr{M}} = P_{\mathscr{M}}(AP_{\mathscr{M}} + P_{\mathscr{M}^\perp})^+$ and generalized Bott-Duffin inverse $A^+_{\mathscr{M},\mathscr{N}} = P_{\mathscr{M},\mathscr{N}}(AP_{\mathscr{M},\mathscr{N}} + P_{\mathscr{N},\mathscr{M}})^+$. Furthermore, the representations for the Bott-Duffin inverse and generalized Bott-Duffin inverse are presented.

1. Introduction

Let $\mathscr{H}$ and $\mathscr{K}$ be Hilbert spaces over the same field. We denote the set of all bounded linear operators from $\mathscr{H}$ into $\mathscr{K}$ by $\mathscr{B}(\mathscr{H},\mathscr{K})$ and by $\mathscr{B}(\mathscr{H})$ when $\mathscr{H} = \mathscr{K}$. For $A \in \mathscr{B}(\mathscr{H},\mathscr{K})$, let A^*, $\mathscr{R}(A)$ and $\mathscr{K}(A)$ be the adjoint, the range and the null space of A, respectively. An operator $P \in \mathscr{B}(\mathscr{H})$ is said to be idempotent if $P^2 = P$. An idempotent P is called an orthogonal projection if $P^2 = P = P^*$. The orthogonal projection onto the closed subspace $\mathscr{M} \subseteq \mathscr{H}$ is denoted by $P_{\mathscr{M}}$. Let $P_{\mathscr{M},\mathscr{N}}$ denote the idempotent with $\mathscr{R}(P_{\mathscr{M},\mathscr{N}}) = \mathscr{M}$ and $\mathscr{K}(P_{\mathscr{M},\mathscr{N}}) = \mathscr{N}$. For closed subspaces $\mathscr{M}$ and $\mathscr{N}$, the direct sum and the orthogonal direct sum are denoted by $\mathscr{M} \oplus \mathscr{N}$ and $\mathscr{M} \oplus^\perp \mathscr{N}$, respectively. It is clear $\mathscr{R}(P_{\mathscr{M}}) + \mathscr{K}(P_{\mathscr{M}}) = \mathscr{M} \oplus^\perp \mathscr{M}^\perp = \mathscr{H}$ and $\mathscr{R}(P_{\mathscr{M},\mathscr{N}}) + \mathscr{K}(P_{\mathscr{M},\mathscr{N}}) = \mathscr{M} \oplus \mathscr{N} = \mathscr{H}$.

The Moore-Penrose inverse (for short, MP inverse) of T is denoted by T^+, and it is the unique solution to the following four operator equations ([5, 16]),

$$TXT = T, \quad XTX = X, \quad TX = (TX)^*, \quad XT = (XT)^*.$$

If $\mathscr{R}(T)$ is closed, then T has MP inverse and the MP inverse of T is unique with $(T^*)^+ = (T^+)^*$, $T^+ = T^*(TT^*)^+ = (T^*T)^+T^*$, $TT^+ = P_{\mathscr{R}(T)}$ and $T^+T = P_{\mathscr{R}(T^*)}$. And T, as an operator from $\mathscr{R}(T^*) \oplus \mathscr{K}(T)$ onto $\mathscr{R}(T) \oplus \mathscr{K}(T^*)$, can be written as $T = T_1 \oplus 0$, where T_1 is invertible. $T^+ = T_1^{-1} \oplus 0 = T^*(TT^* + P_{\mathscr{K}(T^*)})^{-1}$ (see [1]–[3], [5], [11]–[20]).

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For $A \in \mathscr{B}(\mathscr{H})$, the constrained linear equations

$$Ax + y = b, \quad x \in \mathscr{M}, \quad y \in \mathscr{M}^{\perp} \tag{1.1}$$

arise in electrical network theory. It is readily found that the equation is consistent with the linear equation $(AP_{\mathscr{M}} + P_{\mathscr{M}^{\perp}})z = b$ and (x,y) is a solution if and only if $x = P_{\mathscr{M}}z$, $y = P_{\mathscr{M}^{\perp}}z = b - AP_{\mathscr{M}}z$. If $AP_{\mathscr{M}} + P_{\mathscr{M}^{\perp}}$ is invertible, then, for all $b \in \mathscr{H}$, there exists the unique solution

$$x = P_{\mathscr{M}}(AP_{\mathscr{M}} + P_{\mathscr{M}^{\perp}})^{-1}b, \quad y = b - Ax.$$

In general, let $A \in \mathscr{B}(\mathscr{H})$ and $\mathscr{M}$ be a closed subspace of $\mathscr{H}$. If $AP_{\mathscr{M}} + P_{\mathscr{M}^{\perp}}$ is MP invertible, the Bott-Duffin inverse (see [4],[6]–[10],[21]) of A with respect to $\mathscr{M}$, denoted by $A^{+}_{\mathscr{M}}$, is defined by

$$A^{+}_{\mathscr{M}} = P_{\mathscr{M}}(AP_{\mathscr{M}} + P_{\mathscr{M}^{\perp}})^{+}. \tag{1.2}$$

This kind of inverse contains group inverse and Drazin inverse. Ben-Israel and Greville in [2] and G. Wang, Y. Wei and S. Qiao in [16] have mentioned many properties of Bott-Duffin inverse and some applications in constrained linear equations.

In this paper, we will consider the general case. For the idempotent operator $P_{\mathscr{M},\mathscr{N}}$, the generalized Bott-Duffin inverse $A^{+}_{\mathscr{M},\mathscr{N}}$ of A with respect to $\mathscr{M}$ and $\mathscr{N}$ is defined by

$$A^{+}_{\mathscr{M},\mathscr{N}} = P_{\mathscr{M},\mathscr{N}}(AP_{\mathscr{M},\mathscr{N}} + P_{\mathscr{N},\mathscr{M}})^{+}. \tag{1.3}$$

Several authors have considered the problem when the dimension of $\mathscr{H}$ is finite. Chen in [6] and B. Deng et al. in [10] have defined the generalized Bott-Duffin inverse and established some of its properties. In [7, 8] G. Chen, G. Liu and Y. Xue have discussed the perturbation theory of the generalized Bott-Duffin inverse. In this paper, we will study the properties and give the expressions for generalized Bott-Duffin inverse of operators on a Hilbert space. Some relations between $A^{+}_{\mathscr{M}}$ and $A^{+}_{\mathscr{M},\mathscr{N}}$ are obtained.

2. Main results

First, we state one useful result. When consider the MP inverse representation for 2×2 upper-triangular operator matrix $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$, we need the following result.

LEMMA 2.1. ([11, Theorem 6]) *Let B be invertible. The 2 by 2 block operator valued triangular matrices $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ and $\begin{pmatrix} A & 0 \\ D & B \end{pmatrix}$ are MP invertible if and only if $\mathscr{R}(A)$ is closed, in which case*

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}^{+} = \begin{pmatrix} A^{+} - A^{+}C\triangle C^{*}(I - AA^{+}) & -A^{+}C\triangle B^{*} \\ \triangle C^{*}(I - AA^{+}) & \triangle B^{*} \end{pmatrix}, \quad \begin{pmatrix} A & 0 \\ D & B \end{pmatrix}^{+} = \begin{pmatrix} A^{+} - (I - A^{+}A)D^{*}\nabla DA^{+} & (I - A^{+}A)D^{*}\nabla \\ -B^{*}\nabla DA^{+} & B^{*}\nabla \end{pmatrix},$$

where $\triangle = (B^{*}B + C^{*}(I - AA^{+})C)^{-1}, \nabla = (BB^{*} + D(I - A^{+}A)D^{*})^{-1}$.

Recall that any matrix is MP invertible. In an arbitrary Hilbert space, it is not true that every element is MP invertible. For every operator $A \in \mathscr{B}(\mathscr{H})$, we know that $A^+_{\mathscr{M}}$ in (1.2) exists $\Longleftrightarrow$ $(AP_{\mathscr{M}} + P_{\mathscr{M}^\perp})^+$ exists. And $A^+_{\mathscr{M},\mathscr{N}}$ in (1.3) exists $\Longleftrightarrow$ $(AP_{\mathscr{M},\mathscr{N}} + P_{\mathscr{N},\mathscr{M}})^+$ exists. Concerning to background of (1.1), we always give a natural hypothesis that $AM \subseteq N^\perp$. First, we get the following result.

THEOREM 2.1. *Let $P_{\mathscr{M},\mathscr{N}}$ be an idempotent and $A \in \mathscr{B}(\mathscr{H})$ be such that $AM \subseteq N^\perp$. Then*

$$A^+_{\mathscr{M}} \quad \text{exists} \quad \Longleftrightarrow \quad A^+_{\mathscr{M},\mathscr{N}} \quad \text{exists}.$$

Proof. Since $\mathscr{M} = \mathscr{R}(P_{\mathscr{M},\mathscr{N}})$ is closed and $P_{\mathscr{M}}$ is an orthogonal projection on $\mathscr{M}$, we can write A, $P_{\mathscr{M}}$ and $P_{\mathscr{M}^\perp}$ as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad P_{\mathscr{M}} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad P_{\mathscr{M}^\perp} = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \tag{2.1}$$

with respect to the space decomposition $\mathscr{H} = \mathscr{M} \oplus^\perp \mathscr{M}^\perp$. Then, by Lemma 2.1, $AP_{\mathscr{M}} + P_{\mathscr{M}^\perp} = \begin{pmatrix} A_{11} & 0 \\ A_{21} & I \end{pmatrix}$ is MP invertible if and only if A_{11} is MP invertible. As for the idempotents $P_{\mathscr{M},\mathscr{N}}$, $P_{\mathscr{N},\mathscr{M}}$ and orthogonal projection $P_{\mathscr{N}}$, they can be written as

$$P_{\mathscr{M},\mathscr{N}} = \begin{pmatrix} I & P_1 \\ 0 & 0 \end{pmatrix}, \quad P_{\mathscr{N},\mathscr{M}} = \begin{pmatrix} 0 & -P_1 \\ 0 & I \end{pmatrix} \quad \text{and} \quad P_{\mathscr{N}} = \begin{pmatrix} Q_1 & Q_2 \\ Q_2^* & Q_4 \end{pmatrix} \tag{2.2}$$

with respect to the space decomposition $\mathscr{H} = \mathscr{M} \oplus^\perp \mathscr{M}^\perp$. From $P^2_{\mathscr{N}} = P_{\mathscr{N}} = P^*_{\mathscr{N}}$ we get $Q_1^* = Q_1$, $Q_4^* = Q_4$ and

$$Q_1 = Q_1^2 + Q_2Q_2^*, \quad Q_2 = Q_1Q_2 + Q_2Q_4, \quad Q_4 = Q_2^*Q_2 + Q_4^2. \tag{2.3}$$

Since $\mathscr{H} = \mathscr{R}(P_{\mathscr{M},\mathscr{N}}) + \mathscr{K}(P_{\mathscr{M},\mathscr{N}}) = \mathscr{M} + \mathscr{N} = \mathscr{R}(P_{\mathscr{M}}) + \mathscr{R}(P_{\mathscr{N}}) = \mathscr{R}(P_{\mathscr{M}} + P_{\mathscr{N}})$, the positive operator $P_{\mathscr{M}} + P_{\mathscr{N}} = \begin{pmatrix} I+Q_1 & Q_2 \\ Q_2^* & Q_4 \end{pmatrix}$ is invertible. We get Q_4 is invertible. Since

$$P_{\mathscr{M},\mathscr{N}}P_{\mathscr{N}} = \begin{pmatrix} I & P_1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} Q_1 & Q_2 \\ Q_2^* & Q_4 \end{pmatrix} = \begin{pmatrix} Q_1 + P_1Q_2^* & Q_2 + P_1Q_4 \\ 0 & 0 \end{pmatrix} = 0,$$

It follows that $P_1 = -Q_2Q_4^{-1}$ and $Q_1 = Q_2Q_4^{-1}Q_2^*$. The condition $AM \subseteq N^\perp$ implies

$$P_{\mathscr{N}}AP_{\mathscr{M}} = \begin{pmatrix} Q_2Q_4^{-1}Q_2^* & Q_2 \\ Q_2^* & Q_4 \end{pmatrix} \begin{pmatrix} A_{11} & 0 \\ A_{21} & 0 \end{pmatrix} = \begin{pmatrix} Q_2Q_4^{-1}Q_2^*A_{11} + Q_2A_{21} & 0 \\ Q_2^*A_{11} + Q_4A_{21} & 0 \end{pmatrix} = 0.$$

We get $A_{21} = -Q_4^{-1}Q_2^*A_{11}$ and

$$\begin{aligned} AP_{\mathscr{M},\mathscr{N}} + P_{\mathscr{N},\mathscr{M}} &= \begin{pmatrix} A_{11} & A_{12} \\ -Q_4^{-1}Q_2^*A_{11} & A_{22} \end{pmatrix} \begin{pmatrix} I & -Q_2Q_4^{-1} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & Q_2Q_4^{-1} \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} A_{11} & Q_2Q_4^{-1} - A_{11}Q_2Q_4^{-1} \\ -Q_4^{-1}Q_2^*A_{11} & I + Q_4^{-1}Q_2^*A_{11}Q_2Q_4^{-1} \end{pmatrix}. \end{aligned} \tag{2.4}$$

As we know, an operator T is MP invertible if and only if $\mathscr{R}(T)$ is closed. If E and F are invertible such that $ETF = S$, then $\mathscr{R}(T)$ is closed if and only if $\mathscr{R}(S)$ is closed.

Since there exists an invertible operator $S = \begin{pmatrix} I & Q_2Q_4^{-1} \\ 0 & I \end{pmatrix}$ such that

$$S^{-1}\begin{pmatrix} A_{11} & Q_2Q_4^{-1}-A_{11}Q_2Q_4^{-1} \\ -Q_4^{-1}Q_2^*A_{11} & I+Q_4^{-1}Q_2^*A_{11}Q_2Q_4^{-1} \end{pmatrix} S = \begin{pmatrix} A_{11}+Q_2Q_4^{-2}Q_2^*A_{11} & 0 \\ -Q_4^{-1}Q_2^*A_{11} & I \end{pmatrix},$$

we get $AP_{\mathcal{M},\mathcal{N}} + P_{\mathcal{N},\mathcal{M}}$ is MP invertible if and only if $\mathscr{R}\left((I+Q_2Q_4^{-2}Q_2^*)A_{11}\right)$ is closed by Lemma 2.1. Since positive operator $I+Q_2Q_4^{-2}Q_2^*$ is automatically invertible, we obtain that $AP_{\mathcal{M},\mathcal{N}} + P_{\mathcal{N},\mathcal{M}}$ is MP invertible if and only if A_{11} is MP invertible, which gives us the desired result. □

It is clear that $AP_{\mathcal{M},\mathcal{N}} = \left[AP_{\mathcal{M}} + P_{\mathcal{M}^\perp}\right]P_{\mathcal{M},\mathcal{N}}$. If $AP_{\mathcal{M}} + P_{\mathcal{M}^\perp}$ is invertible, we build the following relations between $A^+_{\mathcal{M}}$ and $A^+_{\mathcal{M},\mathcal{N}}$.

THEOREM 2.2. *Let $P_{\mathcal{M},\mathcal{N}}$ be an idempotent and $A \in \mathscr{B}(\mathscr{H})$ be such that $AM \subseteq N^\perp$. Then*

$$AP_{\mathcal{M},\mathcal{N}} + P_{\mathcal{N},\mathcal{M}} \quad \text{is invertible} \quad \Longleftrightarrow \quad AP_{\mathcal{M}} + P_{\mathcal{M}^\perp} \quad \text{is invertible.} \tag{2.5}$$

If $AP_{\mathcal{M}} + P_{\mathcal{M}^\perp}$ is invertible, then $A^+_{\mathcal{M},\mathcal{N}} = A^+_{\mathcal{M}}(P^*_{\mathcal{M},\mathcal{N}})^+ = P_{\mathcal{M}}(AP_{\mathcal{M}} + P_{\mathcal{N}})^{-1}$.

Proof. By the proof in Theorem 2.1, it is easy to obtain that $AP_{\mathcal{M},\mathcal{N}} + P_{\mathcal{N},\mathcal{M}}$ (resp. $AP_{\mathcal{M}} + P_{\mathcal{M}^\perp}$) is invertible if and only if A_{11} is invertible. Hence, (2.5) holds. Note $P_{\mathcal{M}} + P_{\mathcal{N}}$ is always invertible and $P_{\mathcal{M},\mathcal{N}} = P_{\mathcal{M}}(P_{\mathcal{M}} + P_{\mathcal{N}})^{-1}$ for arbitrary idempotent $P_{\mathcal{M},\mathcal{N}}$ and relative orthogonal projections $P_{\mathcal{M}}$ and $P_{\mathcal{N}}$. If $AP_{\mathcal{M}} + P_{\mathcal{M}^\perp}$ is invertible, by the definition of $A^+_{\mathcal{M},\mathcal{N}}$, we know $A^+_{\mathcal{M},\mathcal{N}}$ has the simple representation as

$$\begin{aligned} A^+_{\mathcal{M},\mathcal{N}} &= P_{\mathcal{M},\mathcal{N}}(AP_{\mathcal{M},\mathcal{N}} + P_{\mathcal{N},\mathcal{M}})^{-1} \\ &= P_{\mathcal{M}}(P_{\mathcal{M}} + P_{\mathcal{N}})^{-1}\left[AP_{\mathcal{M}}(P_{\mathcal{M}} + P_{\mathcal{N}})^{-1} + P_{\mathcal{N}}(P_{\mathcal{M}} + P_{\mathcal{N}})^{-1}\right]^{-1} \\ &= P_{\mathcal{M}}(AP_{\mathcal{M}} + P_{\mathcal{N}})^{-1}. \end{aligned}$$

Moreover, by (2.1-2.4), we get $A^+_{\mathcal{M}} = A_{11}^{-1} \oplus 0$ and

$$\begin{aligned} &P_{\mathcal{M},\mathcal{N}}(AP_{\mathcal{M},\mathcal{N}} + P_{\mathcal{N},\mathcal{M}})^{-1} \\ &= \begin{pmatrix} I & -Q_2Q_4^{-1} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A_{11} & Q_2Q_4^{-1}-A_{11}Q_2Q_4^{-1} \\ -Q_4^{-1}Q_2^*A_{11} & I+Q_4^{-1}Q_2^*A_{11}Q_2Q_4^{-1} \end{pmatrix}^{-1} \\ &= \begin{pmatrix} I & -Q_2Q_4^{-1} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} I & Q_2Q_4^{-1} \\ 0 & I \end{pmatrix}\begin{pmatrix} A_{11}+Q_2Q_4^{-2}Q_2^*A_{11} & 0 \\ -Q_4^{-1}Q_2^*A_{11} & I \end{pmatrix}^{-1}\begin{pmatrix} I & -Q_2Q_4^{-1} \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} A_{11}^{-1}(I+Q_2Q_4^{-2}Q_2^*)^{-1} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} I & -Q_2Q_4^{-1} \\ 0 & I \end{pmatrix} \\ &= A^+_{\mathcal{M}}(P_{\mathcal{M},\mathcal{N}}P^*_{\mathcal{M},\mathcal{N}})^+P_{\mathcal{M},\mathcal{N}} \\ &= A^+_{\mathcal{M}}(P^*_{\mathcal{M},\mathcal{N}})^+. \quad \square \end{aligned}$$

It is worth pointing out that $A^{+}_{\mathscr{M},\mathscr{N}}$ in Theorem 2.2 can represent MP inverse, group inverse or Drazin inverse when $\mathscr{M}$ and $\mathscr{N}$ are defined as some different particular subspaces:

Case 1. If A is MP invertible and $AP_{\mathscr{R}(A)}+P_{\mathscr{R}(A)^{\perp}}$ is invertible, then

$$A^{+}=A^{+}_{\mathscr{R}(A^*),\mathscr{K}(A^*)}=P_{\mathscr{R}(A^*)}(AP_{\mathscr{R}(A^*)}+P_{\mathscr{K}(A^*)})^{-1};$$

Case 2. If A is group invertible and $A\mathscr{R}(A)\subseteq\mathscr{R}(A^*)$, then

$$A^{\#}=A^{+}_{\mathscr{R}(A),\mathscr{K}(A)}=P_{\mathscr{R}(A)}(AP_{\mathscr{R}(A)}+P_{\mathscr{K}(A)})^{-1};$$

Case 3. If A is Drazin invertible and $A\mathscr{R}(A^l)\subseteq\mathscr{K}(A^l)^{\perp}$, then

$$A^{D}=A^{+}_{\mathscr{R}(A^l),\mathscr{K}(A^l)}=P_{\mathscr{R}(A^l)}(AP_{\mathscr{R}(A^l)}+P_{\mathscr{K}(A^l)})^{-1},$$

for every $l\geqslant k$ and $\operatorname{ind}(A)=k>1$.

THEOREM 2.3. *Let $A\in\mathscr{B}(\mathscr{H})$ and $P_{\mathscr{M},\mathscr{N}}$ be an idempotent. Then*

$$A^{+}_{\mathscr{M},\mathscr{N}}\ \text{exists}\quad\Longleftrightarrow\quad P_{\mathscr{M},\mathscr{N}}AP_{\mathscr{M}}\ \text{is MP invertible}.$$

Proof. By (2.1) and (2.2), we know that

$$\begin{aligned}AP_{\mathscr{M},\mathscr{N}}+P_{\mathscr{N},\mathscr{M}}&=\begin{pmatrix}A_{11}&A_{12}\\A_{21}&A_{22}\end{pmatrix}\begin{pmatrix}I&-Q_2Q_4^{-1}\\0&0\end{pmatrix}+\begin{pmatrix}0&Q_2Q_4^{-1}\\0&I\end{pmatrix}=\begin{pmatrix}A_{11}&Q_2Q_4^{-1}-A_{11}Q_2Q_4^{-1}\\A_{21}&I-A_{21}Q_2Q_4^{-1}\end{pmatrix}\\&=\begin{pmatrix}I&Q_2Q_4^{-1}\\0&I\end{pmatrix}\begin{pmatrix}A_{11}-Q_2Q_4^{-1}A_{21}&0\\A_{21}&I\end{pmatrix}\begin{pmatrix}I&-Q_2Q_4^{-1}\\0&I\end{pmatrix}\end{aligned}$$

and $P_{\mathscr{M},\mathscr{N}}AP_{\mathscr{M}}=(A_{11}-Q_2Q_4^{-1}A_{21})\oplus 0$. Since $\begin{pmatrix}I&Q_2Q_4^{-1}\\0&I\end{pmatrix}$ is invertible, we derive that $A^{+}_{\mathscr{M},\mathscr{N}}$ exists if and only if $A_{11}-Q_2Q_4^{-1}A_{21}$ is MP invertible by Lemma 2.1, which is equivalent to that $P_{\mathscr{M},\mathscr{N}}AP_{\mathscr{M}}$ is MP invertible. $\square$

We continue to discuss the properties of $A^{+}_{\mathscr{M}}$.

THEOREM 2.4. *Let $A\in\mathscr{B}(\mathscr{H})$ such that $A^{+}_{\mathscr{M}}$ exists. Then the following statements are equivalent:*

(i) $A\left[P_{\mathscr{M}}-(P_{\mathscr{M}}AP_{\mathscr{M}})^{+}(P_{\mathscr{M}}AP_{\mathscr{M}})\right]=0$.

(ii) $A^{+}_{\mathscr{M}}=A^{+}_{\mathscr{M}}P_{\mathscr{M}}$.

(iii) $A^{+}_{\mathscr{M}}A=(P_{\mathscr{M}}AP_{\mathscr{M}})^{+}A$.

(iv) $A^{+}_{\mathscr{M}}=(P_{\mathscr{M}}AP_{\mathscr{M}})^{+}$.

(v) $A\mathscr{M}\cap\mathscr{M}^{\perp}=\{0\}$.

Proof. Let A, $P_{\mathscr{M}}$ and $P_{\mathscr{M},\mathscr{N}}$ have the forms as in (2.1) and (2.2). By Lemma 2.1, we get

$$\begin{aligned} A^+_{\mathscr{M}} &= P_{\mathscr{M}}(AP_{\mathscr{M}}+P_{\mathscr{M}^\perp})^+ = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \left[\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}\right]^+ = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A_{11} & 0 \\ A_{21} & I \end{pmatrix}^+ \\ &= \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A_{11}^+-(I-A_{11}^+A_{11})A_{21}^*\triangle A_{21}A_{11}^+ & (I-A_{11}^+A_{11})A_{21}^*\triangle \\ -\triangle A_{21}A_{11}^+ & \triangle \end{pmatrix} \\ &= \begin{pmatrix} A_{11}^+-(I-A_{11}^+A_{11})A_{21}^*\triangle A_{21}A_{11}^+ & (I-A_{11}^+A_{11})A_{21}^*\triangle \\ 0 & 0 \end{pmatrix}, \end{aligned} \tag{2.6}$$

where $\triangle = \left[I+A_{21}(I-A_{11}^+A_{11})A_{21}^*\right]^{-1}$.

(i)$\Longrightarrow$(ii): Note that

$$A\left[P_{\mathscr{M}}-(P_{\mathscr{M}}AP_{\mathscr{M}})^+(P_{\mathscr{M}}AP_{\mathscr{M}})\right] = \begin{pmatrix} 0 & 0 \\ A_{21}(I-A_{11}^+A_{11}) & 0 \end{pmatrix}.$$

If item (i) holds, then $(I-A_{11}^+A_{11})A_{21}^*\triangle = [\triangle^* A_{21}(I-A_{11}^+A_{11})]^* = 0$. So, by (2.6), item (ii) holds.

(ii)$\Longrightarrow$(iii): If item (ii) holds, it is clear that $A^+_{\mathscr{M}} = (P_{\mathscr{M}}AP_{\mathscr{M}})^+$ by 2.6.

(iii)$\Longrightarrow$(iv): Let $\triangle^{\frac{1}{2}}$ denote the positive square root of positive operator $\triangle$. By (2.6), we have

$$A^+_{\mathscr{M}}A = \begin{pmatrix} A_{11}^+A_{11}+(I-A_{11}^+A_{11})A_{21}^*\triangle A_{21}(I-A_{11}^+A_{11}) & A_{11}^+A_{12}+(I-A_{11}^+A_{11})A_{21}^*\triangle(A_{22}-A_{21}A_{11}^+A_{12}) \\ 0 & 0 \end{pmatrix}$$

and $(P_{\mathscr{M}}A_{\mathscr{M}}P_{\mathscr{M}})^+A = \begin{pmatrix} A_{11}^+A_{11} & A_{11}^+A_{12} \\ 0 & 0 \end{pmatrix}$. Since $A^+_{\mathscr{M}}A = (P_{\mathscr{M}}AP_{\mathscr{M}})^+A$, we derive that

$$\begin{aligned} &(I-A_{11}^+A_{11})A_{21}^*\triangle A_{21}(I-A_{11}^+A_{11}) = 0 \\ \Longrightarrow\ &[\triangle^{\frac{1}{2}}A_{21}(I-A_{11}^+A_{11})]^*[\triangle^{\frac{1}{2}}A_{21}(I-A_{11}^+A_{11})] = 0 \\ \Longrightarrow\ &\triangle^{\frac{1}{2}}A_{21}(I-A_{11}^+A_{11}) = 0 \\ \Longrightarrow\ &A_{21}(I-A_{11}^+A_{11}) = 0. \end{aligned}$$

Hence, by (2.6), item (iv) holds.

(iv)$\Longrightarrow$(v): Let A have the form as in (2.1). Since $\mathscr{M} = \mathscr{R}(A_{11}^*)\oplus\mathscr{K}(A_{11}) = \mathscr{R}(A_{11})\oplus\mathscr{K}(A_{11}^*)$, the operator A_{11} can be decomposed as $A_{11} = A_{11}^0\oplus 0$, where A_{11}^0 as an operator from $\mathscr{R}(A_{11}^*)$ onto $\mathscr{R}(A_{11})$ is invertible. If (iv) holds, then $A_{21}(I-A_{11}^+A_{11}) = 0$. That is $\mathscr{K}(A_{11})\subset\mathscr{K}(A_{21})$ and therefore $AP_{\mathscr{M}}$ has the form as

$$AP_{\mathscr{M}} = \begin{pmatrix} A_{11} & 0 \\ A_{21} & 0 \end{pmatrix}\begin{pmatrix} \mathscr{M} \\ \mathscr{M}^\perp \end{pmatrix} \to \begin{pmatrix} \mathscr{M} \\ \mathscr{M}^\perp \end{pmatrix} = \begin{pmatrix} A_{11}^0 & 0 & 0 \\ 0 & 0 & 0 \\ A_{21}^0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \mathscr{R}(A_{11}^*) \\ \mathscr{K}(A_{11}) \\ \mathscr{M}^\perp \end{pmatrix} \to \begin{pmatrix} \mathscr{R}(A_{11}) \\ \mathscr{K}(A_{11}^*) \\ \mathscr{M}^\perp \end{pmatrix}.$$

The invertibility of A_{11}^0 implies that $A\mathscr{M}\cap\mathscr{M}^\perp = \{0\}$.

(v)$\Longrightarrow$(i): Note that $AP_{\mathscr{M}} = \begin{pmatrix} A_{11} & 0 \\ A_{21} & 0 \end{pmatrix}$. If $A\mathscr{M}\cap\mathscr{M}^\perp = \{0\}$, then $\mathscr{K}(A_{11})\subset\mathscr{K}(A_{21})$. Hence, $A_{21}(I-A_{11}^+A_{11}) = 0$ and therefore

$$A\left[P_{\mathscr{M}}-(P_{\mathscr{M}}AP_{\mathscr{M}})^+(P_{\mathscr{M}}AP_{\mathscr{M}})\right] = \begin{pmatrix} 0 & 0 \\ A_{21}(I-A_{11}^+A_{11}) & 0 \end{pmatrix} = 0. \qquad \square$$

Theorem 2.4 presents a list of equivalent conditions, which help us easier to check that $A^+_{\mathscr{M}} = (P_{\mathscr{M}}AP_{\mathscr{M}})^+$. Moreover, a representation for $A^+_{\mathscr{M}}$ can be derived by the proof of Theorem 2.4.

THEOREM 2.5. *Let $A \in \mathscr{B}(\mathscr{H})$ such that $A^+_{\mathscr{M}}$ exists. Then*

$$A^+_{\mathscr{M}} = (P_{\mathscr{M}}AP_{\mathscr{M}})^+ + \left[P_{\mathscr{M}} - (P_{\mathscr{M}}AP_{\mathscr{M}})^+(P_{\mathscr{M}}AP_{\mathscr{M}})\right] A^*\Phi^+ \left[P_{\mathscr{M}^\perp} - A(P_{\mathscr{M}}AP_{\mathscr{M}})^+\right],$$

where $\Phi = P_{\mathscr{M}^\perp}A\left[P_{\mathscr{M}} - (P_{\mathscr{M}}AP_{\mathscr{M}})^+(P_{\mathscr{M}}AP_{\mathscr{M}})\right]A^*P_{\mathscr{M}^\perp} + P_{\mathscr{M}^\perp}$.

Proof. By the proof of Theorem 2.4, we get

$$A^+_{\mathscr{M}} = P_{\mathscr{M}}(AP_{\mathscr{M}} + P_{\mathscr{M}^\perp})^+ = \begin{pmatrix} A_{11}^+ - (I - A_{11}^+A_{11})A_{21}^*\triangle A_{21}A_{11}^+ & (I - A_{11}^+A_{11})A_{21}^*\triangle \\ 0 & 0 \end{pmatrix},$$

where $\triangle = \left[I + A_{21}(I - A_{11}^+A_{11})A_{21}^*\right]^{-1}$. Note that A, $P_{\mathscr{M}}$ and $P_{\mathscr{M},\mathscr{N}}$ have the forms as in (2.1) and (2.2). The result is obtained from the fact that $\Phi^+ = 0 \oplus \triangle$ and $(P_{\mathscr{M}}AP_{\mathscr{M}})^+ = A_{11}^+ \oplus 0$. $\square$

Representations for the MP inverse for block matrices were given in the literature under certain conditions. In the paper of Miao [13], Tian [14] and Cvetković-Ilić et al. [9], the MP-inverse was considered for the class of matrices $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Baksalary and Styan [1] have given the necessary and sufficient conditions for the representation of the MP-inverse of M by the Banachiewicz-Schur form. The details are given as follows. If $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is a given matrix and $S = D - CA^+B$ is the generalized Schur complement of A in M, then

$$M^+ = \begin{pmatrix} A^+ + A^+BS^+CA^+ & -A^+BS^+ \\ -S^+CA^+ & S^+ \end{pmatrix} \tag{2.7}$$

if and only if

$$B(I - S^+S) = 0, \quad (I - SS^+)C = 0, \quad C(I - A^+A) = 0, \quad (I - AA^+)B = 0. \tag{2.8}$$

Applying this result, we get a representation for the generalized Bott-Duffin inverse.

THEOREM 2.6. *Let $A \in \mathscr{B}(\mathscr{H})$, $\mathscr{M}$ and $\mathscr{N}$ be closed subspaces of $\mathscr{H}$ such that $AM \subseteq N^\perp$ and $\left[P_{\mathscr{M}} - (P_{\mathscr{M}}AP_{\mathscr{M}})(P_{\mathscr{M}}AP_{\mathscr{M}})^+\right]P_{\mathscr{N},\mathscr{M}} = 0$. Then*

$$A^+_{\mathscr{M},\mathscr{N}} = A^+_{\mathscr{M}} - \left[A^+_{\mathscr{M}}(I - A) + I\right]P_{\mathscr{M}}P_{\mathscr{N},\mathscr{M}}P^+_{\mathscr{N},\mathscr{M}}.$$

Proof. Let A, $P_{\mathscr{M}}$ and $P_{\mathscr{M},\mathscr{N}}$ have the forms as in (2.1) and (2.2). In the proof of Theorem 2.1, we obtain that $P_{\mathscr{M},\mathscr{N}} = \begin{pmatrix} I & -Q_2Q_4^{-1} \\ 0 & 0 \end{pmatrix}$, $P_{\mathscr{N},\mathscr{M}} = \begin{pmatrix} 0 & Q_2Q_4^{-1} \\ 0 & I \end{pmatrix}$ and by (2.4)

$$AP_{\mathscr{M},\mathscr{N}} + P_{\mathscr{N},\mathscr{M}} = \begin{pmatrix} A_{11} & (I - A_{11})Q_2Q_4^{-1} \\ -Q_4^{-1}Q_2^*A_{11} & I + Q_4^{-1}Q_2^*A_{11}Q_2Q_4^{-1} \end{pmatrix}.$$

If $\left[P_{\mathscr{M}}-(P_{\mathscr{M}}AP_{\mathscr{M}})(P_{\mathscr{M}}AP_{\mathscr{M}})^{+}\right]P_{\mathscr{N},\mathscr{M}}=0$, then $(I-A_{11}A_{11}^{+})Q_2Q_4^{-1}=0$. So the generalized Schur complement

$$S=I+Q_4^{-1}Q_2^{*}A_{11}Q_2Q_4^{-1}+Q_4^{-1}Q_2^{*}A_{11}A_{11}^{+}\left(Q_2Q_4^{-1}-A_{11}Q_2Q_4^{-1}\right)=I+Q_4^{-1}Q_2^{*}Q_2Q_4^{-1}$$

is invertible and $0\oplus S^{-1}=(P_{\mathscr{N},\mathscr{M}}^{*}P_{\mathscr{N},\mathscr{M}})^{+}$. It is clear that the corresponding conditions in (2.8) hold and, by (2.7),

$$(AP_{\mathscr{M},\mathscr{N}}+P_{\mathscr{N},\mathscr{M}})^{+}=\begin{pmatrix} A_{11}^{+}-A_{11}^{+}(I-A_{11})Q_2Q_4^{-1}S^{-1}Q_4^{-1}Q_2^{*} & -A_{11}^{+}(I-A_{11})Q_2Q_4^{-1}S^{-1} \\ S^{-1}Q_4^{-1}Q_2^{*} & S^{-1}\end{pmatrix}.$$

Since $A_{21}=-Q_4^{-1}Q_2^{*}A_{11}$, we get $(I-A_{11}^{+}A_{11})A_{21}^{*}=\left[A_{21}(I-A_{11}^{+}A_{11})\right]^{*}=0$ and $A_{\mathscr{M}}^{+}=A_{11}^{+}\oplus 0$ by (2.6). Hence,

$$\begin{aligned}
A_{\mathscr{M},\mathscr{N}}^{+}&=P_{\mathscr{M},\mathscr{N}}(P_{\mathscr{M},\mathscr{N}}A+P_{\mathscr{N},\mathscr{M}})^{+}\\
&=\begin{pmatrix} I & -Q_2Q_4^{-1}\\ 0 & 0\end{pmatrix}\begin{pmatrix} A_{11}^{+}-A_{11}^{+}(I-A_{11})Q_2Q_4^{-1}S^{-1}Q_4^{-1}Q_2^{*} & -A_{11}^{+}(I-A_{11})Q_2Q_4^{-1}S^{-1} \\ S^{-1}Q_4^{-1}Q_2^{*} & S^{-1}\end{pmatrix}\\
&=\begin{pmatrix} A_{11}^{+}-(A_{11}^{+}+I-A_{11}^{+}A_{11})Q_2Q_4^{-1}S^{-1}Q_4^{-1}Q_2^{*} & -(A_{11}^{+}+I-A_{11}^{+}A_{11})Q_2Q_4^{-1}S^{-1} \\ 0 & 0\end{pmatrix}\\
&=A_{\mathscr{M}}^{+}-\left[A_{\mathscr{M}}^{+}(I-A)P_{\mathscr{M}}+P_{\mathscr{M}}\right]P_{\mathscr{N},\mathscr{M}}(P_{\mathscr{N},\mathscr{M}}^{*}P_{\mathscr{N},\mathscr{M}})^{+}P_{\mathscr{N},\mathscr{M}}^{*}\\
&=A_{\mathscr{M}}^{+}-\left[A_{\mathscr{M}}^{+}(I-A)+I\right]P_{\mathscr{M}}P_{\mathscr{N},\mathscr{M}}P_{\mathscr{N},\mathscr{M}}^{+}. \qquad \square
\end{aligned}$$

Theorems 2.5 and 2.6 provide some formulas for computing the Bott-Duffin inverse and generalised Bott-Duffin inverse, respectively. The formulas are easy to compute by using projector methods. Moreover, we get the following result.

THEOREM 2.7. *Let $A\in\mathscr{B}(\mathscr{H})$, $P_{\mathscr{M},\mathscr{N}}$ be an idempotent operator such that $\mathscr{R}(P_{\mathscr{M},\mathscr{N}}AP_{\mathscr{M},\mathscr{N}})$ is closed and $AM\subseteq N^{\perp}$. Then*

$$A_{\mathscr{M},\mathscr{N}}^{+}=(P_{\mathscr{M},\mathscr{N}}AP_{\mathscr{M},\mathscr{N}})^{+}\Longleftrightarrow \mathscr{N}=\mathscr{M}^{\perp}.$$

Proof. Sufficiency. If $\mathscr{N}=\mathscr{M}^{\perp}$, then $\mathscr{A}\mathscr{M}\cap\mathscr{M}^{\perp}=\{0\}$ and $P_{\mathscr{M},\mathscr{N}}=P_{\mathscr{M}}$. The result follows immediately by $(iv)\Longleftrightarrow(v)$ in Theorem 2.4.

Necessity. Let $P_{\mathscr{M}}$ and A have the representations as in (2.1), $P_{\mathscr{M},\mathscr{N}}$, $P_{\mathscr{N},\mathscr{M}}$ and $P_{\mathscr{N}}$ have the representations as in (2.2). If $A_{\mathscr{M},\mathscr{N}}^{+}=(P_{\mathscr{M},\mathscr{N}}AP_{\mathscr{M},\mathscr{N}})^{+}$, then

$$\mathscr{R}(A_{\mathscr{M},\mathscr{N}}^{+})=\mathscr{R}(P_{\mathscr{M},\mathscr{N}}^{*}A^{*}P_{\mathscr{M},\mathscr{N}}^{*})\subset\mathscr{R}(P_{\mathscr{M},\mathscr{N}}^{*})\subset\mathscr{N}^{\perp}$$

and therefore $P_{\mathscr{N}}A_{\mathscr{M},\mathscr{N}}^{+}=P_{\mathscr{N}}P_{\mathscr{M},\mathscr{N}}(AP_{\mathscr{M},\mathscr{N}}+P_{\mathscr{N},\mathscr{M}})^{+}=0$.

Hence $P_{\mathscr{N}}P_{\mathscr{M},\mathscr{N}}(P_{\mathscr{M},\mathscr{N}}^{*}A^{*}+P_{\mathscr{N},\mathscr{M}}^{*})=0$. Note that

$$\begin{aligned}
P_{\mathscr{N}}P_{\mathscr{M},\mathscr{N}}P_{\mathscr{M},\mathscr{N}}^{*}A^{*}&=\begin{pmatrix} Q_2Q_4^{-1}Q_2^{*} & Q_2\\ Q_2^{*} & Q_4\end{pmatrix}\begin{pmatrix} I & -Q_2Q_4^{-1}\\ 0 & 0\end{pmatrix}\begin{pmatrix} I & 0\\ -Q_4^{-1}Q_2^{*} & 0\end{pmatrix}\begin{pmatrix} A_{11}^{*} & -A_{11}^{*}Q_2Q_4^{-1}\\ A_{12}^{*} & A_{22}^{*}\end{pmatrix}\\
&=\begin{pmatrix} *** & *** \\ Q_2^{*}(I+Q_2Q_4^{-2}Q_2^{*})A_{11}^{*} & -Q_2^{*}(I+Q_2Q_4^{-2}Q_2^{*})A_{11}^{*}Q_2Q_4^{-1}\end{pmatrix}
\end{aligned}$$

and

$$-P_{\mathscr{N}}P_{\mathscr{M},\mathscr{N}}P^*_{\mathscr{N},\mathscr{M}} = -\begin{pmatrix} Q_2Q_4^{-1}Q_2^* & Q_2 \\ Q_2^* & Q_4 \end{pmatrix}\begin{pmatrix} I & -Q_2Q_4^{-1} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ Q_4^{-1}Q_2^* & I \end{pmatrix}$$
$$= \begin{pmatrix} *** & *** \\ Q_2^*Q_2Q_4^{-2}Q_2^* & Q_2^*Q_2Q_4^{-1} \end{pmatrix},$$

where $***$ can be gotten by the production of relative matrices. Since $P_{\mathscr{N}}P_{\mathscr{M},\mathscr{N}}P^*_{\mathscr{M},\mathscr{N}}A^* = -P_{\mathscr{N}}P_{\mathscr{M},\mathscr{N}}P^*_{\mathscr{N},\mathscr{M}}$, compare the two sides of the above matrices, we get

$$\begin{cases} Q_2^*Q_2 = -Q_2^*(I+Q_2Q_4^{-2}Q_2^*)A_{11}^*Q_2, & (a) \\ Q_2^*Q_2Q_4^{-2}Q_2^* = Q_2^*(I+Q_2Q_4^{-2}Q_2^*)A_{11}^*. & (b) \end{cases}$$

Multiplying Q_2 from right in item (b), we get $Q_2^*Q_2Q_4^{-2}Q_2^*Q_2 = -Q_2^*Q_2$ by item (a). Since $Q_2^*Q_2Q_4^{-2}Q_2^*Q_2 \geqslant 0$ and $Q_2^*Q_2 \geqslant 0$, we get $Q_2^*Q_2 = 0$. That is $Q_2 = 0$. Since Q_4 is invertible and $\mathscr{M}\cap\mathscr{N} = \{0\}$, The orthogonal projestion $P_{\mathscr{N}}$ in (2.2) has the form $P_{\mathscr{N}} = 0\oplus I$. Hence $\mathscr{N} = \mathscr{M}^{\perp}$ and $\mathscr{H} = \mathscr{M}\oplus^{\perp}\mathscr{N}$. □

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