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WEAK MAJORIZATION INEQUALITIES FOR SINGULAR VALUES

LIMIN ZOU AND CHUANJIANG HE

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Abstract. In this paper, we refine an inequality due to Bhatia and Kittaneh [Linear Algebra Appl. 308 (2000) 203–211], and generalize another inequality by Bhatia and Kittaneh [Lett. Math. Phys. 43 (1998) 225–231].

1. Introduction

Let M_n be the space of $n \times n$ complex matrices. Let $\|\cdot\|$ denote any unitarily invariant norm on M_n. We shall always denote the singular values of A by $s_1(A) \geqslant \cdots \geqslant s_n(A) \geqslant 0$. Let M_n^+ be the set of positive semidefinite matrix on M_n.

Let $x = (x_1, \cdots, x_n)$ be an element of R^n. Let $x^\downarrow$ be the vector obtained by rearranging the coordinates of x in the decreasing order. For $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$ belonging to R^n, if

$$\sum_{i=1}^{k} x_i^\downarrow \leqslant \sum_{i=1}^{k} y_i^\downarrow, \quad k = 1, \cdots, n,$$

then we say that x is weakly majorized by y, denoted $x \prec_w y$. If the components of x and y are nonnegative and

$$\prod_{i=1}^{k} x_i^\downarrow \leqslant \prod_{i=1}^{k} y_i^\downarrow, \quad k = 1, \cdots, n,$$

then we say that x is weakly log-majorized by y, denoted $x \prec_{w\log} y$.

It is well known that $x \prec_{w\log} y$ implies $x \prec_w y$. For more information on majorization and matrix inequalities the reader is referred to [1–3].

Let A and B be positive semidefinite. Bhatia and Kittaneh [4, Theorem 1] (see also [2, p. 77]) obtained the following inequality:

$$s(AB) \prec_w s\left(\left(\frac{A+B}{2}\right)^2\right). \tag{1.1}$$

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Zhan [5, Theorem 2.2] proved that for any complex number z,

$$s(A-|z|B) \prec_{w\log} s(A+zB) \prec_{w\log} s(A+|z|B).$$

This is a strengthening of the following inequality:

$$s(A-|z|B) \prec_w s(A+zB) \prec_w s(A+|z|B),$$

which is due to Bhatia and Kittaneh [6, Theorem 2.1]. These authors also proved [6, Theorem 2.2] that for any positive integer m,

$$s(A^m+B^m) \prec_w s((A+B)^m). \tag{1.2}$$

In Section 2, we shall refine (1.1) and generalize (1.2). Section 3 contains some remarks.

2. Main results

In this section, we first refine (1.1).

THEOREM 2.1. *If A, $B \in M_n$ are positive semidefinite, then*

$$s(AB) \prec_w s\left(\int_0^1 A^{1/2+t}B^{3/2-t}\,dt\right) \prec_w s\left(\left(\frac{A+B}{2}\right)^2\right). \tag{2.1}$$

Proof. The well-known arithmetic-geometric mean inequality for singular values due to Bhatia and Kittaneh [7] (see also [1, p. 262]) says that

$$2s_j(PQ^*) \leqslant s_j(P^*P+Q^*Q), \quad j=1,2,\cdots,n \tag{2.2}$$

for any P, $Q \in M_n$. Let

$$P = A^{1/2}(A+B)^{1/2}, \quad Q = B^{1/2}(A+B)^{1/2}.$$

By (2.2), we have

$$2s_j\left(A^{1/2}(A+B)B^{1/2}\right) \leqslant s_j\left((A+B)^2\right), \quad j=1,2,\cdots,n. \tag{2.3}$$

Hiai and Kosaki [8, Corollary 2.3] proved that for all unitarily invariant norms

$$\left\|A^{1/2}XB^{1/2}\right\| \leqslant \left\|\int_0^1 A^tXB^{1-t}\,dt\right\| \leqslant \left\|\frac{AX+XB}{2}\right\|.$$

Putting

$$X = A^{1/2}B^{1/2}$$

in this last inequality, gives

$$\|AB\| \leqslant \left\|\int_0^1 A^{1/2+t}B^{3/2-t}\,dt\right\| \leqslant \left\|\frac{A^{1/2}(A+B)B^{1/2}}{2}\right\|.$$

By Fan's dominance principle, this is equivalent to

$$s(AB) \prec_w s\left(\int_0^1 A^{1/2+t}B^{3/2-t}dt\right) \prec_w s\left(\frac{A^{1/2}(A+B)B^{1/2}}{2}\right). \tag{2.4}$$

It follows from (2.3) and (2.4) that

$$s(AB) \prec_w s\left(\int_0^1 A^{1/2+t}B^{3/2-t}dt\right) \prec_w s\left(\left(\frac{A+B}{2}\right)^2\right).$$

This completes the proof. □

Next, we shall generalize (1.2). To do this, we need the following result [9, Theorem 2.1].

LEMMA 2.1. *Let $A,\, B \in M_n$ be normal and let $f:[0,\,\infty)\to[0,\,\infty)$ be concave. Then, for all unitarily invariant norms,*

$$\|f(|A+B|)\| \leqslant \|f(|A|)+f(|B|)\|. \tag{2.5}$$

THEOREM 2.2. *Let $g(t)=\sum_{k=1}^{m} a_k t^k$ be a polynomial vanishing at 0 and with nonnegative coefficients a_k, $k=1\,,\,\cdots,\,m$. Then for all normal matrices $A,\, B\in M_n$,*

$$s(g(A)+g(B)) \prec_w s(g(|A+B|)). \tag{2.6}$$

In particular,

$$s(A^m+B^m) \prec_w s(|A+B|^m).$$

Proof. Let $X,\,Y$ be any pair of normal matrices in M_n and let $f(t)=g^{-1}(t)$ be the reciprocal function of $g(t)$ for $t\in[0,\,\infty)$. By (2.5), since f is concave, we have

$$s(f(|X+Y|)) \prec_w s(f(|X|)+f(|Y|)).$$

Since g is convex and increasing on $[0,\,\infty)$, it preserves weak majorization on M_n^+, hence the above majorization yields

$$s(|X+Y|) \prec_w s(g(f(|X|)+f(|Y|))).$$

Now, set $X=g(A)$ and $Y=g(B)$. We then have

$$s(|g(A)+g(B)|) \prec_w s(g(f(|g(A)|)+f(|g(B)|))).$$

Since $|g(A)|=g(|A|)$, $|g(B)|=g(|B|)$, and $f(g(t))=t$ on $[0,\,\infty)$, the last majorization completes the proof. □

3. Remarks

REMARK 3.1. The inequality (2.3) has been obtained by Bhatia and Kittaneh [4, p. 206]. Here, we give a simple proof.

REMARK 3.2. Let $A,\ B \in M_n$ be positive semidefinite. Then

$$s_j(AB) \leqslant s_j\left(\left(\frac{A+B}{2}\right)^2\right), \quad 1 \leqslant j \leqslant n. \tag{3.1}$$

This was a question posed by Bhatia and Kittaneh [4](see also [10-11]), and settled in the affirmative by Drury in [12]. In view of (1.1), (2.1) and (3.1), we ask the following: Is it true that

$$s_j(AB) \leqslant s_j\left(\int_0^1 A^{1/2+t}B^{3/2-t}dt\right) \leqslant s_j\left(\left(\frac{A+B}{2}\right)^2\right), \quad 1 \leqslant j \leqslant n?$$

This would be a strengthening of (2.1).

REMARK 3.3. Let $A,\ B \in M_n$ be positive semidefinite. Tao [13, Theorem 3] proved that the following inequality

$$2s_j\left(A^{1/2}(A+B)^r B^{1/2}\right) \leqslant s_j\left((A+B)^{r+1}\right), \quad j = 1, \cdots, n \tag{3.2}$$

holds for any positive integer r. It is a generalization of (2.3). Bhatia and Kittaneh [10, p. 2186] proved that the inequality (3.2) holds for any positive real number r. Now, we give a simple proof of (3.2). In fact, for any $r > 0$, let

$$P = A^{1/2}(A+B)^{r/2}, \quad Q = B^{1/2}(A+B)^{r/2},$$

we obtain the inequality (2.4) from the inequality (2.1).

Moreover, for any $r, r_1, r_2 > 0$, let

$$P = A^{r_1}(A+B)^{r/2}, \quad Q = B^{r_2}(A+B)^{r/2}.$$

Then, for $j = 1, \cdots, n$, we have

$$2s_j\left(A^{r_1}(A+B)^r B^{r_2}\right) \leqslant s_j\left((A+B)^{r/2}\left(A^{2r_1}+B^{2r_2}\right)(A+B)^{r/2}\right).$$

This is a generalization of (3.2).

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Limin Zou
College of Mathematics and Statistics
Chongqing University
Chongqing, 401331, P. R. China
and
School of Mathematics and Statistics
Chongqing Three Gorges University
Chongqing, 404100, P. R. China
e-mail: limin-zou@163.com

Chuanjiang He
Mathematical Sciences Research Institute in Chongqing
Chongqing University
Chongqing, 401331, P. R. China
and
College of Mathematics and Statistics
Chongqing University
Chongqing, 401331, P. R. China