SUBMAXIMAL OPERATOR SPACE STRUCTURES ON BANACH SPACES

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Abstract. Subspaces of maximal operator spaces are called *submaximal* spaces and in general, they need not be maximal. We call those maximal operator spaces with the property that all submaximal spaces turn out to be maximal as *hereditarily maximal* spaces. Any two Banach isomorphic subspaces of a hereditarily maximal space will be completely isomorphic as operator spaces. We derive a characterization of these spaces. We introduce a notion of distance of an operator space to the class of submaximal spaces and discuss some related results.

1. Introduction

Just as one can treat C^*-algebras either concretely as closed *-subalgebras of $\mathscr{B}(\mathscr{H})$ for some Hilbert space $\mathscr{H}$, or abstractly as a Banach algebra satisfying certain properties, operator spaces can also be considered in two different ways. A *(concrete) operator space* X is a closed linear subspace of $\mathscr{B}(\mathscr{H})$. Here, in each *matrix level* $M_n(X)$ (the space of all $n \times n$ matrices with entries from X), we have a norm $\|.\|_n$, induced by the inclusion $M_n(X) \subset M_n(\mathscr{B}(\mathscr{H}))$, where the norm in $M_n(\mathscr{B}(\mathscr{H}))$ is given by the natural identification $M_n(\mathscr{B}(\mathscr{H})) \approx \mathscr{B}(\mathscr{H}^n)$. If X and Y are linear spaces and $\varphi : X \to Y$ is a linear map, $\varphi^{(n)} : M_n(X) \to M_n(Y)$, given by $[x_{ij}] \to [\varphi(x_{ij})]$, with $[x_{ij}] \in M_n(X)$ and $n \in \mathbb{N}$, determines a linear map from $M_n(X)$ to $M_n(Y)$. Suppose that each of the spaces $M_n(X)$ has a given norm $\|.\|_n$, then the *complete bound norm* (in short *cb-norm*) of φ is defined as $\|\varphi\|_{cb} = \sup\left\{\left\|\varphi^{(n)}\right\|; n \in \mathbb{N}\right\}$. φ is *completely bounded* if $\|\varphi\|_{cb} < \infty$. φ is a *complete isometry* if each map $\varphi^{(n)} : M_n(X) \to M_n(Y)$ is an isometry. If φ is a complete isometry, then $\|\varphi\|_{cb} = 1$. If $\|\varphi\|_{cb} \leqslant 1$, φ is said to be a *complete contraction*. If $\varphi : X \to Y$ is a completely bounded linear bijection and if its inverse is also completely bounded, then φ is said to be a *complete isomorphism*.

An *abstract operator space*, is a pair $(X, \{\|.\|_n\}_{n \in \mathbb{N}})$ consisting of a linear space X and a complete norm $\|.\|_n$ on $M_n(X)$ for every $n \in \mathbb{N}$, such that there exists a linear complete isometry $\varphi : X \to \mathscr{B}(\mathscr{H})$ for some Hilbert space $\mathscr{H}$. Two operator spaces are considered to be the same if there is a complete isometric isomorphism from X to Y. i.e., if there is a linear isomorphism $\varphi : X \to Y$ such that $\|\varphi\|_{cb} = \left\|\varphi^{-1}\right\|_{cb} = 1$.

In that case, we write $X \approx Y$ *completely isometrically* . The sequence of matrix norms $\{\|.\|_n\}_{n \in \mathbb{N}}$ is called an *operator space structure* on the linear space X. In 1988, Z-J. Ruan [11], characterized the sequence of matrix norms on a linear space X that defines an operator space structure on X in terms of two properties of matrix norms. This allows us to view an operator space in an abstract way free of any concrete representation on a Hilbert space and so we no longer distinguish between concrete and abstract operator spaces. More information about operator spaces and completely bounded mappings may be found in [3], [8] or [9].

If X is a Banach space, any linear embedding of X into $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, defines an operator space structure on X, by the natural identification $M_n(\mathcal{B}(\mathcal{H})) \approx \mathcal{B}(\mathcal{H}^n)$. If such an embedding preserves the original norm on X, (i.e., if the embedding is isometric), the corresponding operator space structure on X is said to be *admissible* . Blecher and Paulsen [2] observed that the set of all operator space structures admissible on a given Banach space X admits a minimal and maximal element. These structures were further investigated in [6] and [7]. Subspace structure of various maximal operator spaces were studied in [5].

Let X be a Banach space and $K = Ball(X^*)$ the closed unit ball of the dual space X^* of X, with its weak* topology. Then the canonical embedding $J : X \to C(K)$, defined by $J(x)(f) = f(x), x \in X$ and $f \in K$ is a linear isometry. Since by Gelfand-Naimark theorem [4], subspaces of C*-algebras are operator spaces, this identification of X induces matrix norms on $M_n(X)$ that makes X an operator space. The matrix norms on X are given by

$$\left\|[x_{ij}]\right\|_n = \sup\{\left\|[f(x_{ij})]\right\| : f \in K\}$$

for all $[x_{ij}] \in M_n(X)$ and for all $n \in \mathbb{N}$. The above defined operator space structure on X is called the *minimal operator space structure* on X, and we denote this operator space as $Min(X)$. For $[x_{ij}] \in M_n(X)$, we write $\left\|[x_{ij}]\right\|_{Min(X)}$ to denote its norm as an element of $M_n(Min(X))$. This minimal quantization of a normed space is characterized by the property that for any arbitrary operator space Y any bounded linear map $\varphi : Y \to Min(X)$ is completely bounded and satisfies

$$\|\varphi : Y \to Min(X)\|_{cb} = \|\varphi : Y \to X\|.$$

Thus, if X and Y are Banach spaces and $\varphi \in B(X,Y)$, then φ is completely bounded and $\|\varphi\|_{cb} = \|\varphi\|$, when considered as a map from $X \to Min(Y)$. It may be noted that $Min(X)$ is the smallest operator space structure on X. An operator space X is minimal if $Min(X) = X$. Also, an operator space is minimal if and only if it is completely isometric to a subspace of a commutative C*-algebra.

If X is a Banach space, there is a maximal way to consider it as an operator space. The matrix norms given by

$$\left\|[x_{ij}]\right\|_n = \sup\{\left\|[\varphi(x_{ij})]\right\| ; \varphi \in Ball(B(X,Y))\}$$

where the supremum is taken over all operator spaces Y and all linear maps $\varphi \in Ball(B(X,Y))$, makes X an operator space. We denote this operator space as $Max(X)$

and is called the *maximal operator space structure* on X . For $[x_{ij}] \in M_n(X)$, we write $\left\| [x_{ij}] \right\|_{Max(X)}$ to denote its norm as an element of $M_n(Max(X))$. We say that an operator space X is maximal if $Max(X) = X$. By Ruan's theorem [11] , we also have

$$\left\| [x_{ij}] \right\|_{Max(X)} = \sup\{\left\| [\varphi(x_{ij})] \right\| ; \varphi \in Ball(B(X,B(\mathscr{H})))\}$$

where the supremum is taken over all Hilbert spaces $\mathscr{H}$ and all linear maps $\varphi \in Ball(B(X,B(\mathscr{H})))$. By the definition of $Max(X)$, any operator space structure that we can put on X, will be smaller than $Max(X)$. This maximal quantization of a Banach space is characterized by the property that for any arbitrary operator space Y any bounded linear map $\varphi : Max(X) \to Y$ is completely bounded and satisfies

$$\left\| \varphi : Max(X) \to Y \right\|_{cb} = \left\| \varphi : X \to Y \right\|.$$

Thus, if X and Y are Banach spaces and $\varphi \in B(X,Y)$, then φ is completely bounded and $\|\varphi\|_{cb} = \|\varphi\|$, when considered as a map from $Max(X) \to Y$. If X is any operator space, then the identity map on X defines completely contractive maps $Max(X) \to X \to Min(X)$. From the above discussions, we have the following observation.

PROPOSITION 1.1. *Let X and Y be operator spaces and $\varphi : X \to Y$ a bounded linear map, then φ is completely bounded and $\|\varphi\|_{cb} = \|\varphi\|$ if X is given the maximal operator space structure or Y is given the minimal operator space structure.*

Any subspace of a minimal operator space is again minimal, but quotient of a minimal space need not be minimal. Quotients of minimal operator spaces are called Q-spaces. Also, the class of Q-spaces is stable under taking quotients and subspaces. The subspace of a maximal space need not be maximal and such spaces are called *submaximal* spaces [5]. But quotient spaces inherits maximality.

THEOREM 1.2. ([9]) *If X is a maximal operator space and Y a closed subspace then $Max(X/Y) \approx Max(X)/Y$ completely isometrically.*

An operator space Z is *injective* [3] if for any operator spaces X and Y where Y contains X as a closed subspace, and for any completely bounded linear map $\varphi : X \to Z$, there exists a completely bounded extension $\tilde{\varphi} : Y \to Z$ such that $\tilde{\varphi}|_X = \varphi$ and $\|\tilde{\varphi}\|_{cb} = \|\varphi\|_{cb}$. An important class of operator spaces are those $X \subset B(H)$ which are isomorphic (as a Banach space) to a Hilbert space. We call such spaces as *Hilbertian* operator spaces. The spaces $Min(\ell^2)$ and $Max(\ell^2)$ are Hilbertian operator spaces. Blecher and Paulsen [2] observed that any separable infinite dimensional Hilbertian operator space lies between $Min(\ell^2)$ and $Max(\ell^2)$.

We discuss submaximal spaces and some of their properties in the next section. In section 3, we introduce *hereditarily maximal* spaces and give a characterization of these spaces. In section 4, we define the distance $d_{sm}(X)$ of a space to the class of submaximal spaces and make use of this to obtain a necessary and sufficient condition for a space to be submaximal. Also, we compute $d_{sm}(X)$ for some spaces and explore its relation with $d_Q(X)$, the distance of a space X to the class of Q-spaces.

2. Submaximal spaces

We have noted that subspace of a maximal space need not be maximal, i.e., if Y is a subspace of X and if $x_{ij} \in Y$ for $i, j = 1, 2, ..., n$, then the norm of $[x_{ij}]$ in $M_n(Max(Y))$ can be larger than the norm of $[x_{ij}]$ as an element of $M_n(Max(X))$. However, we have the following result.

THEOREM 2.1. ([7]) *Let* $x_{ij} \in X$, $i, j = 1, 2, ..., n$, *then* $\left\| [x_{ij}] \right\|_{Max(X)} = \inf \{ \left\| [x_{ij}] \right\|_{Max(Y)} ; x_{ij} \in Y$ *and* $Y \subset X$, *finite dimensional* $\}$.

Also, if every subspace of $X = Max(X)$ is maximal, then by proposition 1.1, any two Banach isomorphic subspaces of X will be completely isomorphic as operator spaces.

DEFINITION 2.2. ([9]) An operator space X is said to be submaximal if it embeds completely isometrically into a maximal operator space Y.

Paulsen ([6]) observed that every separable submaximal space embeds completely isometrically into a separable maximal space. We now observe that just like every operator space embeds completely isometrically into $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, every submaximal space embeds completely isometrically into $Max(B(\mathcal{H}))$ for some Hilbert space $\mathcal{H}$.

PROPOSITION 2.3. *Let* X *be a submaximal space of a maximal operator space* $Y = Max(Y)$. *If* $\iota : X \to B(\mathcal{H})$ *is a completely isometric inclusion, then* ι *defines a completely isometric inclusion* $\tilde{\iota} : X \to Max(B(\mathcal{H}))$ *and there is a completely contractive extension* $\varphi : Y \to Max(B(\mathcal{H}))$.

Proof. Let $\iota : X \to B(\mathcal{H})$ be a complete isometric inclusion. Since $X \hookrightarrow Y = Max(Y)$ and since $B(\mathcal{H})$ is injective, $\iota : X \to B(\mathcal{H})$ extends to a complete contraction $\varphi : Y \to B(\mathcal{H})$. The maximal operator space structure of Y implies that

$$\| \varphi : Y \to Max(B(\mathcal{H})) \|_{cb} = \| \varphi : Y \to Max(B(\mathcal{H})) \| \leqslant \| \varphi : Y \to B(\mathcal{H}) \|_{cb} \leqslant 1$$

Thus, $\| \tilde{\iota} = \varphi|_X : X \to Max(B(\mathcal{H})) \|_{cb} \leqslant 1$. If $\tilde{X} = \tilde{\iota}(X)$, then $\| \tilde{\iota}^{-1} : \tilde{X} \to X \|_{cb} \leqslant 1$, so that $\tilde{\iota}$ is a complete isometric inclusion of X into $Max(B(\mathcal{H}))$. $\square$

Using the above result, we now derive a characterization of submaximal spaces.

THEOREM 2.4. *An operator space* X *is a submaximal space if and only if any complete contraction* $u : X \to B(\mathcal{H})$ *defines a complete contraction* $\tilde{u} = u : X \to Max(B(\mathcal{H}))$.

Proof. Assume that X is a submaximal space and let $X \subset Max(Y)$. Let $u : X \to B(\mathcal{H})$ be a complete contraction. As in proposition 2.3, there exists a complete contractive extension $\phi : Max(Y) \to Max(B(\mathcal{H}))$ such that $\| \tilde{u} = \phi|_X : X \to$

$Max(B(\mathscr{H}))\|_{cb} \leqslant \|\phi : Max(Y) \to Max(B(\mathscr{H}))\|_{cb} \leqslant 1$. Conversely assume that $\iota : X \to B(\mathscr{H})$ be a complete isometric inclusion of X in $B(\mathscr{H})$, then by assumption, $\tilde{\iota} = \iota : X \to Max(B(\mathscr{H}))$ is a complete contraction. Also, $\|\tilde{\iota}^{-1} : \tilde{\iota}(X) \to X\|_{cb} \leqslant 1$, so that $\tilde{\iota}$ defines a complete isometric inclusion of X in $Max(B(\mathscr{H}))$, so that X is completely isometrically isomorphic to the submaximal space $\tilde{\iota}(X) \subset Max(B(\mathscr{H}))$.   □

Now we identify the completely bounded maps between two submaximal spaces.

THEOREM 2.5. *Let $X \subset Max(B(\mathscr{H}_1))$ and $Y \subset Max(B(\mathscr{H}_2))$ be submaximal spaces. Then any completely bounded map $\varphi : X \to Y$ extends to a bounded linear map from $B(\mathscr{H}_1)$ to $B(\mathscr{H}_2)$. Also, any bounded linear map $\varphi : B(\mathscr{H}_1) \to B(\mathscr{H}_2)$ with $\varphi(X) \subset Y$ induces a completely bounded map from X to Y.*

Proof. Let $\varphi : X \to Y$ be completely bounded. It is known from proposition 2.3 that $\tilde{\varphi} = \varphi : X \to Max(B(\mathscr{H}_2))$ is also completely bounded and $\|\tilde{\varphi}\|_{cb} \leqslant \|\varphi\|_{cb}$. Since $B(\mathscr{H}_2)$ is injective, we have an extension $\phi : Max(B(\mathscr{H}_1)) \to Max(B(\mathscr{H}_2))$ such that $\phi|_X = \tilde{\varphi}$ and $\|\phi\|_{cb} = \|\tilde{\varphi}\|_{cb}$. Thus, $\|\phi\| \leqslant \|\phi\|_{cb} = \|\tilde{\varphi}\|_{cb} \leqslant \|\varphi\|_{cb} < \infty$. This implies that ϕ is a bounded linear map from $B(\mathscr{H}_1)$ to $B(\mathscr{H}_2)$. Now let $\varphi : B(\mathscr{H}_1) \to B(\mathscr{H}_2)$ be a bounded linear map such that $\varphi(X) \subset Y$. Regarding φ as a map $\tilde{\varphi} = \varphi : Max(B(\mathscr{H}_1)) \to Max(B(\mathscr{H}_2))$, we see that $\tilde{\varphi}$ is completely bounded and $\|\tilde{\varphi}\|_{cb} = \|\varphi : B(\mathscr{H}_1) \to B(\mathscr{H}_2)\|$, so that $\|\tilde{\varphi}|_X : X \to Y\|_{cb} \leqslant \|\tilde{\varphi}\|_{cb} = \|\varphi\| < \infty$.   □

REMARK 2.6. The class of submaximal spaces is closed under taking closed subspaces and quotients. To see this, if X is a submaximal space, it embeds completely isometrically into a maximal operator space Y; say $\varphi : X \hookrightarrow Max(Y)$. If Z is a closed subspace of X, then the restriction of φ to Z will be a complete isometric embedding of Z in $Max(Y)$. Also, in this case, $X/Z \subset Max(Y)/Z$ and from theorem 1.2, we have $Max(Y/Z) \approx Max(Y)/Z$ completely isometrically. This shows that the quotient space X/Z is a submaximal space.

Now we show that submaximal spaces are stable under ℓ^1-sums.

THEOREM 2.7. *Let $\{X_i\}_{i \in I}$ be a family of submaximal operator spaces. Then $\ell^1(\{X_i; i \in I\})$ is submaximal.*

Proof. First, we note that ℓ^1-sum of maximal spaces is again maximal. Let X_i's be maximal, so that $Max(X_i) = X_i$ for all $i \in I$. Let $u : \ell^1(\{X_i; i \in I\}) \to B(H)$ be a bounded linear map. Corresponding to this u, there exists bounded linear maps $u_i : X_i \to B(H)$ for every $i \in I$. Then $\|u\| = \sup\{\|u_i\| ; i \in I\}$. But by the definition of operator space structure on $\ell^1(\{X_i; i \in I\})$, and using the fact that X_i has maximal operator space structure, we see that $\|u\|_{cb} = \sup\{\|u_i\|_{cb} ; i \in I\} = \sup\{\|u_i\| ; i \in I\} = \|u\|$. This shows that $\ell^1(\{X_i; i \in I\})$ is maximal.

If $X_i's$ are submaximal, we have $X_i \subset Max(Y_i)$, for every $i \in I$, so that $\ell^1(\{X_i; i \in I\}) \subset \ell^1(\{Max(Y_i); i \in I\})$. Hence $\ell^1(\{X_i; i \in I\})$ is a submaximal space.   □

Now we show that submaximality of a space will be reflected in its bidual also.

THEOREM 2.8. *An operator space X is submaximal if and only if its bidual X^{**} is submaximal.*

Proof. Let $X \subset Max(Y)$. Then $X^{**} \subset (Max(Y))^{**}$. But from the duality relations [1], $(Max(Y))^* = Min(Y^*)$, so that $(Max(Y))^{**} = (Min(Y^*))^* = Max(Y^{**})$. Thus X^{**} is submaximal. The converse part follows from the fact that $X \subset X^{**}$ and from the remark 2.6. □

3. Hereditarily maximal spaces

It is known that submaximal operator space structure on a Banach space X need not equals with the maximal operator space structure on X. But we show that in some spaces this will happen. We call such spaces as *hereditarily maximal*. (This name was suggested by Prof. Gilles Pisier in a private communication with us.)

DEFINITION 3.1. A maximal operator space X is said to be hereditarily maximal if any closed subspace of X is again a maximal operator space. i.e., for any closed subspace $Y \subset X = Max(X)$, we have $Max(Y) = Y$.

Note that any two closed subspaces of a hereditarily maximal space which are isomorphic as Banach spaces, are in fact completely isomorphic as operator spaces.
Now we derive a characterization of these spaces.

THEOREM 3.2. *A maximal operator space X is hereditarily maximal if and only if the space X has the following extension property: For any closed subspace $Y \subset X$, and for any bounded linear map $\varphi : Y \to B(\mathcal{H})$, there exists a bounded extension $\tilde{\varphi} : X \to B(\mathcal{H})$ such that $\|\tilde{\varphi}\| = \|\varphi\|$.*

Proof. Assume that X is hereditarily maximal. Let $Y \subset X$ be a closed subspace. Then $Max(Y) = Y$. Let $\varphi : Y \to B(\mathcal{H})$ be a bounded linear map. Since $Y = Max(Y)$, we see that φ is completely bounded and $\|\varphi\|_{cb} = \|\varphi\|$. Since $B(\mathcal{H})$ is injective, there exists a mapping $\tilde{\varphi} : X \to B(\mathcal{H})$ such that $\tilde{\varphi}|_X = \varphi$ and $\|\tilde{\varphi}\|_{cb} = \|\varphi\|_{cb}$. Since X has maximal structure, $\|\tilde{\varphi}\|_{cb} = \|\tilde{\varphi}\|$. Thus $\|\tilde{\varphi}\| = \|\tilde{\varphi}\|_{cb} = \|\varphi\|_{cb} = \|\varphi\|$.
Conversely, assume that X has the above described extension property. We have to show that X is hereditarily maximal. Let $Y \subset X = Max(X)$. Let $u : Y \to Max(Y) \subset B(\mathcal{H})$ be the isometric inclusion mapping. Then u is bounded. By assumption it has an extension $\tilde{u} : X \to B(\mathcal{H})$ such that $\tilde{u}|_Y = u$ and $\|\tilde{u}\| = \|u\|$. Then $\|u\|_{cb} = \|\tilde{u}|_Y\|_{cb} \leqslant \|\tilde{u}\|_{cb} = \|\tilde{u}\| = \|u\|$. The second last equality follows from the fact that $X = Max(X)$. This shows that $\|u\|_{cb} = \|u\|$. Thus the formal isometric inclusion mapping of Y to $Max(Y)$ is a complete isometry, which implies that $Max(Y) = Y$. □

The following theorem will serve us some examples for these type of spaces.

THEOREM 3.3. *If X is a Hilbertian operator space and Y be a closed subspace of $Max(X)$, then $Y = Max(Y)$.*

Proof. Let $p : X \to Y$ be the orthogonal projection. Then $p : Max(X) \to Max(Y)$ is a complete contraction. Note that the inclusion map $j : Y \hookrightarrow Max(X)$ is completely bounded. Therefore, $pj : Y \to Max(Y)$ is completely contractive. Thus $Max(Y) = Y$. $\square$

REMARK 3.4. The above theorem shows that every maximal Hilbertian operator space is hereditarily maximal. This can also be obtained by noting that a Hilbertian operator space X has the bounded extension property described in theorem 3.2. Let Y be a closed subspace of X and let u be any bounded linear map $u : Y \to B(\mathcal{H})$. Then Y admits an orthogonal projection $p : X \to Y$. Now the composition map $\tilde{u} = u \circ p : X \to B(\mathcal{H})$ is bounded and $\tilde{u}|_Y = u$. Also, $\|\tilde{u}\| = \|u \circ p\| \leqslant \|u\|$. Since $\tilde{u}$ is an extension of u, this shows that $\|\tilde{u}\| = \|u\|$.

4. Distance to the class of submaximal spaces

We now introduce a notion of distance of an operator space to the class of submaximal spaces, and by using it we derive a characterization of submaximal spaces. Note that completely bounded Banach-Mazur distance between two operator spaces X and Y is defined as $d_{cb}(X,Y) = \inf\{\|\varphi\|_{cb} \|\varphi^{-1}\|_{cb} : \varphi : X \to Y \text{ a complete isomorphism}\}$.

DEFINITION 4.1. For a given operator space X, we define $d_{sm}(X) = \inf\{d_{cb}(X,Y); Y \text{ a submaximal space}\}$

We make use of this concept to characterize a submaximal space up to complete isometric isomorphism.

THEOREM 4.2. *An operator space X is a submaximal space, up to complete isometric isomorphism, if and only if $d_{sm}(X) = 1$.*

Proof. If X is a submaximal space, then clearly $d_{sm}(X) = 1$. For proving the other direction we use the fact that submaximal spaces are stable under ultra products. To see this, note if $X_i \subset Max(Y_i)$, and if $\mathscr{U}$ is a nontrivial ultra filter on the indexing set I, then $\prod X_i / \mathscr{U} \subset \prod Max(Y_i) / \mathscr{U}$, which is completely isometrically isomorphic to $Max(\prod Y_i / \mathscr{U})$ [5]. If $d_{sm}(X) = 1$, then for any $n \in \mathbb{N}$, there exists a submaximal space X_n such that $d_{cb}(X,X_n) < 1 + 1/n$, and a complete isomorphism $u_n : X \to X_n$ such that $\|u_n\|_{cb} < 1 + 1/n$ and $\|u_n^{-1}\|_{cb} = 1$. Choose a nontrivial ultra filter $\mathscr{U}$ on $\mathbb{N}$ such that $lim_{\mathscr{U}} \|u_n\|_{cb} = 1$. Then the ultra product $\prod X_n / \mathscr{U}$ is a submaximal space and $u : X \to \prod X_n / \mathscr{U}$ defined by $u = (u_n)_{n \in \mathbb{N}}$ is a complete isometric isomorphism. This shows that X is a submaximal space. $\square$

Operator spaces X and Y are said to be *C-completely isomorphic* if there exists a linear isomorphism $\varphi : X \to Y$ such that $\|\varphi\|_{cb} \|\varphi^{-1}\|_{cb} \leqslant C$. The following result gives another characterization of submaximal spaces upto C-complete isomorphism. This result is implicitly contained in [5].

THEOREM 4.3. *Let X be an operator space and $C > 0$. Then X is C-completely isomorphic to a submaximal space Y if and only if for every complete contraction $u : X \to B(\mathcal{H})$, we have $\|u : X \to Max(B(\mathcal{H}))\|_{cb} \leqslant C$*

Proof. Assume that X is C-completely isomorphic to a submaximal space Y. Let $\varphi : X \to Y$ be such that $\|\varphi\|_{cb} \|\varphi^{-1}\|_{cb} \leqslant C$. Since Y is submaximal, $Y \subset Max(Z)$, for some operator space Z. Let $u : X \to B(\mathcal{H})$ be a complete contraction. Then the map $v = u\varphi^{-1} : Y \to B(\mathcal{H})$ is completely bounded. As in proposition 2.3, there exists a completely bounded extension $\tilde{v} =: Max(Z) \to Max(B(\mathcal{H}))$ such that $\tilde{v}|_Y = u\varphi^{-1} : Y \to Max(B(\mathcal{H}))$ is completely bounded. Therefore,

$$
\begin{aligned}
\|u : X \to Max(B(\mathcal{H}))\|_{cb} &= \|u\varphi^{-1}\varphi : X \to Max(B(\mathcal{H}))\|_{cb} \\
&\leqslant \|\varphi\|_{cb} \|u\varphi^{-1} : Y \to Max(B(\mathcal{H}))\|_{cb} \\
&\leqslant \|\varphi\|_{cb} \|\varphi^{-1}\|_{cb} \leqslant C
\end{aligned}
$$

Conversely, let $\iota : X \to B(\mathcal{H})$ be a complete isometric inclusion, then by assumption, $\|\iota : X \to Max(B(\mathcal{H}))\|_{cb} \leqslant C$. Then $\iota(X) \subset Max(B(\mathcal{H}))$ is a submaximal space and since $\iota^{-1} : \iota(X) \to X$ is a complete contraction, ι defines a C-complete isomorphism from X onto the submaximal space $\iota(X) \subset Max(B(\mathcal{H}))$. □

COROLLARY 4.4. *For an operator space X, $d_{sm}(X)$ is the smallest $C > 0$ such that for any complete contraction $u : X \to B(\mathcal{H})$, $\|u : X \to Max(B(\mathcal{H}))\|_{cb} \leqslant C$*

As an illustration, we compute $d_{sm}(X)$ for some finite dimensional operator spaces.

PROPOSITION 4.5. *Let R_n and C_n denotes the n-dimensional row and column Hilbert spaces and M_n denotes the space of $n \times n$ scalar matrices. Then, we have $d_{sm}(R_n) = d_{sm}(C_n) = \sqrt{n}$ and $d_{sm}(M_n) = n$.*

Proof. Consider the formal identity map $id : R_n \to \ell_2^n$. From the factorization of the identity map from R_n to C_n, along $Max(\ell_2^n)$, $R_n \to Max(\ell_2^n) \to C_n$, we have

$$
\|id : R_n \to Max(\ell_2^n)\|_{cb} \geqslant \|id : R_n \to C_n\|_{cb} = \|id : R_n \to C_n\|_{HS} = \sqrt{n}.
$$

Therefore by the above corollary 4.4, $d_{sm}(R_n) \geqslant \sqrt{n}$. Since $Max(\ell_2^n)$ itself is a submaximal space, and since $\|id : R_n \to Max(\ell_2^n)\|_{cb} = \sqrt{n}$ and $\|id^{-1} : Max(\ell_2^n) \to R_n\|_{cb} = 1$, from the definition of $d_{sm}(X)$, we have $d_{sm}(R_n) \leqslant \sqrt{n}$. Thus $d_{sm}(R_n) = \sqrt{n}$. Similarly $d_{sm}(C_n) = \sqrt{n}$. Note that $id : M_n \to Max(M_n)$ has the cb-norm n, so that $d_{sm}(M_n) \geqslant n$. But $Max(M_n)$ itself is a submaximal space, hence we have $d_{sm}(M_n) = n$. □

THEOREM 4.6. *Let X be an operator space. Then X is C-complete isomorphic to a submaximal space, for some $C > 0$, if and only if $d_{sm}(Y) \leqslant C$ for any finite dimensional subspace Y of X.*

Proof. If X is C-complete isomorphic to a submaximal space, for some $C > 0$, then there exists a submaximal space Z such that $\varphi : X \to Z$ satisfies $\|\varphi\|_{cb} \|\varphi^{-1}\|_{cb} \leqslant C$. Then for any subspace Y of X, $\varphi|_Y : Y \to \varphi(Y) \subset Z$, is such that $\|\varphi|_Y\|_{cb} \|\varphi|_Y^{-1}\|_{cb} \leqslant C$. This shows that $d_{sm}(Y) \leqslant C$.

Conversely assume that $d_{sm}(Y) \leqslant C$ for any finite dimensional subspace Y of X for some $C > 0$. If X is not C-completely isomorphic to any submaximal space, then by theorem 4.3, there exists a Hilbert space $\mathscr{H}$ and a complete contraction $\varphi : X \to Max(B(\mathscr{H}))$, with $\|\varphi : X \to Max(B(\mathscr{H}))\|_{cb} > C$. Therefore, by definition of cb-norm, there exists a finite set of elements $\{x_{i,j}; 1 \leqslant i, j \leqslant n\}$ for some $n \in \mathbb{N}$, in X such that $\|[x_{i,j}]\| \leqslant 1$ and $\|[\varphi(x_{i,j})]\| > C$. Choose Y as the linear span of $\{x_{i,j}; 1 \leqslant i, j \leqslant n\}$. Then Y is a finite dimensional subspace of X, and $\|\varphi|_Y : Y \to Max(B(\mathscr{H}))\|_{cb} > C$. So by theorem 4.3, Y is not C-complete isomorphic to a submaximal space. This implies that $d_{sm}(Y) > C$, which is a contradiction. $\square$

A Q-space ([10]) is an (operator) quotient of a minimal space. In a similar way of defining $d_{sm}(X)$, we can have $d_Q(X) = \inf\{d_{cb}(X,Y); Y$ is a Q-space$\}$ which measures the distance of a given operator space X to the class of Q-spaces. Also, X is a Q-space if and only if $d_Q(X) = 1$. Note that if X is a Q-space, then X^* is a submaximal space. More precisely, if X is C-completely isomorphic to a Q-space, then X^* is C-completely isomorphic to a submaximal space. Conversely, the dual of a submaximal space is a Q-space. Thus we have the following observation.

THEOREM 4.7. *For any operator space X, we have $d_{sm}(X) = d_Q(X^*)$.*

COROLLARY 4.8. *For $n \in \mathbb{N}$, we have: $d_Q(R_n) = d_Q(C_n) = \sqrt{n}$ and $d_Q(S_1^n) = n$, where $S_1^n = M_n^*$, the space of trace class operators.*

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