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NORMAL MATRIX COMPRESSIONS

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Abstract. There has been longstanding interest in the problem of characterizing normal compressions of normal matrices. Indeed, the solution to the Hermitian case goes back to the Cauchy interlacing theorem, and its converse (due to Fan and Pall). More recently, the theory of higher–rank numerical ranges has included the solution in the case of scalar compressions. Here we take steps towards a similar treatment of the general case. We develop some natural necessary conditions on the eigenvalues as well as some convenient sufficient conditions, showing by a study of the 2x2 compressions of 4x4 normals that the necessary conditions are not sufficient. We also give a new proof of the Choi–Kribs–Życzkowski conjecture for 2x2 compressions by means of a powerful extension of that result. The CKŻ conjecture (more recently a theorem) may be stated as follows: given an $N \times N$ normal matrix M with eigenvalues $\lambda_1, \ldots, \lambda_N$, the set of $a \in \mathbb{C}$ for which the scalar matrix aI_k is a compression of M is precisely

$$\Omega_k(M) = \bigcap_{\#(J)=N-k+1} \operatorname{conv}\{\lambda_j : j \in J\}.$$

Thus, for $k = 2$ we see that $a \in \Omega_2(M)$ implies that $\operatorname{diag}(a,a)$ is a compression of M (the reverse implication is relatively straightforward). We show that, in fact, for *any pair* $a,b \in \Omega_2(M)$, $\operatorname{diag}(a,b)$ is a compression of M. Our proof is independent of the earlier results and depends on a novel approach. We also study the continuity of the map $a \to B(a)$, where $B(a)$ denotes the set of all $b \in \mathbb{C}$ such that $\operatorname{diag}(a,b)$ is a compression of M.

1. Introduction

Given a linear operator T on a complex Hilbert space $\mathbb{H}$, and any orthogonal projection P, we say that $PT|_{P\mathbb{H}}$ is a *compression* of T. If $\mathbb{H} = \mathbb{C}^N$ and T is represented by a matrix $M \in \mathbb{M}_N$ (the $N \times N$ complex matrices), a second matrix C represents a compression of T (or a compression of M) iff there is a unitary matrix U such that C is a NW corner of UMU^*. If C is $k \times k$ we say it is a rank–k compression of M. There is a rich history of results that allow us to identify compressions by means of intrinsic criteria. A classic example is the Cauchy interlacing theorem [3], along with its converse [10], which may be expressed as follows.

THEOREM 1. *If $M \in \mathbb{M}_N$ is Hermitian, with eigenvalues*

$$a_1 \leqslant a_2 \leqslant \ldots \leqslant a_N,$$

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then C is a rank–k compression of M iff C is Hermitian with eigenvalues b_j satisfying

$$a_1 \leqslant b_1 \leqslant a_{N-k+1}, a_2 \leqslant b_2 \leqslant a_{N-k+2}, \dots, a_k \leqslant b_k \leqslant a_N.$$

In particular, C is a rank $N-1$ compression iff

$$a_1 \leqslant b_1 \leqslant a_2 \leqslant b_2 \leqslant a_3 \leqslant \ldots \leqslant a_{N-1} \leqslant b_{N-1} \leqslant a_N,$$

the classic "interlacing" of eigenvalues.

A much more recent example is provided by the theory of *higher–rank numerical ranges*.

The striking development of this theory was motivated originally by problems in quantum information theory. Since the introduction of this concept by Choi, Kribs, and Życzkowski [4, 5] only a few years ago, it has indeed been effectively applied in the area of quantum information (see [14, 15, 16, 20], for example). It has also inspired a remarkable development of its purely mathematical aspects (see, for example, [6, 7, 26, 18, 17, 9]). From this point of view the theory of the higher–rank numerical ranges may be described as a highly successful analysis of *scalar* compressions of arbitrary matrices $M \in \mathbb{M}_N$. This suggests a more general program: characterize the *normal* (diagonal) compressions of M. Among other approaches to this program we may mention [1], [2] (where the program is included among a "treasure trove of open problems"; see Problem 6), [8], [11], [19], [22], [23], [24], and [25].

Added in proof. Two papers of Gau and Wu are also relevant in this context; see Linear and Multilinear Algebra 52:3, 195–201, and Linear Algebra Appl. 390, 121–136.

In the present paper we make a detailed study of 2×2 compressions $\operatorname{diag}(a,b)$ of normal M. As a result we obtain, for the first time, examples where the natural necessary conditions on the spectrum of a compression are *not* sufficient: see Figure 3 and Proposition 13. We also see parts of the theory of higher–rank numerical ranges from a new angle: we give a novel proof of the Choi–Kribs–Życzkowski conjecture (more recently a theorem) for the rank–2 numerical range. That result says that $\operatorname{diag}(a,a)$ is a compression of normal M having spectrum $\{\lambda_1,\dots,\lambda_N\}$ provided a lies in

$$\Omega_2(M) = \bigcap_{k=1}^{N} \operatorname{conv}\{\lambda_j : j \neq k\}.$$

We prove a significantly stronger result (and by an argument that is independent of earlier work): in fact, for *any pair* $a,b \in \Omega_2(M)$, $\operatorname{diag}(a,b)$ is a compression of M. See the discussion of Figure 1 below, and Proposition 15.

The rank–k numerical range of M, usually denoted in the literature by $\Lambda_k(M)$, was defined by Choi, Kribs, and Życzkowski as the set of those complex λ such that for some rank–k orthogonal projection P we have

$$PMP = \lambda P.$$

In terms of compressions, we see that $\lambda \in \Lambda_k(M)$ iff λI_k is a (matrix) compression of M. Thus the following fundamental result of Li and Sze [18] may be placed in the same family as the Cauchy interlacing theorem (and, in fact, the interlacing theorem plays a role in the argument of Li and Sze).

THEOREM 2. *Given $M \in \mathbb{M}_N$, let $\lambda_j(\theta)$ be an enumeration of the eigenvalues of the (Hermitian)*

$$Re(e^{i\theta}M) = (e^{i\theta}M + e^{-i\theta}M^*)/2$$

such that

$$\lambda_1(\theta) \leqslant \lambda_2(\theta) \leqslant \ldots \leqslant \lambda_N(\theta).$$

For each real θ, let the half–plane $H(M,\theta)$ be defined by

$$H(M,\theta) = e^{i\theta}\{z : Re(z) \leqslant \lambda_{N-k+1}(-\theta)\}.$$

Then

$$\Lambda_k(M) = \bigcap\{H(M,\theta) : \theta \in [0,2\pi]\}. \tag{1}$$

Our more general program seeks to describe all *normal* compressions of M, ie to describe those complex $a_1,\ldots,a_k$ such that $\mathrm{diag}(a_1,\ldots,a_k)$ is a compression of M. Equivalently, we ask when there exist orthonormal

$$u_1, u_2, \ldots, u_k$$

such that $(Mu_i,u_i) = a_i$ for each i and $(Mu_i,u_j) = 0$ whenever $i \neq j$; in particular, $\Lambda_1(M)$ is nothing but the classical numerical range

$$W(M) = \{(Mu,u) : \|u\| = 1\}$$

(hence the "higher–rank numerical range" terminology). In this work we usually restrict our attention to the case where M itself is also normal, although we occasionally comment on cases where either M or its compression may not be normal.

Note that for normal $M \in \mathbb{M}_N(\mathbb{C})$ Theorem 2 shows that $\Lambda_k(M)$ can be explicitly described in terms of the eigenvalues $z_1,\ldots,z_N$ of M:

$$\Lambda_k(M) = \Omega_k(M), \tag{2}$$

where

$$\Omega_k(M) = \bigcap_{\#(J)=N-k+1} \mathrm{conv}\{z_j : j \in J\}. \tag{3}$$

We shall refer to this result, first proposed by Choi, Kribs, and Życzkowski, as the CKŻ conjecture, although it is now a theorem. The CKŻ conjecture played an important role in the development of the theory of higher–rank numerical ranges. For example, while Li and Sze gave an effective description of $\Lambda_k(M)$ for non–normal M (Theorem 2), their proof of the CKŻ conjecture was a key step towards the general result. Of course, the case $k=1$ of (2) is easy and well–known: for normal M, $W(M) = \mathrm{conv}\{z_1,\ldots,z_N\}$.

The following observation is often useful.

PROPOSITION 3. *For every $M \in \mathbb{M}_N$, if $k \leqslant N$, C is a rank-k compression of M, and Q is a compression of rank $N-k+1$, then*

$$W(C) \cap W(Q) \neq \emptyset.$$

Proof. Let S and T be the subspaces corresponding to compressions C and Q. Since the dimensions add to more than N, S and T must intersect non–trivially; let u be a unit vector in $S \cap T$. Then

$$(Mu,u) = (Mu,P_S u) = (P_S Mu,u) = (Cu,u) \in W(C),$$

and similarly $(Mu,u) \in W(Q)$. $\square$

Applying this observation to the normal case, we see that part of the CKŻ conjecture is straightforward.

PROPOSITION 4. *If $M \in \mathbb{M}_N$ is normal with eigenvalues $z_1,\dots,z_N$, and the rank–k compression C is normal with eigenvalues $c_1,\dots,c_k$, then for every index set J having $\#(J) = N-k+1$*

$$conv\{c_1,\dots,c_k\} \cap conv\{z_j : j \in J\} \neq \emptyset.$$

In particular,

$$\Lambda_k(M) \subseteq \bigcap_{\#(J)=N-k+1} conv\{z_j : j \in J\}$$

(compare (2)).

Proof. We have noted that for normal (finite–dimensional) operators the numerical range is just the convex hull of the eigenvalues. Thus $W(C) = \text{conv}\{c_1,\dots,c_k\}$. On the other hand, let Q be the compression to the span of eigenvectors corresponding to $\{z_j : j \in J\}$; then Q is normal and $W(Q) = \text{conv}\{z_j : j \in J\}$. Apply Proposition 3. In particular, for points $\lambda \in \Lambda_k(M)$ we may let $c_1 = c_2 = \ldots = c_k = \lambda$. $\square$

On the other hand, the fact that $\Lambda_k(M)$ completely fills the RHS of (2) is more subtle, in general, although for certain combinations of N and k it is relatively easy to see. To illustrate this, and to introduce some of the methods of the present paper, consider the case $N = 5$, $k = 2$. In Figure 1 we see the eigenvalues $z_1,\dots,z_5$ of a normal (in fact, unitary) M as the outer points of the blue pentagram. It is easy to see that (2) implies that $\Lambda_2(M)$ is the inner pentagon. As far as we know, there is no simple proof that $\Lambda_k(M)$ fills this pentagon, but three markedly disparate arguments may be found in the literature:

(1) in [6] there is an argument based in part on topological concepts such as simple connectivity and winding number;

(2) as it is easy to conclude (see section 2) that the vertices of the inner pentagon are in $\Lambda_2(M)$, the fact that (whether or not M is normal) $\Lambda_k(M)$ is convex (see [7] and [26])) – a striking extension of the classical Toeplitz–Hausdorff Theorem for $W(M)$ – may be used;

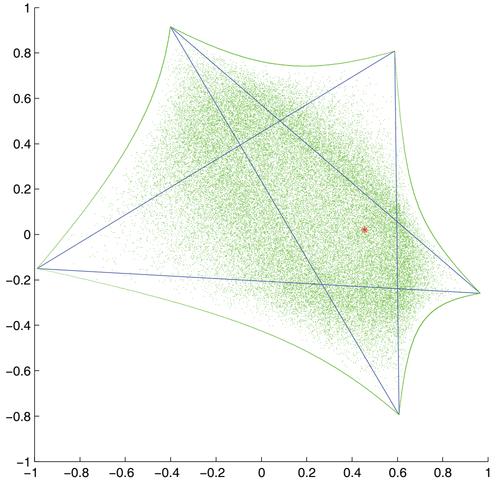

Figure 1: *Choosing a (red asterisk) at random in $\Lambda_2(M)$ (the inner pentagon), we see that $B(a)$ includes a "starfish" that covers $\Lambda_2(M)$ and more.*

(3) as we have noted, (2) is a direct consequence of the Li and Sze result Theorem 2.

A fourth, and quite different yet again, approach can be obtained by considering those eigenvalue pairs a, b that can belong to rank–2 normal compressions of M. Given $a \in \mathbb{C}$ we denote by $B(a)$ the set of b that match a in this sense. We shall prove in section 3 that for a in the inner pentagon $B(a)$ includes a "starfish" (outlined in green for the example of Figure 1) covering the (filled) pentagon (our conjecture, in addition, is that the starfish is precisely $B(a)$). Since $a \in B(a)$ says that $a \in \Lambda_2(M)$, we conclude once again that $\Lambda_2(M)$ fills the pentagon.

Plan of the paper: section 2 has some general results, section 3 treats the case $k = 2$, section 4 examines continuity of $B(\cdot)$, and section 5 discusses non–normal compressions.

NOTE. Since detailed proofs of our results are readily available online via the preprint [13], we'll often merely refer the interested reader to that resource.

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2. Some general results (arbitrary k, N)

Note that if C is a rank–k compression of $M \in \mathbb{M}_N$ and C' is a rank–k' compression of C, then C' is a rank–k' compression of M. Thus Proposition 3 has the following consequence.

PROPOSITION 5. *If C is a compression of $M \in \mathbb{M}_N$ then*

$$W(C) \subseteq W(M).$$

Proof. Regard $z \in W(C)$ as a rank–1 compression C' of C, hence of M and apply Proposition 3 with $k = 1$, C replaced by C' and $Q = M$. $\square$

Whereas Proposition 4 supplies a *necessary* condition on the eigenvalues $c_1, \ldots, c_k$ of a normal compression C of normal M, the following proposition points out a *sufficient* condition that is sometimes useful. An interesting analysis of such necessary vs sufficient conditions may be found in [23].

PROPOSITION 6. *If $M \in \mathbb{M}_N$ is normal with eigenvalues $z_1, \ldots, z_N$ then $c_1, \ldots, c_k \in \mathbb{C}$ are eigenvalues of a normal compression C of M provided that there exists a partition $J_1, \ldots, J_k$ of $\{1, 2, \ldots, N\}$ such that for each $i = 1, \ldots, k$*

$$c_i \in conv\{z_j : j \in J_i\}.$$

Proof. See proof of Proposition 6 in [13]. $\square$

In [4] Choi, Kribs, and Życzkowski identified explicitly the higher–rank numerical ranges of Hermitian matrices, and their argument may be viewed, along the lines of the proof of our next proposition, as an illustration of the combined force of the necessary condition from Proposition 4 with the sufficient condition from Proposition 6. Note that the result might also have been obtained as a special case of the Fan–Pall result, Theorem 1 (taking $b_1 = b_2 = \ldots = b_k$).

PROPOSITION 7. *If $M \in \mathbb{M}_N$ is Hermitian with (real) eigenvalues*

$$a_1 \leqslant a_2 \leqslant \ldots \leqslant a_N,$$

then for each $k \leqslant N/2$ we have

$$\Lambda_k(M) = [a_k, a_{N-k+1}].$$

If $a_{N-k+1} < a_k$, then $\Lambda_K(M) = \emptyset$.

Proof. See proof of Proposition 7 in [13]. $\square$

As another example of such general arguments we treat the normal compression problem for the case $k = N - 1$. This result goes back to Fan-Pall [10]; their proof is algebraic in character whereas ours is more geometric. We restrict to the case where the matrix and its compression have no common eigenvalues since this is where our general principles are most pertinent; Fan and Pall also treat the general case by means of a direct sum construction.

PROPOSITION 8. *Let $z_1,\ldots,z_N$ and $c_1,\ldots,c_{N-1}$ be two collections of complex numbers having no elements in common. Then there is a normal $M \in \mathbb{M}_N$ with eigenvalues z_j having a rank–$(N-1)$ normal compression C with eigenvalues c_j iff the z_j are collinear and alternate with the c_j (in some order) along the common line.*

Proof. Let us first show that if such M,C exist then the z_j must be collinear. Label the z_j lying on the boundary of $W(M)$ in counterclockwise order: $z_1,\ldots,z_p$. If the z_j are not collinear there must be some z_{k-1},z_k,z_{k+1} that are not collinear, as in Figure 2. Proposition 4 requires that $[z_{k-1},z_k]$ meets $W(C)$ at some λ closest to z_k; this λ is extreme in $W(C)$ and so must be an eigenvalue of C. Similarly we have an eigenvalue μ of C in $[z_k,z_{k+1}]$, as in Figure 2. Note that Proposition 4 also tells us that z_k cannot be a repeated eigenvalue of M, since it would then coincide with an eigenvalue of C.

Let $u_1,\ldots,u_N$ be an orthonormal set of eigenvectors of M, with $Mu_j = z_j u_j$, and let orthonormal v,w be eigenvectors of C with $Cv = \lambda v$ and $Cw = \mu w$. Expand v,w in terms of the u_j:

$$v = \sum_{j=1}^{N} a_j u_j, \quad w = \sum_{j=1}^{N} b_j u_j;$$

then

$$\lambda = (Cv,v) = (Mv,v) = \sum_{j=1}^{N} |a_j|^2 z_j,$$

so that $a_j = 0$ unless z_j lies on the line through z_{k-1},z_k. Similarly $b_j = 0$ unless z_j lies on the line through z_k,z_{k+1}. Since z_k is the only common point,

$$0 = (v,w) = a_k \overline{b_k}.$$

If $a_k = 0$ we have $\lambda = z_{k-1}$, which we have ruled out, while if $b_k = 0$ we have $\mu = z_{k+1}$, also ruled out.

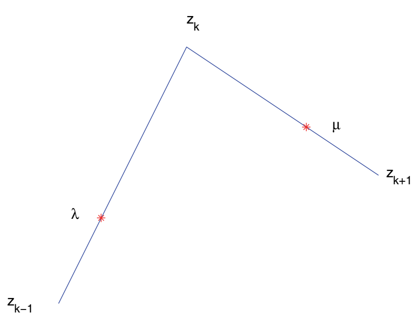

Figure 2: *An example of the eigenvalue geometry ruled out in the proof of Proposition 8.*

Thus the eigenvalues all lie on a common line and by an affine map $M \to \alpha I_N + \beta M$ this common line can be $\mathbb{R}$, ie we are in the Hermitian case. Proposition 1 then completes the argument, giving the interlacing property.

On the other hand, if the collinearity and interlacing conditions are met, the same sort of affine map and Proposition 1 establish the existence of M and C. □

3. Results for $k = 2$ and small N

For 2×2 normal compressions $\mathrm{diag}(a,b)$, we can give a more detailed account of the ab–geometry, leading up to an understanding of the “starfish” seen in Figure 1.

Recall that, given normal $M \in \mathbb{M}_N$ and complex a, we denote by $B(a)$ the set of complex b such that $\mathrm{diag}(a,b)$ is a compression of M. Of course, in order that $B(a)$ should be nonempty we must have

$$a \in \mathrm{conv}\{z_1, z_2, \ldots, z_N\},$$

where the z_j are the eigenvalues of M. Note that Proposition 4 also requires that for $b \in B(a)$ we require that the line segment $[a,b]$ intersect

$$\mathrm{conv}\{z_j : j \neq i\}$$

for each $i = 1,\ldots,N$.

The simplest case to consider: $N = 3$ and the eigenvalues of M form a nontrivial triangle.

PROPOSITION 9. *Suppose that the eigenvalues z_1, z_2, z_3 of a normal $M \in \mathbb{M}_3$ are not collinear. Then $b \in B(a)$ iff either a is one of these eigenvalues, say $a = z_1$ and $b \in [z_2, z_3]$ (the opposite side of the triangle formed by z_1, z_2, z_3) or a is in one of the sides, say $[z_2, z_3]$, and $b = z_1$.*

Proof. Since $[a,b]$ must meet each of the triangle’s sides, the necessity of the condition is clear. On the other hand, Proposition 6 shows that these conditions suffice for a,b to be the eigenvalues of a normal compression. □

REMARK. Here we have a very simple case of the result of Fan and Pall [10] where they characterize in general the case $k = N-1$.

When $N = 4$ we encounter more complex behaviour, such as that seen in Figure 3, where $B(a)$ is a curve interior to $\mathrm{conv}\{z_1, z_2, z_3, z_4\}$ (except for endpoints).

To analyse such behaviour, it will be convenient to assume in what follows that the eigenvalues of M are generic in the sense that no three are collinear. We may also assume that $M = \mathrm{diag}(z_1,\ldots,z_N)$, so that the eigenvectors of M are the standard basis vectors e_j.

Note that if $b \in B(a)$ we have orthonormal u, w such that

$$(Mu,u) = a, (Mw,w) = b, \text{ and } (Mu,w) = (Mw,u) = 0.$$

Thus $a = \sum_1^N |u_j|^2 z_j$, a convex combination. Let Δ_N denote the N–dimensional simplex, ie $\mathrm{conv}\{e_1,\ldots,e_N\}$; then $|u|^2$ (where the operations are performed componentwise) belongs to

$$C(a) = \{t \in \Delta_N : a = \sum_1^N t_j z_j\}.$$

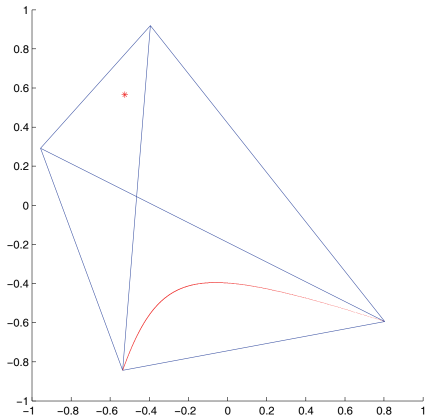

Figure 3: *For a (red asterisk) strictly inside the upper quadrant (case (a)), we see that $B(a)$ is a curve in the opposite quadrant.*

By exchanging complex arguments between the components of u and w we may assume that $u \geqslant 0$; then the possible u lie in $\{\sqrt{t} : t \in C(a)\}$. The conditions on $w \in \mathbb{C}^N$ are then given by

$$\|w\| = 1, w \perp u, w \perp z \circ u, \text{ and } w \perp \overline{z} \circ u,$$

where $\circ$ indicates Schur (componentwise) multiplication, so that

$$z \circ u = (z_1 u_1, \ldots, z_N u_N)',$$

with $'$ indicating transpose.

We may thus describe $B(a)$ as follows.

PROPOSITION 10. *Given $a \in W(M) (= conv\{z_1, \ldots, z_N\})$,*

$$B(a) = \bigcup_{t \in C(a)} B(a,t),$$

where

$$B(a,t) = \{\sum_1^N |w_j|^2 z_j : \|w\| = 1, w \perp \sqrt{t}, z \circ \sqrt{t}, \overline{z} \circ \sqrt{t}\}.$$

Proof. To the discussion above we need only add the observation that

$$b = (Mw, w) = \sum_1^N |w_j|^2 z_j. \quad \square$$

Clearly $C(a)$ is a compact convex subset of Δ_N. It is therefore the convex hull of its extreme points, which are identified in the following result.

PROPOSITION 11. *The extreme points of $C(a)$ are those $t \in C(a)$ such that at most three $t_k > 0$.*

Proof. See proof of Proposition 11 in [13]. $\square$

For distinct indices i,j,l, let $t(i,j,l)$ denote the element of $C(a)$ (if it exists) such that $t_k(i,j,l) = 0$ whenever $k \neq i,j,l$. Note that such elements are uniquely determined since

$$a = t_i(i,j,l)z_i + t_j(i,j,l)z_j + t_l(i,j,l)z_l$$

represents a uniquely as a point in the triangle $\operatorname{conv}\{z_i, z_j, z_l\}$; here again we use the assumption that no three of the eigenvalues z_j are collinear. Thus

$$C(a) = \operatorname{conv}\{t(i,j,l) : i,j,l \text{ are distinct and } a \in \operatorname{conv}\{z_i,z_j,z_l\}\}. \tag{4}$$

The complexity of $B(a,t)$ increases with the number of nonzero t_k. For example, if only one $t_k > 0$, then $t_k = 1$ and $a = z_k$. Here the simple sufficient condition of Proposition 6 is also necessary:

$$B(a,t) = \operatorname{conv}\{z_j : j \neq k\}.$$

We see this as follows. Evidently, with $u = \sqrt{t} = e_k$, u,w are orthonormal exactly when $w = \sum_{j\neq k} \alpha_j e_j$ with $\sum_{j\neq k} |\alpha_j|^2 = 1$; then

$$b = (Nw,w) = \sum_{j\neq k} |\alpha_j|^2 z_j \in \operatorname{conv}\{z_j : j \neq k\},$$

and any $b \in \operatorname{conv}\{z_j : j \neq k\}$ can be obtained in this way.

The same sort of simplification occurs if only two or three $t_k > 0$.

PROPOSITION 12. *(a) If $t \in C(a)$ has exactly two positive components, say $t_1,t_2 > 0$, then*

$$B(a,t) = conv\{z_j : j > 2\}.$$

(b) If $t \in C(a)$ has exactly three positive components, say $t_1,t_2,t_3 > 0$, then

$$B(a,t) = conv\{z_j : j > 3\}.$$

Proof. See proof of Proposition 12 in [13]. $\square$

We are now in a position to understand the features of Figure 3 and, indeed, to analyse all the possibilities when $N = 4$. We treat in detail the case where z_1,z_2,z_3,z_4 are all extreme in $\operatorname{conv}\{z_1,z_2,z_3,z_4\}$; the case where one of the eigenvalues lies in the interior of $W(M)$ (eg $z_4 \in \operatorname{conv}\{z_1,z_2,z_3\}$) can be treated similarly.

PROPOSITION 13. *Let $N = 4$ and suppose that z_1,z_2,z_3,z_4 are all extreme in $W(M)$ and are numbered in counterclockwise order. The diagonals $[z_1,z_3]$ and $[z_2,z_4]$ meet at q and divide $W(M)$ into four quadrants. Consider $a \in W(M)$; the possibilities for $B(a)$ are as follows.*

(a) See figure 3: a lies in the interior of one of the quadrants. For convenience, assume that $a \in conv\{z_1, z_2, q\}$; let $x = t(1,2,3)$, $y = t(1,2,4)$. Then $B(a)$ is the curve traced out by the function $b(r)$ defined for $0 < r < 1$ by

$$b(r) = \sum_{k=1}^{4} \frac{(x_k - y_k)^2}{(1-r)x_k + ry_k} z_k \Big/ \sum_{k=1}^{4} \frac{(x_k - y_k)^2}{(1-r)x_k + ry_k}.$$

Note that $x_4 = 0$ and $y_3 = 0$ so that

$$\lim_{r\to 0} b(r) = z_4, \quad \lim_{r\to 1} b(r) = z_3,$$

and we obtain a continuous curve parametrized on $[0,1]$ when we interpret $b(0)$ as z_4 and $b(1)$ as z_3. Except for these endpoints, the curve lies in the interior of the opposite quadrant $conv\{z_3, z_4, q\}$.

(b) If a lies in the interior of one of the sides of $W(M)$ then $B(a)$ is the opposite side (eg if a is inside $[z_1, z_2]$ then $B(a) = [z_3, z_4]$). If $a = z_k$ then $B(a)$ is the opposite triangle $conv\{z_j : j \neq k\}$.

(c) See Figure 4: a lies interior to the diagonals but is not q; say a is interior to $[z_1, q]$. Then $B(a)$ is the T–shaped object $[z_2, z_4] \cup [q, z_3]$.

(d) If $a = q$ then $B(a)$ is the union of the two diagonals.

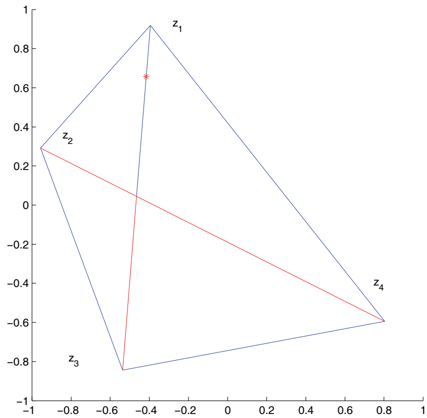

Figure 4: *For a (red asterisk) strictly inside the segment $[z_1, q]$ (case (c)), we see that $B(a)$ is the T–shaped object consisting of $[z_2, z_4] \cup [q, z_3]$.*

Proof. See proof of Proposition 13 in [13]. □

We now have the tools to continue the theme of Proposition 12, treating the case when exactly *four* of the components of $t \in C(a)$ are positive.

PROPOSITION 14. *Suppose that $N > 4$ and that $t \in C(a)$ has exactly four positive components; for convenience, assume that $t_1, t_2, t_3, t_4 > 0$ and that a lies in the upper* *quadrant relative to* $Q = conv\{z_1, z_2, z_3, z_4\}$*, ie* a *is interior to* $conv\{z_1, z_2, q\}$ *(see Figure 3, with the understanding that it is now intended to show only the relation of* a *to* z_1, z_2, z_3, z_4*, and Proposition 13). Let* β *be the curve traced out by* $b(\cdot)$ *of Proposition 13(a) (and shown in Figure 3). Then*

$$B(a,t) = conv\{\beta, z_5, z_6, \ldots, z_N\}.$$

Proof. See proof of Proposition 14 in [13]. □

Proposition 14 allows us to understand, in large part, the phenomenon illustrated in Figure 1. Let $N = 5$ and suppose that each eigenvalue z_k is an extreme point of $W(M) = \mathrm{conv}\{z_1, \ldots, z_5\}$ (eg whenever M is unitary). For convenience, label the z_k in counterclockwise order. Suppose that a lies strictly inside the central pentagon, ie $\Omega_2(M)$ as defined in (3). For each k let β_k denote the curve obtained as in Proposition 14 by regarding a as an element of the quadrilateral $Q_k = \mathrm{conv}\{z_j : j \neq k\}$. Note that β_k connects z_{k+2} and z_{k+3} (numbering modulo 5) and lies in the quadrant of Q_k opposite to the one containing a. We claim that (as illustrated in Figure 1) $B(a)$ includes the whole "starfish" region bounded by $\beta_1, \beta_2, \ldots, \beta_5$.

To see this note that the starfish is the union of the wedges $W_k = \mathrm{conv}\{\beta_k, z_k\}$, so it suffices to show that each $W_k \subseteq B(a)$. Since $a \in Q_k$ there is $t \in C(a)$ such that $t_k = 0$. Then Proposition 14 tells us that $B(a,c) = W_k$.

Figure 1 was obtained by first computing $C(a)$ via the relation (4) as

$$\mathrm{conv}\{t(k, k+2, k+3) : k = 1, 2, \ldots, 5\}$$

(note that for a in the inner pentagon, the only eigenvalue triangles containing a correspond to the triples z_k, z_{k+2}, z_{k+3}). To generate each of the thousands of b's in $B(a)$, plotted as green points in Figure 1, our MATLAB program first chose a "random" point $t \in C(a)$ (ie a random convex combination of the five $c(k, k+2, k+3)$), put $u = \sqrt{t}$, then computed $b = (Nw, w)$ where w was chosen "randomly" in

$$\mathbb{C}^5 \ominus \mathrm{span}\{u, u \circ \mathrm{Re}(z), u \circ \mathrm{Im}(z)\}$$

(and normalized so that $\|w\| = 1$). The curves β_k were added using the formula of Proposition 13(a). Such simulations strongly suggest the following "starfish conjecture", since no green dots fall outside the starfish: in such a situation (and in particular when $N = 5$ and M is unitary), $B(a)$ not only contains the starfish but is equal to it.

We have seen in the discussion of Figure 1 that for $N = 5$ and $a, b \in \Omega_2(M)$ we always have a, b as eigenvalues of a normal compression of M. The following proposition points out that this is true for any N – and that $N = 5$ is, in fact, the only subtle case.

PROPOSITION 15. *Let* M *be normal in* $\mathbb{M}_N$ *and such that the eigenvalues* $z_1, \ldots, z_N$ *are distinct and each is an extreme point of* $W(M)$ *(eg* M *unitary). Then* $a, b \in \Omega_2(M)$ *implies that* $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ *is a compression of* M*.*

Proof. For $N \leqslant 3$, $\Omega_2(M) = \emptyset$. For even $N \geqslant 4$, the relation (3) tells us that $\Omega_2(M)$ is the "inner N–gon" cut off by the line segments $[z_j, z_{j+2}]$ (indexing modulo N). Thus for even $N \geqslant 4$

$$\Omega_2(M) = \mathrm{conv}\{z_j : j \text{ odd}\} \cap \mathrm{conv}\{z_j : j \text{ even}\},$$

and Proposition 6 suffices. For $N = 5$ the "starfish" discussion proves our assertion. For odd $N \geqslant 7$ we see that $\mathrm{conv}\{z_j : j \text{ odd}\} \supseteq \Omega_2(M)$ and $\mathrm{conv}\{z_j : j \text{ even}\}$ covers all of $\Omega_2(M)$ except that part lying in $Q = \mathrm{conv}\{z_1, z_2, z_{N-1}, z_N\}$. Hence Proposition 6 suffices for $a \notin Q, b \in \Omega_2(M)$. The same argument applies for $a \notin \tilde{Q} = \mathrm{conv}\{z_2.z_3, z_4, z_5\}$ and because $N > 5$ this covers any $a \in Q$. □

4. Continuity of $B(\cdot)$

A natural assertion of "continuity" for $B(\cdot)$ might be that $d_H(B(a'), B(a)) \to 0$ as $a' \to a$, where $d_H(X,Y)$ is the Hausdorff distance between compact nonempty sets $X, Y \subset \mathbb{C}$. Recall that

$$d_H(X,Y) = \max\{\hat{d}_H(X,Y), \hat{d}_H(Y,X)\},$$

where

$$\hat{d}_H(X,Y) = \max_{x \in X}(\min_{y \in Y} |x - y|).$$

However, we have seen simple examples where this fails: recall the analysis of $B(a)$ for various $a \in \mathrm{conv}\{z_1, z_2, z_3, z_4\}$ that was provided by Proposition 13. If a' lies in the interior of $[z_1, z_2]$ and $a' \to a = z_1$, then $B(a') = [z_3, z_4]$ "jumps" to $B(a) = \mathrm{conv}\{z_2, z_3, z_4\}$. A perhaps more surprising example: let a be interior to $[z_1, q]$ as in Figure 4; for a' approaching a from the interior of $\mathrm{conv}\{z_1, z_2, q\}$ we see $B(a')$ as a curve joining z_3 and z_4 in $\mathrm{conv}\{z_3, z_4, q\}$, whereas for a' approaching a from the interior of $\mathrm{conv}\{z_1, z_4, q\}$ we see $B(a')$ as a curve joining z_2 and z_3 in $\mathrm{conv}\{z_2, z_3, q\}$.

In spite of such "failures" we'll show that $B(\cdot)$ is continuous with respect to Hausdorff distance at most points of $W(M)$ and enjoys a "one–sided" Hausdorff continuity in general.

Our standard set–up for this discussion is as in section 3, ie we assume M is normal in $\mathbb{M}_N$ and is in diagonal form: $M = \mathrm{diag}(z)$, where no three eigenvalues are collinear. Thus $W(M) = \mathrm{conv}\{z_1, \ldots, z_N\}$ and $B(a') = \emptyset$ if $a' \notin W(M)$. Seeking continuity, we restrict attention to $a' \to a$ with $a', a \in W(M)$. Note that if $N = 3$ and a' is interior to $W(M) = \mathrm{conv}\{z_1, z_2, z_3\}$, we again have $B(a') = \emptyset$, since $b \in B(a')$ and Proposition 4 would require that $[a', b]$ meet each side of the triangle $W(M)$. We therefore restrict also to cases where $N \geqslant 4$.

PROPOSITION 16. *If $N \geqslant 4$, $B(a)$ is a compact nonempty set for any $a \in W(M)$.*

Proof. See proof of Proposition 16 in [13]. □

A related argument shows that, in general, $B(\cdot)$ is continuous in a one–sided Hausdorff sense.

PROPOSITION 17. *If $a, a_n \in W(M)$ and $a_n \to a$, then*

$$\hat{d}_H(B(a_n), B(a)) \to_n 0. \tag{5}$$

Proof. See proof of Proposition 17 in [13]. □

In terms of the obvious extension of Hausdorff distance to compact nonempty subsets of Δ_N, we note that $C(\cdot)$ is continuous and in fact satisfies a Lipschitz condition for each fixed M.

PROPOSITION 18. *There is a constant $K < \infty$ depending only on M such that for all $a, a' \in W(M)$*

$$d_H(C(a), C(a')) \leqslant K|a - a'|.$$

Proof. See proof of Proposition 18 in [13]. □

Next we show that $B(\cdot)$ is d_H–continuous at any point that is "off the grid", and that continuity is uniform if we stay bounded away from the grid.

PROPOSITION 19. *If $a \in W(M)$ but a does not lie on any line segment $[z_i, z_j]$, then $a' \to a$ implies that*

$$d_H(B(a'), B(a)) \to 0.$$

In fact, on any subset $S(d) \subset W(M)$ that is a positive distance d from the grid

$$G = \bigcup \{[z_i, z_j] : i, j = 1, \ldots, N\},$$

so that

$$S(d) = \{a \in W(M) : \min_{g \in G} |a - g| \geqslant d\},$$

the map $a \mapsto B(a)$ is uniformly continuous.

Proof. See proof of Proposition 19 in [13]. □

Note that *sometimes* $B(\cdot)$ is continuous even at points that are on the grid. For example, from Proposition 13(a) and 13(b) we can see that there is continuity everywhere on the boundary segments $[z_i, z_{i+1}]$ except at the endpoints.

5. Related results

We offer some remarks on the apparently more difficult problem of characterizing *arbitrary* compressions of a normal matrix M. Suppose again that M is $N \times N$, and is represented by the diagonal matrix $\mathrm{diag}(z)$ and that X is a rank–k compression of M, ie there is a k–dimensional subspace S such that $X = P_S M|_S$. From Proposition 3 we obtain a *necessary* condition on X: the (classical) numerical range $W(X)$ of X must intersect the convex hull of any subset of the eigenvalues z_j having size $N - k + 1$.

When $k = 2$, ie X is represented by a 2×2 matrix, the numerical range $W(X)$ determines X uniquely as an operator. Indeed, $W(X)$ is a (filled–in) ellipse in this case with the eigenvalues of X as foci and the length of the minor axis is the modulus of the off–diagonal entry of any upper–triangular matrix for X. Let's consider the problem of characterizing such compressions X geometrically via the elliptical $W(X)$ in the cases where $N = 3$ and $N = 4$.

When $N = 3$, the necessary condition of above tells us that $W(X)$ must be tangent to each of the three sides of $\operatorname{conv}\{z_1, z_2, z_3\}$ (recall that Proposition 5 tells us that in general we must have $W(X) \subseteq W(M) = \operatorname{conv}\{z_j : j = 1, \ldots, n\}$). In fact, Williams showed long ago that the necessary condition is also sufficient when $N = 3$ (see [25]).

When $N = 4$ we consider the case where the eigenvalues z_j form a quadrilateral Q. The necessary condition above tells us that $W(X)$ must intersect each of the four triangles $T_i = \operatorname{conv}\{z_j : j \neq i\}$. Thus $W(X)$ must intersect each of the quadrants $T_i \cap T_k$. This phenomenon is borne out by numerical experiments such as Figure 5 illustrates, but it is not clear what additional conditions must be satisfied by $W(X)$, even in this $N = 4$ case. Of course, if by chance $W(X)$ is tangent to all three sides of some T_i, then Williams' result tells us that X is indeed a 2–dimensional compression.

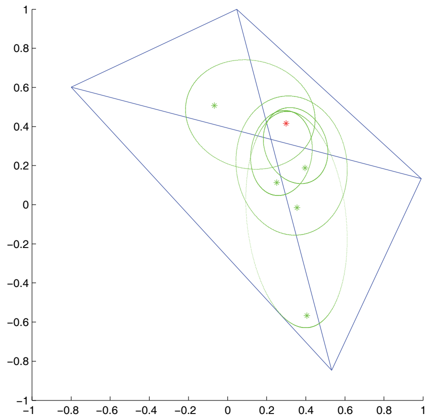

Figure 5: *Shows the (elliptical) boundaries of the numerical ranges of several (nonnormal) compressions of a* 4×4 *normal* M*, each compression having a (red asterisk) as an eigenvalue (therefore seen as one of the foci of each ellipse)*

REFERENCES

[1] W. ARVESON, *Unitary invariants for compact operators*, Bull, Amer. Math. Soc. 76, 88–91, 1970.

[2] K. AUDENAERT AND F. KITTANEH, *Problems and conjectures in matrix and operator inequalities*, arXiv:1201.5232v2 [math.FA], 2012.

[3] A. L. CAUCHY, *Sur l'équation à l'aide de laquelle on détermine les inégalités séculaires des mouvements des planètes*, Oeuvres complètes, Second Ser., IX, 174–195.

[4] M.-D. CHOI, D. W. KRIBS, AND K. ŻYCZKOWSKI, *Higher–rank numerical ranges and compression problems*, Linear Algebra Appl. 418, 828–839, 2006.

[5] M.-D. CHOI, D. W. KRIBS, AND K. ŻYCZKOWSKI, *Quantum error correcting codes from the compression formalism*, Rep. Math. Phys. 58, 77–91, 2006.

[6] M.-D. CHOI, J. A. HOLBROOK, D. W. KRIBS, AND K. ŻYCZKOWSKI, *Higher–rank numerical ranges of unitary and normal matrices*, Operators and Matrices 1, 409–426, 2007.

[7] M.-D. CHOI, M. GIESINGER, J. A. HOLBROOK, AND D. W. KRIBS, *Geometry of higher–rank numerical ranges*, Linear and Multilinear Algebra 56, 53–64, 2008.

[8] W. E. CHUM, *The unitary equivalence of compact operators*, Glasgow Math. J. 26, 145–149, 1985.

[9] C. F. DUNKL, P. GAWRON, J. HOLBROOK, Z. PUCHALA, AND K. ŻYCZKOWSKI, *Numerical shadows: measures and densities on the numerical range*, Linear Algebra Appl. 434, 2042–2080, 2011.

[10] K. FAN AND G. PALL, *Imbedding conditions for Hermitian and normal matrices*, Canadian J. Math. 9, 298–304, 1957.

[11] D. R. FARENICK, *Normal compressions and restrictions to 2–dimensional subspaces, and the Berberian spectral extension, Special Issue: The numerical range and numerical radius*, Linear and Multilinear Algebra 37, #1–3, 131–138, 1994.

[12] P. GAWRON, Z. PUCHALA, J. MISZCZAK, L. SKOWRONEK, AND K. ŻYCZKOWSKI, *Restricted numerical range: a versatile tool in the theory of quantum information*, J. Math. Physics 51, 2010.

[13] J. A. HOLBROOK, N. MUDALIGE, AND R. PEREIRA, *Normal matrix compressions*, arXiv:1203.3908v2 [math.FA], 2012.

[14] D. W. KRIBS, A. PASIEKA, M. LAFOREST, C. RYAN, AND M. P. DA SILVA, *Research problems on numerical ranges in quantum computing*, Linear and Multilinear Algebra 57, 491–502, 2009.

[15] C.-K. LI AND Y.-T. POON, *Generalized numerical ranges and quantum error correction*, J. Operator Theory 66, 335–351, 2011.

[16] C.-K. LI, Y.-T. POON, AND N.-S. SZE, *Higher rank numerical ranges and low rank perturbations of quantum channels*, J. Math. Analysis Appl. 348, 843–855, 2008.

[17] C.-K. LI, Y.-T. POON, AND N.-S. SZE, *Condition for the higher rank numerical range to be non–empty*, Linear and Multilinear Algebra 57, 365–368, 2009.

[18] C.-K. LI AND N.-S. SZE, *Canonical forms, higher–rank numerical ranges, totally isotropic subspaces, and matrix equations*, Proc. Amer. Math. Soc. 136, 3013–3023, 2008.

[19] C.-K. LI AND N.-K. TSING, *On the kth matrix numerical range*, Linear and Multilinear Algebra 28, #4, 229–239, 1991.

[20] K. MAJGIER, H. MAASSEN, AND K. ŻYCZKOWSKI, *Protected subspaces in quantum information*, Quantum Information Processing 9, 343–367, 2010.

[21] N. MUDALIGE, *Higher Rank Numerical Ranges of Normal Operators*, MSc thesis, U of Guelph, 2010.

[22] F. M. POLLACK, *Properties of the matrix range of an operator*, Indiana Univ. Math. J. 22, 419–427, 1972/73.

[23] J. F. QUEIRÓ AND A. L. DUARTE, *Imbedding conditions for normal matrices*, Linear Algebra Appl. 430, 1806–1811, 2009.

[24] R. C. THOMPSON, *Research problem: The matrix numerical range*, Linear and Multilinear Algebra 21, 321–323, 1987.

[25] J. P. WILLIAMS, *On compressions of matrices*, J. London Math. Soc. (2) 3, 526–530, 1971.

[26] H. WOERDEMAN, *The higher–rank numerical range is convex*, Linear and Multilinear Algebra 56, 65–67, 2008.

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