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MATRICES WITH DEFECT INDEX ONE

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Abstract. In this paper, we give some characterizations of matrices which have defect index one. Recall that an n-by-n matrix A is said to be of class $\mathscr{S}_n$ (resp., $\mathscr{S}_n^{-1}$) if its eigenvalues are all in the open unit disc (resp., in the complement of closed unit disc) and rank $(I_n - A^*A) = 1$. We show that an n-by-n matrix A is of defect index one if and only if A is unitarily equivalent to $U \oplus C$, where U is a k-by-k unitary matrix, $0 \leqslant k < n$, and C is either of class $\mathscr{S}_{n-k}$ or of class $\mathscr{S}_{n-k}^{-1}$. We also give a complete characterization of polar decompositions, norms and defect indices of powers of $\mathscr{S}_n^{-1}$-matrices. Finally, we consider the numerical ranges of $\mathscr{S}_n^{-1}$-matrices and $\mathscr{S}_n$-matrices, and give a generalization of a result of Chien and Nakazato on tridiagonal matrices (cf. [3, Theorem 7]).

1. Introduction

Let M_n be the algebra of n-by-n complex matrices and $A \in M_n$. The *defect index* d_A of A is, by definition, rank $(I_n - A^*A)$, that is, the dimension of the range of $I_n - A^*A$. It is a way to measure how far A is from the unitary matrices. In this paper, we give some characterizations of matrices which have defect index one.

Recall that a matrix $A \in M_n$ is said to be of *class* $\mathscr{S}_n$ if its eigenvalues are all in the open unit disc $\mathbb{D}(\equiv \{z \in \mathbb{C} : |z| < 1\})$ and $d_A = 1$. The n-by-n *Jordan block*

$$J_n = \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}$$

is one example. Such operators and their infinite-dimensional analogues $S(\phi)$ (ϕ an inner function) were first studied by Sarason [16]. They play the role of the building blocks of the Jordan model for C_0 contractions [1, 15]. In particular, if an $\mathscr{S}_n$-matrix A is invertible, then

$$d_{A^{-1}} = \operatorname{rank}(I_n - (A^{-1})^*(A^{-1})) = \operatorname{rank}((A^{-1})^*(A^*A - I_n)(A^{-1})) = 1,$$

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and all eigenvalues of A^{-1} are in $\mathbb{C}\setminus\overline{\mathbb{D}}$, the complement of the closed unit disc. Therefore, we recall that a matrix $A \in M_n$ is said to be of *class* $\mathcal{S}_n^{-1}$ if its eigenvalues are all in $\mathbb{C}\setminus\overline{\mathbb{D}}$ and $d_A = 1$. It is easily seen that if A is in $\mathcal{S}_n^{-1}$ (resp., $\mathcal{S}_n$), then A^* and $e^{i\theta}A$ are also in $\mathcal{S}_n^{-1}$ (resp., $\mathcal{S}_n$). Moreover, if A is in $\mathcal{S}_n^{-1}$, then A has no unitary part, A is invertible, and A^{-1} is in $\mathcal{S}_n$.

In Section 2, we first give a complete characterization of matrices which have defect index one. We show that a matrix $A \in M_n$ is of defect index one if and only if A is unitarily equivalent to $U \oplus C$, where $U \in M_k$, $0 \leqslant k < n$, is unitary, and C is either in $\mathcal{S}_{n-k}$ or in $\mathcal{S}_{n-k}^{-1}$. In recent years, properties of $\mathcal{S}_n$-matrices have been intensely studied (cf. [5, 6, 8, 9, 13, 14, 17]). Therefore, we will restrict our attention to $\mathcal{S}_n^{-1}$-matrices in the rest of this section. We will give a complete characterization of polar decompositions, norms and defect indices of powers of $\mathcal{S}_n^{-1}$-matrices.

In Section 3, we take up the numerical ranges of $\mathcal{S}_n^{-1}$-matrices and $\mathcal{S}_n$-matrices. From Proposition 2.4 (e), an $\mathcal{S}_n^{-1}$-matrix A is unitarily equivalent to a polar decomposition UD_t, where U is unitary and $D_t = \operatorname{diag}(t, 1, \cdots, 1)$ for some $t > 1$. We show that if $0 \in W(U)$, then $W(UD_{t_1}) \subseteq W(UD_{t_2})$ for $1 \leqslant t_1 \leqslant t_2$. Among other things, recall that an operator A in $\mathcal{S}_n$ always has the matrix representation $[f_1 \cdots f_n]$ so that $\|f_j\| = 1$ $(1 \leqslant j \leqslant n-1)$, $\|f_n\| < 1$ and $f_i \perp f_j$ $(1 \leqslant i \neq j \leqslant n)$. We show that if $B = [f_1 \cdots f_{n-1}]$, then the numerical range of the 2-by-2 block matrix

$$\begin{bmatrix} 0 & I_n' + B \\ -I_n'^* + B^* & 0 \end{bmatrix} \in M_{2n-1}$$

is the convex hull of two ellipses, where I_n' is the n-by-$(n-1)$ submatrix of I_n obtained by deleting its last column. This generalizes a result of Chien and Nakazato on tridiagonal matrices (cf. [3, Theorem 7]).

2. Defect indices of powers, polar decompositions and norms

We start by giving a complete characterization of matrices which have defect index one. For abbreviation, the notation $A \cong B$ means that A is unitarily equivalent to B for any $A, B \in M_n$.

THEOREM 2.1. *Let A be an n-by-n matrix. Then $d_A = 1$ if and only if A is unitarily equivalent to $U \oplus C$, where $U \in M_k$, $0 \leqslant k < n$, is unitary, and C is either in $\mathcal{S}_{n-k}$ or in $\mathcal{S}_{n-k}^{-1}$.*

The proof depends on the following lemma.

LEMMA 2.2. *Let A be an n-by-n matrix with $d_A \leqslant 1$.*

(a) If $A = \begin{bmatrix} A' & B \\ 0 & C \end{bmatrix}$, where $A' \in M_k$, $1 \leqslant k \leqslant n$, then $d_{A'} \leqslant 1$ and $d_C \leqslant 1$.

(b) If $n = 2$ and $A = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}$, then $|a_{12}|^2 = (1-|a_{11}|^2)(1-|a_{22}|^2)$.

(c) If $A = \begin{bmatrix} U & B \\ 0 & C \end{bmatrix}$, where $U \in M_k$, $1 \leqslant k < n$, is unitary, then $B = 0$.

(d) If $A = [t_{ij}]_{i,j=1}^n$ is an upper triangular matrix with $|t_{ii}| = 1$ for all i, $1 \leqslant i \leqslant n$, then $t_{ij} = 0$ for all $i \neq j$.

Proof. (a) It is easily seen that

$$I_n - A^*A = \begin{bmatrix} I_k - A'^*A' & * \\ * & * \end{bmatrix} \quad \text{and} \quad I_n - AA^* = \begin{bmatrix} * & * \\ * & I_{n-k} - CC^* \end{bmatrix}.$$

Hence $d_{A'} = \operatorname{rank}(I_k - A'^*A') \leqslant \operatorname{rank}(I_n - A^*A) = d_A \leqslant 1$ and $d_C = d_{C^*} \leqslant d_{A^*} = d_A \leqslant 1$ as asserted.

(b) A simple computation shows that

$$I_2 - A^*A = \begin{bmatrix} 1 - |a_{11}|^2 & \overline{a_{11}}a_{12} \\ a_{11}\overline{a_{12}} & 1 - (|a_{12}|^2 + |a_{22}|^2) \end{bmatrix}.$$

Since $d_A \leqslant 1$, $I_2 - A^*A$ is not invertible. Thus

$$0 = \det(I_2 - A^*A) = (1 - |a_{11}|^2)(1 - |a_{22}|^2) - |a_{12}|^2.$$

Hence $|a_{12}|^2 = (1 - |a_{11}|^2)(1 - |a_{22}|^2)$ as asserted.

(c) Note that

$$I_n - A^*A = \begin{bmatrix} 0 & -U^*B \\ -B^*U & I_{n-k} - (B^*B + C^*C) \end{bmatrix}.$$

Since $\operatorname{rank}(I_n - A^*A) = d_A \leqslant 1$, it implies that every column of $I_n - A^*A$ is a scalar multiple of the first column of $I_n - A^*A$. Hence we conclude that $U^*B = 0$ or $B = 0$, since U is unitary.

(d) Let $A_k = [t_{ij}]_{i,j=1}^k$ for $k = 1, \cdots, n$. From (a), we have $d_{A_k} \leqslant 1$ for all k. Since $d_{A_2} \leqslant 1$, by (b), we obtain that $t_{12} = 0$. Thus $A_2 = \begin{bmatrix} t_{11} & 0 \\ 0 & t_{22} \end{bmatrix}$ is unitary. Since $d_{A_3} \leqslant 1$ and A_2 is unitary, by (c), we deduce that $A_3 = \operatorname{diag}(t_{11}, t_{22}, t_{33})$ is unitary. Repeating this argument gives us $A = A_n = \operatorname{diag}(t_{11}, \cdots, t_{nn})$. This completes the proof. $\square$

Proof of Theorem 2.1. Assume that $d_A = 1$. Let $\sigma(A) = \{\lambda_1, \cdots, \lambda_n\}$. We want to show that either $|\lambda_j| \leqslant 1$ for all j, or $|\lambda_j| \geqslant 1$ for all j. Indeed, if there exist $|\lambda_{i_0}| > 1$ and $|\lambda_{j_0}| < 1$, then A is unitarily equivalent to an upper triangular matrix $[a_{ij}]_{i,j=1}^n$ such that $a_{11} = \lambda_{i_0}$, $a_{22} = \lambda_{j_0}$ and $a_{ij} = 0$ for all $1 \leqslant j < i \leqslant n$. Let $A_2 = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}$ be the 2-by-2 principal submatrix of $[a_{ij}]_{i,j=1}^n$. Then $A \cong \begin{bmatrix} A_2 & * \\ 0 & * \end{bmatrix}$. By Lemma 2.2 (b), we have

$$0 \leqslant |a_{12}|^2 = (1 - |a_{11}|^2)(1 - |a_{22}|^2) = (1 - |\lambda_{i_0}|^2)(1 - |\lambda_{j_0}|^2) < 0,$$

a contradiction, since $|\lambda_{i_0}| > 1$ and $|\lambda_{j_0}| < 1$. Hence we conclude that either $\sigma(A) \subseteq \overline{\mathbb{D}}$ or $\sigma(A) \subseteq \mathbb{C} \setminus \mathbb{D}$.

Now, if $\sigma(A) \subseteq \mathbb{D}$ (resp., $\sigma(A) \subseteq \mathbb{C} \setminus \overline{\mathbb{D}}$), since $d_A = 1$, then $A \in \mathscr{S}_n$ (resp., $A \in \mathscr{S}_n^{-1}$) as required. Therefore, we may assume that A is unitarily equivalent to an upper triangular matrix $[t_{ij}]_{i,j=1}^n$ such that $|t_{ii}| = 1$ for all $1 \leqslant i \leqslant k$ $(1 \leqslant k \leqslant n)$ and $|t_{jj}| \neq 1$ for all $k+1 \leqslant j \leqslant n$. Write $[t_{ij}]_{i,j=1}^n = \begin{bmatrix} U & B \\ 0 & C \end{bmatrix}$, where $U = [t_{ij}]_{i,j=1}^k \in M_k$. From Lemma 2.2 (a), we have $d_U \leqslant 1$. Since $|t_{ii}| = 1$ for all $1 \leqslant i \leqslant k$, by Lemma 2.2 (d), we infer that $U = \operatorname{diag}(t_{11}, \cdots, t_{kk})$ and $k < n$, because $d_A = 1 \neq 0$. Moreover, by Lemma 2.2 (c), we obtain that $B = 0$. Therefore, $A \cong U \oplus C$ and $d_C = d_A = 1$. If $\sigma(A) \subseteq \overline{\mathbb{D}}$, then $|t_{jj}| < 1$ for all j, $k+1 \leqslant j \leqslant n$, and it follows that $C = [t_{ij}]_{i,j=k+1}^n \in \mathscr{S}_{n-k}$. On the other hand, $\sigma(A) \subseteq \mathbb{C} \setminus \mathbb{D}$ implies that $|t_{jj}| > 1$ for all j, $k+1 \leqslant j \leqslant n$. Hence C is in $\mathscr{S}_{n-k}^{-1}$ as asserted.

The converse is trivial. $\square$

For an n-by-n matrix A with $d_A = 1$, if A has no unitary part, then A is either in $\mathscr{S}_n$ or in $\mathscr{S}_n^{-1}$ from Theorem 2.1. In recent years, properties of $\mathscr{S}_n$-matrices have been intensely studied (cf. [5, 6, 8, 9, 13, 14, 17]). Therefore, we will restrict our attention to $\mathscr{S}_n^{-1}$-matrices in the rest of this section. We generalize some known results about $\mathscr{S}_n$-matrices to $\mathscr{S}_n^{-1}$-matrices.

In [6], Gau and Wu gave an upper triangular matrix representation for $\mathscr{S}_n$-matrices. In [4], the author gave an upper triangular matrix representation for $\mathscr{S}_n^{-1}$-matrix without proof, because the proof is the same as the one in [6, Corollary 1.3]. We present it here for easy reference. For its detailed proof, the reader may consult [18, Theorem 3.8].

PROPOSITION 2.3. *An operator is in $\mathscr{S}_n^{-1}$ if and only if it has the upper triangular matrix representation $[t_{ij}]_{i,j=1}^n$, where $|t_{ii}| > 1$ for all i and $t_{ij} = s_{ij}(|t_{ii}|^2 - 1)^{1/2}(|t_{jj}|^2 - 1)^{1/2}$ for $i < j$ with*

$$s_{ij} = \begin{cases} \prod_{k=i+1}^{j-1}(\bar{t}_{kk}) & \text{if } j > i+1 \\ 1 & \text{if } j = i+1. \end{cases}$$

Wu gave a complete characterization of the polar decomposition of an $\mathscr{S}_n$-matrix [17]. Here, we prove an analogue of Wu's result for $\mathscr{S}_n^{-1}$-matrices.

PROPOSITION 2.4. *The following are equivalent for an n-by-n matrix A:*

(a) A is an $\mathscr{S}_n^{-1}$-matrix;

(b) $A = U(I_n + sxx^*)$, where U is a unitary matrix with distinct eigenvalues, $s > 0$ and x is a unit cyclic vector for U;

(c) A is unitarily equivalent to $U'(I_n + sx'x'^*)$, where U' is a diagonal unitary matrix with distinct eigenvalues, $s > 0$ and x' is a unit vector with all components nonzero;

(d) $A = U(I_n + sP)$, where U is a unitary matrix, $s > 0$ and P is a rank-one (orthogonal) projection whose kernel contains no eigenvector of U and whose range contains a cyclic vector of U;

(e) A is unitarily equivalent to VD, where V is a unitary matrix such that all its eigenvectors have a nonzero first component and it has $[1\ 0 \cdots 0]^T$ as a cyclic vector, and D is the diagonal matrix $\operatorname{diag}(t, 1, \ldots, 1)$ *with $t > 1$.*

Proof. (a)$\Rightarrow$(b): Notice that for any $A \in \mathscr{S}_n^{-1}$, A is invertible and $A^{-1} \in \mathscr{S}_n$. It follows that $(A^{-1})^* \in \mathscr{S}_n$. By [17, Proposition 3.4], there exists a unitary matrix U with distinct eigenvalues, $0 < r \leqslant 1$ and a unit cyclic vector x for U such that

$$(A^{-1})^* = U(I_n - rxx^*).$$

Thus $A = U(I_n - rxx^*)^{-1}$. Since $I_n - rxx^*$ is invertible and x is a unit vector, we deduce that $0 < r < 1$ and

$$(I_n - rxx^*)^{-1} = I_n + \sum_{j=1}^{\infty}(rxx^*)^j = I_n + \sum_{j=1}^{\infty} r^j xx^* = I_n + \frac{r}{1-r}xx^*.$$

Hence $A = U(I_n + sxx^*)$, where $s = r/(1-r) > 0$.

(b)$\Rightarrow$(a): If $A = U(I_n + sxx^*)$ as in (b), then $A^*A = I_n + (2s+s^2)xx^*$. Since xx^* is a rank one matrix, we have $d_A = \operatorname{rank}((2s+s^2)xx^*) = 1$. We will check that $\sigma(A) \subseteq \mathbb{C} \setminus \overline{\mathbb{D}}$. On the contrary, suppose that there is a unit vector $y \in \mathbb{C}^n$ such that $Ay = \lambda y$ for some $|\lambda| \leqslant 1$. Then

$$\begin{aligned} 1 \geqslant \|\lambda y\|^2 &= \|Ay\|^2 = \|(I_n + sxx^*)y\|^2 \\ &= \langle (I_n + sxx^*)^2 y, y\rangle = \|y\|^2 + (s^2+2s)|\langle x,y\rangle|^2 \\ &= 1 + (s^2+2s)|\langle x,y\rangle|^2 \geqslant 1. \end{aligned}$$

We thus get $\langle x,y\rangle = 0$, since $s > 0$. Moreover,

$$\lambda y = Ay = U(I_n + sxx^*)y = Uy.$$

Since y is an eigenvector of U, it follows that $\langle x,y\rangle \neq 0$ because x is a cyclic vector for U. This is a contradiction. Hence we conclude that A is an $\mathscr{S}_n^{-1}$-matrix.

For the other equivalences, the proofs are essentially the same as in [17, Proposition 3.4]. Hence we omit the proofs. $\square$

The following proposition shows how the characteristic polynomial of an $\mathscr{S}_n^{-1}$-matrix A can be expressed in terms of s and the entries of U' and x' in Proposition 2.4 (c). It is an analogue of [17, Proposition 3.5] for $\mathscr{S}_n^{-1}$-matrices, and its proof is omitted because it is essentially the same as the one for [17, Proposition 3.5].

PROPOSITION 2.5. *Let A be an $\mathscr{S}_n^{-1}$-matrix with polar decomposition $U'(I_n + sx'x'^*)$ as in Proposition 2.4 (c). If U' has eigenvalues $u_1,\cdots,u_n$ and $x' = [x_1 \cdots x_n]^T$, then the characteristic polynomial of A is given by*

$$\sum_{j=1}^{n} |x_j|^2 (z-u_1)\cdots(z-(1+s)u_j)\cdots(z-u_n).$$

An $\mathscr{S}_n$-matrix A may be not invertible, hence its polar decomposition is not unique (cf. [17, Proposition 3.6]). But every $\mathscr{S}_n^{-1}$-matrix is invertible, and thus its polar decomposition is unique. The following proposition shows this simple fact.

PROPOSITION 2.6. *Let A be an $\mathscr{S}_n^{-1}$-matrix with polar decomposition $A = U_1(I_n + s_1x_1x_1^*) = U_2(I_n + s_2x_2x_2^*)$ as in Proposition 2.4 (b). Then*

(a) $s_1 = s_2$,

(b) $x_1 = \lambda x_2$ for some λ, $|\lambda| = 1$, and

(c) $U_1 = U_2$.

Proof. Since $I_n + s_1x_1x_1^* = (A^*A)^{1/2} = I_n + s_2x_2x_2^*$, (a) and (b) follow easily. Note that since $I_n + s_1x_1x_1^*$ is positive definite, it is invertible. Hence

$$U_1 = U_2(I_n + s_2x_2x_2^*)(I_n + s_1x_1x_1^*)^{-1} = U_2$$

as asserted. □

Proposition 2.3 says that an $\mathscr{S}_n^{-1}$-matrix is completely determined by its eigenvalues. Therefore, we give the norm of an $\mathscr{S}_n^{-1}$-matrix in terms of its eigenvalues in the next corollary. It is an easy consequence of Proposition 2.4 (e). Among other things, it's well-known that $\|A\| = 1$ for all $A \in \mathscr{S}_n$.

COROLLARY 2.7. *Let A be an $\mathscr{S}_n^{-1}$-matrix with eigenvalues $\lambda_1, \cdots, \lambda_n$. Then $\|A\| = |\lambda_1 \cdots \lambda_n|$.*

Proof. From Proposition 2.4 (e), A can be written as a polar decomposition $A = UD$, where U is unitary and $D = \operatorname{diag}(t, 1, \cdots, 1)$ for some $t > 1$. Hence

$$\|A\| = \|UD\| = \|D\| = t = \det D = |\det UD| = |\det A| = |\lambda_1 \cdots \lambda_n|$$

as asserted. □

For a matrix $A \in M_n$, $\operatorname{Re} A = (A + A^*)/2$ and $\operatorname{Im} A = (A - A^*)/(2i)$ are the *real* and *imaginary* parts of A, respectively.

In [5, Corollary 2.7], Gau and Wu show that if A is in $\mathscr{S}_n$, then all eigenvalues of $\operatorname{Re} A$ and $\operatorname{Im} A$ are simple. Moreover, Gau [4, Theorem 2.5] shows that if A is in $\mathscr{S}_n^{-1}$, then the maximal eigenvalue of $\operatorname{Re} A$ is simple. Here, we prove an analogue of Gau and Wu's result for $\mathscr{S}_n^{-1}$-matrices.

THEOREM 2.8. *If A is in $\mathscr{S}_n^{-1}$, then both* $\operatorname{Re} A$ *and* $\operatorname{Im} A$ *have simple eigenvalues.*

Let $A = [t_{ij}]_{i,j=1}^n$ be an $\mathscr{S}_n^{-1}$-matrix represented as in Proposition 2.3, and let $A_k = [t_{ij}]_{i,j=1}^k$ for $k = 1, \cdots, n$. Then A_k is in $\mathscr{S}_k^{-1}$ from Proposition 2.3. Moreover, $e^{i\theta}A$ is also in $\mathscr{S}_n^{-1}$ for all $\theta \in \mathbb{R}$. Since $\operatorname{Im} A = \operatorname{Re}(-iA)$, we need only to prove the result for $\operatorname{Re} A$. Let r be an eigenvalue of $\operatorname{Re} A$ and $K = \ker(rI_n - \operatorname{Re} A)$. If $\dim K \geqslant 2$, then there exists a nonzero vector $x \in K$ such that the nth entry of x is zero. Therefore, Theorem 2.8 can be proven by the following lemma.

LEMMA 2.9. *Let A be an $\mathscr{S}_n^{-1}$-matrix represented as in Proposition 2.3 and let r be an eigenvalue of* $\operatorname{Re} A$. *If $x = [x_1 \cdots x_n]^T \in \mathbb{C}^n$ is an eigenvector of* $\operatorname{Re} A$ *corresponding to the eigenvalue r, then $x_n \neq 0$.*

Proof. The proof is by induction on n. For $n=2$, then $r=\max\sigma(\mathrm{Re}A)$ or $r=\min\sigma(\mathrm{Re}A)=-\max\sigma(\mathrm{Re}(-A))$. Hence our assertion follows from [4, Lemma 2.6].

Assume the assertion holds for $n-1$. We will prove it for n. Suppose that $x=[x_1\cdots x_n]^T$ is a unit eigenvector of $\mathrm{Re}A$ corresponding to r. From [4, Lemma 2.6], we may assume that r is neither the maximal eigenvalue nor the minimal eigenvalue of $\mathrm{Re}A$. We now show that $x_n\neq 0$. On the contrary, suppose that $x_n=0$. It implies that $(\mathrm{Re}A_{n-1})y=ry$, where $A_{n-1}=[t_{ij}]_{i,j=1}^{n-1}$ is the $(n-1)$-by-$(n-1)$ principal submatrix of A and $y=[x_1\cdots x_{n-1}]^T\in\mathbb{C}^{n-1}$. Thus y is an eigenvector of $\mathrm{Re}A_{n-1}$ corresponding to the eigenvalue r. On the other hand, let us compute the nth and $(n-1)$th entries of $(rI_n-\mathrm{Re}A)x$, we have

$$-\frac{1}{2}\sum_{j=1}^{n-2}x_j\bar{t}_{j,n}-\frac{1}{2}x_{n-1}\bar{t}_{n-1,n}=0 \tag{1}$$

and

$$-\frac{1}{2}\sum_{j=1}^{n-2}x_j\bar{t}_{j,n-1}+x_{n-1}(r-\mathrm{Re}\,t_{n-1,n-1})=0. \tag{2}$$

By Proposition 2.3, we have

$$t_{j,n}=t_{j,n-1}\cdot\frac{\sqrt{|t_{n,n}|^2-1}}{\sqrt{|t_{n-1,n-1}|^2-1}}\cdot\bar{t}_{n-1,n-1}, \tag{3}$$

for $1\leqslant j\leqslant n-2$. Substituting (3) into (1) yields

$$0=-\frac{\sqrt{|t_{n,n}|^2-1}}{2\sqrt{|t_{n-1,n-1}|^2-1}}\cdot t_{n-1,n-1}\sum_{j=1}^{n-2}x_j\bar{t}_{j,n-1}-\frac{1}{2}x_{n-1}\bar{t}_{n-1,n} \tag{4}$$

Substituting (2) into (4) we obtain

$$0=-\frac{\sqrt{|t_{nn}|^2-1}}{\sqrt{|t_{n-1,n-1}|^2-1}}t_{n-1,n-1}x_{n-1}(r-\mathrm{Re}\ t_{n-1,n-1})-\frac{1}{2}x_{n-1}\bar{t}_{n-1,n}.$$

By induction hypothesis, we have $x_{n-1}\neq 0$, which implies that

$$-t_{n-1,n-1}\frac{\sqrt{|t_{nn}|^2-1}}{\sqrt{|t_{n-1,n-1}|^2-1}}(r-\mathrm{Re}\ t_{n-1,n-1})=\frac{1}{2}\bar{t}_{n-1,n}.$$

Thus $-t_{n-1,n-1}(r-\mathrm{Re}\ t_{n-1,n-1})=\frac{1}{2}(|t_{n-1,n-1}|^2-1)$, and hence $t_{n-1,n-1}$ is real and $t_{n-1,n-1}=r\pm\sqrt{r^2-1}$. Since $t_{n-1,n-1}$ is real, it implies that $r^2\geqslant 1$ or $|r|\geqslant 1$. The result [4, Lemma 2.9 (1)] says that if λ_j is the jth largest eigenvalue of $\mathrm{Re}A$, then $\lambda_2\leqslant 1$. Note that $-\lambda_{n-1}$ is the second largest eigenvalue of $\mathrm{Re}(-A)$ and $-A\in\mathscr{S}_n^{-1}$, so by [4, Lemma 2.9 (1)] again, we have $-\lambda_{n-1}\leqslant 1$ or $\lambda_{n-1}\geqslant -1$. Now, since r is neither the maximal eigenvalue nor the minimal eigenvalue of $\mathrm{Re}A$, by [4, Lemma

2.9 (1)], we have $-1 \leqslant \lambda_{n-1} \leqslant r \leqslant \lambda_2 \leqslant 1$. Thus $|r| = 1$. Consequently, we obtain $|t_{n-1,n-1}| = 1$, a contradiction. Hence $x_n \neq 0$ as asserted. □

We now restrict our attention to the defect indices of powers of an $\mathscr{S}_n^{-1}$-matrix. In recent years, the defect indices of powers of a contraction have been intensely studied (cf. [7, 9, 10, 13]). In particular, Gau and Wu [7, Theorem 3.1] have shown that for an n-by-n contraction A ($\|A\| \leqslant 1$), the following conditions are equivalent: (a) A is in $\mathscr{S}_n$; (b) $\|A\| = \|A^{n-1}\| = 1$ and $\|A^n\| < 1$; (c) $d_{A^k} = k$ for all k, $1 \leqslant k \leqslant n$; (d) $d_{A^k} = k$ for $k = n$ and for k equal to some k_0, $1 \leqslant k_0 < n$. Notice that the norm of an $\mathscr{S}_n^{-1}$-matrix is greater than one, that is, it is not a contraction. Here, we prove an analogue of [7, Theorem 3.1] for $\mathscr{S}_n^{-1}$-matrices.

THEOREM 2.10. *Let A be an n-by-n matrix with $\|A\| > 1$. Then the following conditions are equivalent:*

(a) $A \in \mathscr{S}_n^{-1}$;

(b) $d_{A^k} = k$ for all k, $1 \leqslant k \leqslant n$;

(c) $d_{A^n} = n$ and $d_A = 1$.

Proof. (a)⇒(b): Since A^{-1} is in $\mathscr{S}_n$, by [7, theorem 3.1], we have $d_{A^{-k}} = k$ for all k, $1 \leqslant k \leqslant n$. On the other hand, since A^k is invertible and

$$I_n - A^{k*}A^k = A^{k*}[A^{-k*}A^{-k} - I_n]A^k = -A^{k*}[I_n - A^{-k*}A^{-k}]A^k,$$

we deduce that $d_{A^k} = k$ for all k, $1 \leqslant k \leqslant n$.

(b)⇒(c): This is trivial.

(c)⇒(a): Since $\|A\| > 1$ and $d_A = 1$, by Theorem 2.1, we may assume that A is unitarily equivalent to $U \oplus C$, where $U \in M_k$ is unitary, $0 \leqslant k < n$, and $C \in S_{n-k}^{-1}$. We obtain that

$$A^n \cong \begin{bmatrix} U^n & 0 \\ 0 & C^n \end{bmatrix},$$

and hence

$$I_n - A^{n*}A^n \cong \begin{bmatrix} 0 & 0 \\ 0 & I_{n-k} - C^{n*}C^n \end{bmatrix}.$$

But $d_{A^n} = n$, which clearly forces $A \cong C$. □

For an n-by-n matrix A, $d_{A^n} = n$ means that A has no unitary part. Moreover, if A is a contraction with $d_{A^n} = n$, [7, Theorem 3.1] shows that $d_A = 1$ if and only if $d_{A^k} = k$ for some k, $1 \leqslant k < n$. The following examples show that it is not the case for an n-by-n matrix A with $\|A\| > 1$.

EXAMPLE 2.11. Let

$$A = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix},$$

then $\|A\| > 1$. After a simple calculation, we have

$$I_3 - A^*A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{4} & 0 \\ 0 & 0 & -3 \end{bmatrix}, \qquad I_3 - A^{2*}A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and $I_3 - A^{3*}A^3 = I_3$. Thus, we have $d_{A^2} = 2$ and $d_{A^3} = 3$. But $d_A = 3 \neq 1$.

Next, we construct a 4-by-4 matrix $A = B \oplus C$ such that $B^2 \in \mathscr{S}_2^{-1}$ and $C^2 \in \mathscr{S}_2$. Then $d_{A^2} = d_{B^2} + d_{C^2} = 1 + 1 = 2$ and $d_{A^4} = d_{B^4} + d_{C^4} = 2 + 2 = 4$. But, it is easily seen that $d_A \neq 1$.

EXAMPLE 2.12. Let

$$A = \begin{bmatrix} \sqrt{2} & \frac{3}{2\sqrt{2}} & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{3}{4\sqrt{2}} \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Then $\begin{bmatrix} \sqrt{2} & \frac{3}{2\sqrt{2}} \\ 0 & \sqrt{2} \end{bmatrix}^2 = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} \in \mathscr{S}_2^{-1}$, $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{3}{4\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}^2 = \begin{bmatrix} \frac{1}{2} & \frac{3}{4} \\ 0 & \frac{1}{2} \end{bmatrix} \in \mathscr{S}_2$ and $d_A = 4 \neq 1$. But $d_{A^2} = 2$ and $d_{A^4} = 4$.

Let A be an n-by-n contraction and $H_j(A) = \ker(I_n - A^{j*}A^j)$ for $j = 1, \cdots, n$. Since $\|A\| \leqslant 1$, we have $x \in H_j(A)$ if and only if $\|A^j x\| = \|x\|$. Moreover, $\|A^j y\| \leqslant \|A^{j-1}y\| \leqslant \cdots \leqslant \|Ay\| \leqslant \|y\|$ for all $y \in \mathbb{C}^n$. Therefore, $H_j(A) \subseteq H_{j-1}(A)$ for all j. On the other hand, let $V_j = H_j(A) \cap H_j(A^*)$, it is clear that $V_j \subseteq V_{j-1}$ for all j. Let $V_\infty = \bigcap_{j=1}^\infty V_j$, then $A|_{V_\infty}$ is the unitary part of A. Therefore, A has no unitary part if and only if $V_k = \{0\}$ for some k. Let $k(A) = \min\{k : 0 \leqslant k \leqslant \infty, V_k = V_\infty\}$. It was shown independently by Gau-Wu [9] and Li [13] that $k(A) \leqslant \lceil n/2 \rceil$. Moreover, they also showed that the equality $k(A) = \lceil n/2 \rceil$ holds if and only if one of the following holds. (a) $A \in \mathscr{S}_n$. (b) n is even and A is unitarily equivalent to $[e^{it}] \oplus A_1$ with $t \in \mathbb{R}$ and $A_1 \in \mathscr{S}_{n-1}$. (c) n is even, $\|A^{n-2}\| = 1 > \|A^{n-1}\|$. The following theorem provides an analogue of the above result for $\mathscr{S}_n^{-1}$-matrices. Its proof is inspired by that of [13, Theorem 2].

THEOREM 2.13. *Let A be an n-by-n ($n \geqslant 2$) matrix with $\|A\| > 1$ and $H_j(A) = \ker(I_n - A^{j*}A^j)$ for $j = 1, 2, \cdots, n$. The following statements are equivalent:*

(a) $A \in \mathscr{S}_n^{-1}$;

(b) $\dim[H_k(A) \cap H_k(A^*)] = n - 2k$, *for all* $1 \leqslant k \leqslant \lfloor n/2 \rfloor$, *and* $\dim[H_{k_0}(A) \cap H_{k_0}(A^*)] = 0$, *where* $k_0 = \lceil n/2 \rceil$;

(c) $\dim[H_1(A) \cap H_1(A^*)] = n - 2$ *and* $\dim[H_{k_0}(A) \cap H_{k_0}(A^*)] = 0$, *where* $k_0 = \lceil n/2 \rceil$.

For its proof, we need the following lemmas.

LEMMA 2.14. *Let A be an n-by-n matrix with* $\ker(I_n - A^*A) = \ker(I_n - AA^*)$. *If*

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \quad \text{on } \mathbb{C}^n = \ker(I_n - A^*A) \oplus \operatorname{ran}(I_n - A^*A),$$

then $A_2 = 0 = A_3$ *and* A_1 *is unitary.*

Proof. By polar decomposition, write $A = UD$ where U is unitary and $D = (A^*A)^{1/2}$. Write $k = \dim\ker(I_n - A^*A)$,

$$U = \begin{bmatrix} U_1 & U_2 \\ U_3 & U_4 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} I_k & D_2 \\ D_2^* & D_4 \end{bmatrix}$$

on $\mathbb{C}^n = \ker(I_n - A^*A) \oplus \operatorname{ran}(I_n - A^*A)$. Note that $\ker(I_n - A^*A) = \ker(I_n - AA^*)$. Thus

$$\operatorname{ran}(I_n - AA^*) = \ker(I_n - AA^*)^\perp = \ker(I_n - A^*A)^\perp = \operatorname{ran}(I_n - A^*A),$$

where $S^\perp$ denotes the orthogonal complement of $S \subseteq \mathbb{C}^n$.

Since $U(I_n - A^*A) = (I_n - AA^*)U$, we have

$$U(\operatorname{ran}(I_n - A^*A)) \subseteq \operatorname{ran}(I_n - AA^*) = \operatorname{ran}(I_n - A^*A)$$

and

$$U(\ker(I_n - A^*A)) \subseteq \ker(I_n - AA^*) = \ker(I_n - A^*A).$$

Hence $U_2 = 0 = U_3$, and U_1 and U_4 are unitary. Moreover,

$$A = \begin{bmatrix} U_1 & 0 \\ 0 & U_4 \end{bmatrix} \begin{bmatrix} I_k & D_2 \\ D_2^* & D_4 \end{bmatrix} = \begin{bmatrix} U_1 & U_1D_2 \\ U_4D_2^* & U_4D_4 \end{bmatrix}.$$

Since $A(I_n - A^*A) = (I_n - AA^*)A$, it follows that $A(\operatorname{ran}(I_n - A^*A)) \subseteq \operatorname{ran}(I_n - AA^*) = \operatorname{ran}(I_n - A^*A)$. Therefore, $U_1D_2 = 0$ or $D_2 = 0$, since U_1 is unitary. Hence

$$\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} = \begin{bmatrix} U_1 & 0 \\ 0 & U_4D_4 \end{bmatrix}.$$

It is clear that $A_2 = 0 = A_3$ and $A_1 = U_1$ is unitary. $\square$

LEMMA 2.15. *If* $A \in \mathscr{S}_n^{-1}$, *then* $\ker(I_n - A^*A) = \ker(I_n - AA^*)$ *if and only if* $n = 1$.

Proof. If $n = 1$, then $\ker(I_n - A^*A) = \{0\} = \ker(I_n - AA^*)$.

Conversely, if $\ker(I_n - A^*A) = \ker(I_n - AA^*)$, by Lemma 2.14, the restriction of A on $\ker(I_n - A^*A)$ is unitary. Since $A \in \mathscr{S}_n^{-1}$ implies A has no unitary part, it follows that $\ker(I_n - A^*A) = \{0\}$. Since $\operatorname{rank}(I_n - A^*A) = d_A = 1$ implies

$$n = \dim(\ker(I_n - A^*A)) + \operatorname{rank}(I_n - A^*A) = 1,$$

the proof is complete. $\square$

Notice that for any $x \in \mathbb{C}^n$ and $A \in \mathscr{S}_n^{-1}$, we have $\|Ax\| \geqslant \|x\|$. Indeed, by Proposition 2.4 (e), we may assume that $A = VD$, where $V \in M_n$ is unitary and $D = \operatorname{diag}(t, 1, \ldots, 1) \in M_n$, $t > 1$. Thus $\|Ax\| = \|Dx\| \geqslant \|x\|$. Moreover, if $x \in H_k(A)$ for some $k \geqslant 1$, then

$$\|x\| = \|A^k x\| \geqslant \|A^{k-1}x\| \geqslant \cdots \geqslant \|Ax\| \geqslant \|x\|.$$

Thus $x \in H_j(A)$ for all $1 \leqslant j \leqslant k$, and hence $H_k(A) \subseteq H_{k-1}(A)$ for all k. Among other things, since $A^*A - I_n = D^2 - I_n$ is positive semi-definite, it follows that a vector x is in $H_k(A)$ if and only if $\|A^k x\| = \|x\|$ for $k = 1, \ldots, n$. We are now ready to prove Theorem 2.13.

Proof of Theorem 2.13. (a)$\Rightarrow$(b): Let $V_k = H_k(A) \cap H_k(A^*)$ for $k = 1, \ldots, n$. Since $A \in \mathscr{S}_n^{-1}$ implies $A^* \in \mathscr{S}_n^{-1}$, by the above paragraph, we have $V_{k+1} \subseteq V_k$ for all k. Moreover, we also deduce that a vector x is in V_k if and only if $\|A^k x\| = \|A^{*k} x\| = \|x\|$ for $k = 1, \ldots, n$.

We first show that $AV_{k+1} \subseteq V_k$. Suppose $\{x_1, \ldots, x_l\}$ is an orthonormal basis for V_{k+1}. Since $V_{k+1} \subseteq V_k$, we let $\{x_1, \ldots, x_p\}$ be an orthonormal basis for V_k, where $p \geqslant l$. Let x be a unit vector in V_{k+1}, then $\|A^k(Ax)\| = 1$. Since $V_{k+1} \subseteq V_k \subseteq \cdots \subseteq V_1$, it implies $A^*Ax = x$ and

$$\|A^{k*}(Ax)\| = \|(A^{k-1})^*(A^*A)x\| = \|(A^{k-1})^* x\| = 1.$$

Hence $Ax \in V_k$ and $AV_{k+1} \subseteq V_k$. Similarly, we have $A^*V_{k+1} \subseteq V_k$.

We claim that $l \leqslant \max\{0, p-2\}$.

If $l = p$, then $AV_k = AV_{k+1} \subseteq V_k$. If $\alpha_1 Ax_1 + \cdots + \alpha_p Ax_p = 0$ for some scalars $\alpha_1, \ldots, \alpha_p$, then $A^*(\alpha_1 Ax_1 + \cdots + \alpha_p Ax_p) = 0$. It follows that $\alpha_1 x_1 + \cdots + \alpha_p x_p = 0$ since $A^*Ax_j = x_j$ for $j = 1, \cdots, p$. But $\{x_1, \ldots, x_p\}$ is orthonormal, which implies that $\alpha_1 = \cdots = \alpha_p = 0$. Hence $\{Ax_1, \cdots, Ax_p\}$ is linearly independent and $A(V_k) = V_k$. Interchanging A and A^* in the preceding arguments yields $A^*(V_k) = V_k$. Therefore A is unitarily equivalent to $\begin{bmatrix} A_{11} & 0 \\ 0 & * \end{bmatrix}$ on $\mathbb{C}^n = V_k \oplus V_k^\perp$, and $A_{11} \in M_p$ is unitary, which contradicts the fact that A has no unitary part.

If $l = p-1$, let $U \in M_n$ be a unitary matrix such that $x_1, \ldots, x_p$ are the first p columns of U. We have

$$\widetilde{A} \equiv U^*AU = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

on $\mathbb{C}^n = \mathbb{C}^p \oplus \mathbb{C}^{n-p}$. Let $A_{11} = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix} \in M_p$ on $\mathbb{C}^p = \mathbb{C}^{p-1} \oplus \mathbb{C}$. Since $AV_{k+1} \subseteq V_k$, $A^*V_{k+1} \subseteq V_k$, and $V_{k+1} = \bigvee\{x_1, \cdots, x_{p-1}\}$, we obtain that

$$\widetilde{A} = \begin{bmatrix} C_1 & C_2 & 0 \\ C_3 & C_4 & y^T \\ 0 & x & * \end{bmatrix} \quad \text{and} \quad \widetilde{A}^*\widetilde{A} = \begin{bmatrix} A_{11}^*A_{11} + J & * \\ * & * \end{bmatrix},$$

where $x, y \in \mathbb{C}^{n-p}$ are nonzero vectors and

$$J = A_{21}^*A_{21} = \begin{bmatrix} 0 \\ x^* \end{bmatrix} \begin{bmatrix} 0 & x \end{bmatrix} = \operatorname{diag}(0, \ldots, 0, \|x\|^2).$$

Since $V_k \subseteq \ker(I_n - A^*A)$ and $x_j \in V_k$ for $j = 1, \cdots, p$, it follows that $\widetilde{A}^*\widetilde{A} = I_p \oplus B$ for some $B \in M_{n-p}$, and consequently

$$A_{11}^*A_{11} = \begin{bmatrix} I_{p-1} & 0 \\ 0 & 1 - \|x\|^2 \end{bmatrix}.$$

Symmetrically, we have

$$A_{11}A_{11}^* = \begin{bmatrix} I_{p-1} & 0 \\ 0 & 1-\|y\|^2 \end{bmatrix}.$$

We now prove that C_2 and C_3 are zero matrices. Note that

$$A_{11}^*A_{11} = \begin{bmatrix} C_1^*C_1 + C_3^*C_3 & C_1^*C_2 + C_3^*C_4 \\ C_2^*C_1 + \overline{C_4}C_3 & C_2^*C_2 + \overline{C_4}C_4 \end{bmatrix}$$

and

$$A_{11}A_{11}^* = \begin{bmatrix} C_1C_1^* + C_2C_2^* & C_1C_3^* + C_2\overline{C_4} \\ C_3C_1^* + C_4C_2^* & C_3C_3^* + C_4\overline{C_4} \end{bmatrix}.$$

It follows that $C_2^*C_1 + \overline{C_4}C_3 = 0$, and hence $C_2^*C_1C_1^* + \overline{C_4}C_3C_1^* = 0$. Combining $C_1C_1^* + C_2C_2^* = I_{p-1}$ with $C_3C_1^* + C_4C_2^* = 0$, we obtain that

$$C_2^*(I_{p-1} - C_2C_2^*) - |C_4|^2C_2^* = 0$$

or $(1-\|C_2\|^2 - |C_4|^2)C_2^* = 0$. It implies that $C_2 = 0$, since $\|C_2\|^2 + |C_4|^2 = 1 - \|x\|^2 \neq 1$. Similarly, since $C_3C_1^*C_1 + C_4C_2^*C_1 = 0$ and $\|C_3\|^2 + |C_4|^2 = 1 - \|y\|^2 \neq 1$, we have $C_3 = 0$. Hence $\widetilde{A} = \begin{bmatrix} C_1 & 0 \\ 0 & * \end{bmatrix}$ and $C_1C_1^* = I_{p-1}$. It implies that C_1 is a unitary direct summand of $\widetilde{A}$, which is a contradiction.

By the above, we see that $l \leqslant \max\{0, p-2\}$. By Lemma 2.15, we have $\ker(I_n - A^*A) \neq \ker(I_n - AA^*)$. Hence $n-1 < \dim(\ker(I_n - A^*A) + \ker(I_n - AA^*))$ and

$$\begin{aligned} \dim V_1 &= \dim(\ker(I_n - A^*A)) + \dim(\ker(I_n - AA^*)) - \dim[\ker(I_n - A^*A) + \ker(I_n - AA^*)] \\ &= (n-1) + (n-1) - n = n-2. \end{aligned}$$

Moreover,

$$n-2 = \dim(V_1) \geqslant \dim(V_2) + 2 \geqslant \cdots \geqslant \dim(V_k) + (2k-2) \geqslant \cdots. \tag{5}$$

Therefore, if $k_0 = \lceil n/2 \rceil$, then $\dim V_{k_0} = 0$ and $\dim V_j = 0$ for $j \geqslant k_0$ since $V_j \subseteq V_{k_0}$.

On the other hand, $\dim(\ker(I_n - (A^{n-1})^*A^{n-1})) = 1$, so there is a unit vector $x \in C^n$ such that $\|A^{n-1}x\| = 1$. Since A is an $\mathscr{S}_n^{-1}$-matrix, we have

$$1 = \|A^{n-1}x\| \geqslant \|A^{n-2}x\| \geqslant \cdots \geqslant \|Ax\| \geqslant \|x\| = 1.$$

It follows that $x, Ax, \cdots, A^{n-1}x$ are unit vectors.

If $n = 2k_0 - 1$, then $A^{k_0-1}(A^{k_0-1}x) = A^{2k_0-2}x$ and $(A^*)^{k_0-1}A^{k_0-1}x = x$ are unit vectors. So $A^{k_0-1}x \in V_{k_0-1}$ and $V_{k_0-1} \neq \{0\}$. Similarly, if $n = 2k_0$, then $A^{k_0-1}(A^{k_0-1}x) = A^{2k_0-2}x$ and $(A^*)^{k_0-1}A^{k_0-1}x = x$ are unit vectors, and hence $A^{k_0-1}x \in V_{k_0-1} \neq \{0\}$. This implies that k_0 is the smallest integer satisfying $V_k = \{0\}$.

Now, we are going to show that $l = \max\{0, p-2\}$. If $n = 2k_0 - 1$, by (5), $\dim(V_{k_0-1}) \leqslant 1$, and hence $\dim(V_{k_0-1}) = 1$ since $V_{k_0-1} \neq \{0\}$. If $n = 2k_0$, by (5), $\dim(V_{k_0-1}) \leqslant 2$. On the other hand,

$$\begin{aligned} \dim(V_{k_0-1}) &= \dim H_{k_0-1}(A) + \dim H_{k_0-1}(A^*) - \dim[H_{k_0-1}(A) + H_{k_0-1}(A^*))] \\ &\geqslant (n-k_0+1) + (n-k_0+1) - n = 2. \end{aligned}$$

Thus $\dim(V_{k_0-1}) = 2$ and in both cases

$$n-2 = \dim(V_1) \geqslant \dim(V_2)+2 \geqslant \cdots \geqslant \dim(V_{k_0-1})+2k_0-4 = n-2.$$

Therefore, $\dim(V_j) = n-2j$ for $1 \leqslant j \leqslant \lfloor n/2 \rfloor$, and $\dim(V_j) = 0$ for $j \geqslant k_o$.

(b)$\Rightarrow$(c): This is trivial.

(c)$\Rightarrow$(a): We want to show that $\dim(\ker(I_n - A^*A)) = n-1$. Indeed, since $\dim[\ker(I_n - A^*A) \cap \ker(I_n - AA^*)] = \dim[H_1(A) \cap H_1(A^*)] = n-2$, it follows that $n-2 \leqslant \dim\ker(I_n - A^*A) \leqslant n$. If $\dim(\ker(I_n - A^*A)) = n$, then A is a unitary matrix, but this contradicts the fact that $\dim[H_{k_0}(A) \cap H_{k_0}(A^*)] = 0$, where $k_0 = \lceil n/2 \rceil$. On the other hand, if $\dim(\ker(I_n - A^*A)) = n-2 = \dim[\ker(I_n - A^*A) \cap \ker(I_n - AA^*)]$, then $\ker(I_n - A^*A) \subseteq \ker(I_n - AA^*)$. Note that for every finite matrix T, the dimensions of $\ker(I - T^*T)$ and $\ker(I - TT^*)$ are the same. Hence we deduce that $\ker(I_n - A^*A) = \ker(I_n - AA^*)$. Lemma 2.14 now yields that $A \cong \begin{bmatrix} A_1 & 0 \\ 0 & A_4 \end{bmatrix}$ on $\mathbb{C}^n = \ker(I_n - A^*A) \oplus \operatorname{ran}(I_n - A^*A)$, where A_1 is unitary. This contradicts the fact that $\dim[H_{k_0}(A) \cap H_{k_0}(A^*)] = 0$, where $k_0 = \lceil n/2 \rceil$. Therefore, we infer that $\dim\ker(I_n - A^*A) = n-1$ or $\operatorname{rank}(I_n - A^*A) = 1$. Moreover, $\dim[H_{k_0}(A) \cap H_{k_0}(A^*)] = 0$ implies that A has no unitary part. Hence, by Theorem 2.1 and $\|A\| > 1$, we conclude that $A \in \mathscr{S}_n^{-1}$ as desired. $\square$

3. Numerical ranges

Recall that the numerical range $W(A)$ of any n-by-n matrix A is the subset

$$W(A) = \{\langle Ax, x\rangle : x \in \mathbb{C}^n, \|x\| = 1\}$$

of the plane, where $\langle \cdot,\cdot \rangle$ denotes the usual inner product in $\mathbb{C}^n$. Properties of the numerical range can be found in [12, Chapter 1].

From Proposition 2.4 (e), an $\mathscr{S}_n^{-1}$-matrix A can be written as a polar decomposition $A = UD_t$, where U is unitary and $D_t = \operatorname{diag}(t, 1, \cdots, 1)$ for some $t > 1$. The following theorem shows that if $0 \in W(U)$, then $W(UD_{t_1}) \subseteq W(UD_{t_2})$ for $1 \leqslant t_1 \leqslant t_2$.

THEOREM 3.1. *Let A_t be in $\mathscr{S}_n^{-1}$ with $A_t = UD_t$, where U is a unitary matrix and $D_t = diag(t, 1, \cdots, 1)$, $t > 1$. If 0 is in $W(U)$, then*

(a) $0 \in W(A_t)$ for all $t > 1$,

(b) $W(U) \subseteq W(A_t)$, and

(c) $W(A_{t_1}) \subseteq W(A_{t_2})$ for $1 \leqslant t_1 \leqslant t_2$.

Proof. (a) Assume that $0 \notin W(A_t)$ for some $t > 1$. By convexity of $W(A_t)$, there exists a $0 \leqslant \theta \leqslant 2\pi$ such that $\operatorname{Re} w > 0$ for all $w \in W(e^{-i\theta}A_t)$. Without loss of generality, we may assume $\theta = 0$. We will show that $\operatorname{Re}\lambda > 0$ for all $\lambda \in \sigma(U)$. Then $W(U) \subseteq \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ which contradicts the fact that $0 \in W(U)$.

Let x be a unit eigenvector of U corresponding to $\lambda \in \sigma(U)$, then

$$\langle A_t x, x\rangle = \langle Ux, x\rangle + \langle U(D_t - I_n)x, x\rangle = \lambda(1 + \langle (D_t - I_n)x, x\rangle).$$

Note that $D_t - I_n$ is a positive semi-definite matrix. Thus $1 + \langle (D_t - I_n)x, x\rangle > 0$. Since $\operatorname{Re}\langle A_t x, x\rangle > 0$, it follows that $\operatorname{Re}\lambda > 0$ as required.

(b) As in the proof of (a), for any $\lambda \in \sigma(U)$ and a unit eigenvector x of U corresponding to λ, we have

$$\lambda = \langle A_t x, x\rangle \cdot \frac{1}{1 + \langle (D_t - I_n)x, x\rangle} + 0 \cdot \frac{\langle (D_t - I_n)x, x\rangle}{1 + \langle (D_t - I_n)x, x\rangle} \in W(A_t),$$

since $\langle (D_t - I_n)x, x\rangle \geqslant 0$ and $0 \in W(A_t)$.

Hence $\sigma(U) \subseteq W(A)$ and $W(U) = \text{convex hull}(\sigma(U)) \subseteq W(A)$ as required.

(c) Let $A_{t_1} = UD_{t_1}$ and $A_{t_2} = UD_{t_2}$, $t_2 > t_1 > 1$. Let λ be the maximal eigenvalue of $\operatorname{Re} A_{t_1}$. We want to show that $\lambda \leqslant \max\sigma(\operatorname{Re} A_{t_2})$. Indeed, let x be a unit eigenvector of $\operatorname{Re} A_{t_1}$ corresponding to λ, then

$$\frac{1}{2}(UD_{t_1} + D_{t_1}U^*)x = \lambda x.$$

Therefore,

$$\begin{aligned}\lambda &= \frac{1}{2}[\langle UD_{t_1}x, x\rangle + \langle D_{t_1}U^*x, x\rangle]\\ &= \frac{1}{2}[\langle D_{t_1}x, U^*x\rangle + \langle U^*x, D_{t_1}x\rangle]\\ &= \operatorname{Re}\langle D_{t_1}x, U^*x\rangle.\end{aligned}$$

Write $U = [u_1 \cdots u_n]$, $x = [x_1 \cdots x_n]^T$ and $D_{t_1} = I_n + T_1$, where $T_1 = \operatorname{diag}(t_1 - 1, 0, \cdots, 0)$. Hence

$$\begin{aligned}\lambda &= \operatorname{Re}\langle (I_n + T_1)x, U^*x\rangle\\ &= \operatorname{Re}\langle Ux, x\rangle + (t_1 - 1)\operatorname{Re}(x_1\langle u_1, x\rangle).\end{aligned}$$

Note that $\lambda = \max\sigma(\operatorname{Re} A_{t_1}) > 0$, because $0 \in W(A_{t_1})$. Since $t_1 - 1 > 0$ and $W(U) \subseteq W(A_{t_1})$, it follows that $\operatorname{Re}(x_1\langle u_1, x\rangle) > 0$.

Similarly, we have

$$\langle (\operatorname{Re} A_{t_2})x, x\rangle = \operatorname{Re}\langle Ux, x\rangle + (t_2 - 1)\operatorname{Re}(x_1\langle u_1, x\rangle).$$

Therefore, $\langle (\operatorname{Re} A_{t_2})x, x\rangle - \lambda = (t_2 - t_1)\operatorname{Re}(x_1\langle u_1, x\rangle) > 0$. That is,

$$\lambda < \langle (\operatorname{Re} A_{t_2})x, x\rangle \leqslant \max\sigma(\operatorname{Re} A_{t_2}).$$

Now, for each $\theta \in [0, 2\pi)$, let $U' = e^{i\theta}U$, then

$$e^{i\theta}A_{t_1} = U'D_{t_1} \quad \text{and} \quad e^{i\theta}A_{t_2} = U'D_{t_2}.$$

From the above result that we have proven, we deduce that

$$\max\sigma(\operatorname{Re}(e^{i\theta}A_{t_1})) \leqslant \max\sigma(\operatorname{Re}(e^{i\theta}A_{t_2}))$$

for all $\theta \in [0, 2\pi)$. Hence $W(A_{t_1}) \subseteq W(A_{t_2})$. $\square$

The next example shows that the condition $0 \in W(U)$ in Theorem 3.1 is essential.

EXAMPLE 3.2. Let

$$U = \begin{bmatrix} 0.8986+0.1493i & -0.0996+0.0911i & 0.2597+0.0792i & 0.2680+0.0797i \\ -0.0996+0.0911i & 0.8986+0.1493i & 0.2680+0.0797i & 0.2597+0.0792i \\ -0.2597-0.0792i & -0.2680-0.0797 & 0.8986+0.1493i & -0.0996+0.0911i \\ -0.2680-0.0797 & -0.2597-0.0792i & -0.0996+0.0911i & 0.8986+0.1493i \end{bmatrix}$$

and $B_k = UD_k$ for $k = 1, 2$, where $D_1 = \operatorname{diag}(1.1, 1, 1, 1)$ and $D_2 = \operatorname{diag}(1.2, 1, 1, 1)$. By computing, we have $\sigma(U) \subseteq \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$, that is, $0 \notin W(U)$, and

$$\min \sigma(\operatorname{Re} B_1) \approx 0.6522, \qquad \min \sigma(\operatorname{Re} B_2) \approx 0.6587,$$

and

$$\max \sigma(\operatorname{Re} B_1) \approx 1.067, \qquad \max \sigma(\operatorname{Re} B_2) \approx 1.149.$$

Hence $W(B_1) \nsubseteq W(B_2)$ and $W(B_2) \nsubseteq W(B_1)$.

We remark that Theorem 3.1 does not hold for $\mathscr{S}_n$-matrices. In fact, [5, Lemma 4.2] says that if T_1 and T_2 are in $\mathscr{S}_n$, then $T_1 \cong T_2$ if and only if $W(T_1) \subseteq W(T_2)$. Hence $W(A_{t_1}) \nsubseteq W(A_{t_2})$ for any $0 \leqslant t_1 \neq t_2 < 1$.

In the end of this paper, we give a generalization of Chien and Nakazato's result [3]. In [3], they study the numerical range of the tridiagonal matrix

$$A = A(n,r) = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ r & 0 & 1 & 0 & \cdots & 0 \\ 0 & r^2 & 0 & 1 & \cdots & 0 \\ 0 & 0 & r^3 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & r^{n-1} & 0 \end{bmatrix}.$$

In particular, they examined more details on the numerical range of $A(n, -1)$. For $n \geqslant 4$, they show that $W(A(n, -1))$ is contained in the square

$$Q = \{z \in \mathbb{C} : |\operatorname{Re} z| \leqslant 1 \text{ and } |\operatorname{Im} z| \leqslant 1\}.$$

Moreover, if n is even, they show that the numerical range $W(A(n, -1))$ has 4 flat portions on its boundary $\partial W(A(n, -1))$ (cf. [3, Theorem 8]). In fact, these 4 flat portions lie on the boundary ∂Q of the square Q. Note that if $\{e_1, \ldots, e_{2k}\}$ denotes the standard basis for $\mathbb{C}^{2k}$ and P is the $2k$-by-$2k$ permutation matrix so that $Pe_{2j-1} = e_j$ and $Pe_{2j} = e_{k+j}$ for $1 \leqslant j \leqslant k$, then

$$PA(2k, -1)P^* = \begin{bmatrix} 0 & I_k + J_k^* \\ -I_k + J_k & 0 \end{bmatrix},$$

where J_k is the k-by-k Jordan block as the form

$$\begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}.$$

The next theorem generalizes the Chien and Nakazato's result about $A(2k,-1)$ to the 2-by-2 block matrix $\begin{bmatrix} 0 & I_k+B^* \\ -I_k+B & 0 \end{bmatrix} \in M_{2k}$ for general contraction $B \in M_k$.

THEOREM 3.3. *For any $B \in M_k$ with $\|B\| \leqslant 1$ and $k \geqslant 1$, let*

$$A = \begin{bmatrix} 0 & I_k+B^* \\ -I_k+B & 0 \end{bmatrix} \in M_{2k}.$$

Then the numerical range of A is contained in the square

$$Q = \{z \in \mathbb{C} : |\mathrm{Re}\, z| \leqslant 1 \text{ and } |\mathrm{Im}\, z| \leqslant 1\}.$$

Moreover, the flat portions on $\partial W(A)$ are

$$\{\pm(t+i) : t \in W(\mathrm{Im}\, B)\} \quad \text{and} \quad \{\pm(1+it) : t \in W(\mathrm{Im}\, B_M)\},$$

where $M = \ker(I_k - B^*B)$ and B_M is the compression of B on M.

Proof. Note that

$$\mathrm{Re}\, A = \begin{bmatrix} 0 & B^* \\ B & 0 \end{bmatrix} \quad \text{and} \quad \mathrm{Im}\, A = \begin{bmatrix} 0 & -iI_k \\ iI_k & 0 \end{bmatrix}.$$

Since

$$\begin{bmatrix} 0 & B^* \\ B & 0 \end{bmatrix}\begin{bmatrix} 0 & B^* \\ B & 0 \end{bmatrix} = \begin{bmatrix} B^*B & 0 \\ 0 & BB^* \end{bmatrix}$$

and $\|B^*B\| = \|BB^*\| = \|B\|^2 \leqslant 1$, we have $\|\mathrm{Re}\, A\| \leqslant 1$ and $W(\mathrm{Re}\, A) \subseteq [-1,1]$. On the other hand, since

$$\begin{bmatrix} 0 & -iI_k^* \\ iI_k^* & 0 \end{bmatrix}\begin{bmatrix} 0 & -iI_k \\ iI_k & 0 \end{bmatrix} = \begin{bmatrix} I_k & 0 \\ 0 & I_k \end{bmatrix},$$

it is obviously that $\|\mathrm{Im}\, A\| = 1$ and $W(\mathrm{Im}\, A) \subseteq [-1,1]$. Hence $W(A)$ is contained in the square Q.

For any $x, y \in \mathbb{C}^k$ with $\|x\|^2 + \|y\|^2 = 1$,

$$\begin{aligned} &\langle A \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix} \rangle = \langle (\mathrm{Re}\, A + i\,\mathrm{Im}\, A) \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix} \rangle \\ &= \langle \begin{bmatrix} 0 & B^* \\ B & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix} \rangle + i \langle \begin{bmatrix} 0 & -iI_k \\ iI_k & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix} \rangle \\ &= 2\,\mathrm{Re}\, \langle Bx, y \rangle + i\, 2\, \mathrm{Im}\, \langle y, x \rangle. \end{aligned} \tag{6}$$

We first prove that $2\,\mathrm{Im}\,\langle y,x\rangle = 1$ if and only if $y = ix$ and $\|x\| = \|y\| = 1/\sqrt{2}$. Assume $2\,\mathrm{Im}\,\langle y,x\rangle = 1$, then

$$1 = 2\,\mathrm{Im}\,\langle y,x\rangle \leqslant 2|\langle y,x\rangle| \leqslant 2\|y\|\|x\| \leqslant 2\cdot\frac{\|x\|^2+\|y\|^2}{2} = 1.$$

It follows that $|\langle y,x\rangle| = \|y\|\|x\|$ and $y = e^{i\theta}x$. Therefore,

$$1 = 2\,\mathrm{Im}\,\langle e^{i\theta}x,x\rangle = 2\,\mathrm{Im}\,e^{i\theta}\|x\|^2 = \mathrm{Im}\,(\cos\theta + i\sin\theta) = \sin\theta.$$

This implies that $y = ix$. Moreover,

$$2\,\mathrm{Re}\,\langle Bx,y\rangle = 2\,\mathrm{Re}\,(-i)\|x\|^2\langle B\frac{x}{\|x\|},\frac{x}{\|x\|}\rangle = \mathrm{Re}\,(-i)\langle B\frac{x}{\|x\|},\frac{x}{\|x\|}\rangle \in W(\mathrm{Im}\,B),$$

and applying (6), we deduce that

$$W(A)\cap\{z\in\mathbb{C} : \mathrm{Im}\,z = 1\} \subseteq \{t+i : t\in W(\mathrm{Im}\,B)\}.$$

Conversely, for any $t\in W(\mathrm{Im}\,B)$, let $t = \langle(\mathrm{Im}\,B)x,x\rangle$ for some unit vector $x\in\mathbb{C}^k$. Replace $\begin{bmatrix} x\\ y\end{bmatrix}$ by $\frac{1}{\sqrt{2}}\begin{bmatrix} x\\ ix\end{bmatrix}$ in (6), we obtain that

$$\langle A\begin{bmatrix} \frac{x}{\sqrt{2}}\\ \frac{ix}{\sqrt{2}}\end{bmatrix},\begin{bmatrix} \frac{x}{\sqrt{2}}\\ \frac{ix}{\sqrt{2}}\end{bmatrix}\rangle = t+i,$$

and hence

$$W(A)\cap\{z\in\mathbb{C} : \mathrm{Im}\,z = 1\} = \{t+i : t\in W(\mathrm{Im}\,B)\}.$$

For any $x,y\in\mathbb{C}^k$ with $\|x\|^2+\|y\|^2 = 1$, we now check that $2\,\mathrm{Re}\,\langle Bx,y\rangle = 1$ if and only if $y = Bx, x\in M$ and $\|x\| = \|y\| = 1/\sqrt{2}$. Suppose that $2\,\mathrm{Re}\,\langle Bx,y\rangle = 1$, then

$$1 = 2\,\mathrm{Re}\,\langle Bx,y\rangle \leqslant 2|\langle Bx,y\rangle| \leqslant 2\|Bx\|\|y\| \leqslant 2\|x\|\|y\| \leqslant 2\cdot\frac{\|x\|^2+\|y\|^2}{2} = 1.$$

We obtain that $\|x\| = \|y\| = 1/\sqrt{2}$, $\|B\| = 1$ and $\|Bx\| = \|B\|\|x\| = \|x\|$. This implies that

$$x\in\ker(I - B^*B) = M.$$

On the other hand, since $|\langle Bx,y\rangle| = \|Bx\|\|y\|$, it follows that $y = e^{i\theta}Bx$. Hence

$$1 = 2\,\mathrm{Re}\,\langle Bx,y\rangle = 2\,\mathrm{Re}\,(e^{-i\theta}\|Bx\|^2) = 2\,\mathrm{Re}\,(e^{-i\theta}\|x\|^2) = \mathrm{Re}\,(\cos\theta - i\sin\theta) = \cos\theta.$$

We thus get $y = Bx$. Moreover,

$$2\,\mathrm{Im}\,\langle y,x\rangle = 2\,\mathrm{Im}\,\|x\|^2\langle B\frac{x}{\|x\|},\frac{x}{\|x\|}\rangle = \mathrm{Im}\,\langle B\frac{x}{\|x\|},\frac{x}{\|x\|}\rangle \in W(\mathrm{Im}\,(P_M B|_M)),$$

and

$$W(A)\cap\{z\in\mathbb{C} : \mathrm{Re}\,z = 1\} \subseteq \{1+it : t\in W(\mathrm{Im}\,B_M)\}.$$

Conversely, for any unit vector $x \in M$, then $\|Bx\| = \|x\| = 1$. A simple computation shows that

$$\langle A \begin{bmatrix} \frac{x}{\sqrt{2}} \\ \frac{Bx}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{x}{\sqrt{2}} \\ \frac{Bx}{\sqrt{2}} \end{bmatrix} \rangle = 1 + i\langle (\mathrm{Im} B_M)x, x\rangle,$$

and hence

$$W(A) \cap \{z \in \mathbb{C} : \mathrm{Re}\, z = 1\} = \{1 + it : t \in W(\mathrm{Im} B_M)\}.$$

Note that

$$\begin{bmatrix} I_k & 0 \\ 0 & -I_k \end{bmatrix} A \begin{bmatrix} I_k & 0 \\ 0 & -I_k \end{bmatrix} = -A.$$

Hence our assertion follows from the fact that $W(A) = W(-A)$. $\square$

We remark that if $B = J_k$, then $W(\mathrm{Im} B) = [-\cos(\pi/(k+1)), \cos(\pi/(k+1))]$ and $W(\mathrm{Im} B_M) = [-\cos(\pi/k), \cos(\pi/k)]$. Therefore, the numerical range $W(A(2k,-1))$ can be described clearly.

For odd $n = 2k-1$, $k \geqslant 3$, Chien and Nakazato show that $W(A(2k-1,-1))$ is the convex hull of the two ellipses $e^{i\pi/4}E$ and $e^{3i\pi/4}E$, where E is the ellipse given by the equation: $x^2/(1+\cos(\pi/k)) + y^2/(1-\cos(\pi/k)) = 1$ (cf. [3, Theorem 7]). Now we can see that

$$e^{i\pi/4}E = \partial W\left(\begin{bmatrix} (1+i)\sqrt{\cos(\pi/k)} & 2\sqrt{1-\cos(\pi/k)} \\ 0 & (-1-i)\sqrt{\cos(\pi/k)} \end{bmatrix}\right)$$

and

$$e^{3i\pi/4}E = \partial W\left(\begin{bmatrix} (-1+i)\sqrt{\cos(\pi/k)} & 2\sqrt{1-\cos(\pi/k)} \\ 0 & (1-i)\sqrt{\cos(\pi/k)} \end{bmatrix}\right).$$

Moreover, these two ellipses $e^{i\pi/4}E$ and $e^{3i\pi/4}E$ are inscribed in the square Q. Notice that if B is the k-by-$(k-1)$ submatrix of J_k^* obtained by deleting its last column, then $B^*B = I_{k-1}$. On the other hand, if $\{e_1, \ldots, e_{2k-1}\}$ denotes the standard basis for $\mathbb{C}^{2k-1}$ and P is the $(2k-1)$-by-$(2k-1)$ permutation matrix so that $Pe_{2j-1} = e_j$ for $1 \leqslant j \leqslant k$ and $Pe_{2j} = e_{k+j}$ for $1 \leqslant j \leqslant k-1$, then

$$PA(2k-1,-1)P^* = \begin{bmatrix} 0 & I_k' + B \\ -I_k' + B^* & 0 \end{bmatrix},$$

where I_k' is the k-by-$(k-1)$ submatrix of I_k obtained by deleting its last column. Therefore, we are interested in the numerical ranges of such a 2-by-2 block matrices for any k-by-$(k-1)$ matrix B with $B^*B = I_{k-1}$. The next theorem shows that the numerical range of such 2-by-2 block matrix is also the convex hull of two ellipses.

Among other things, the k-by-$(k-1)$ matrix B with $B^*B = I_{k-1}$ is a submatrix of an $\mathscr{S}_k$-matrix obtained by deleting its last column. Indeed, let T be an operator in $\mathscr{S}_k$. We will consider a special matrix representation for T. Since $K = \ker(I_k - T^*T)$ has codimension 1, there is an orthonormal basis $\{h_1, \cdots, h_k\}$ of $\mathbb{C}^k$ such that $\{h_1, \cdots, h_{k-1}\}$ forms a basis for K. Let T have the matrix representation $[f_1 \cdots f_k]$ with respect to this basis, where each $f_j = Th_j$ represents a column vector. Since

K consists of all vectors x in $\mathbb{C}^k$ with the property $\|Tx\| = \|x\|$, we have $\|f_j\| = 1$ $(1 \leqslant j \leqslant k-1)$, $\|f_k\| < 1$ and $f_i \perp f_j$ $(1 \leqslant i \neq j \leqslant k)$. Let $B = [f_1 \cdots f_{k-1}]$. It is clear that $B^*B = I_{k-1}$. We can see that $J_k^* \in \mathscr{S}_k$ and the standard basis $\{e_1, \cdots, e_k\}$ for $\mathbb{C}^k$ satisfies $\vee\{e_1, \cdots, e_{k-1}\} = \ker(I_k - J_k J_k^*)$. Hence the result of Chien and Nakazato [3, Theorem 7] is a special case of the following theorem.

THEOREM 3.4. *Suppose that B is an k-by-$(k-1)$ $(k \geqslant 3)$ matrix with $B^*B = I_{k-1}$ and I_k' is the k-by-$(k-1)$ submatrix of I_k obtained by deleting its last column. Let*

$$A = \begin{bmatrix} 0 & I_k' + B \\ -I_k'^* + B^* & 0 \end{bmatrix} \in M_{2k-1},$$

then the numerical range

$$W(A) = W\left(\begin{bmatrix} \sqrt{\beta}(-1+i) & 2\sqrt{1-\beta} \\ 0 & \sqrt{\beta}(1-i) \end{bmatrix} \oplus \begin{bmatrix} \sqrt{(-\alpha)}(1+i) & 2\sqrt{1+\alpha} \\ 0 & \sqrt{(-\alpha)}(-1-i) \end{bmatrix}\right),$$

where $\alpha = \min \sigma(\mathrm{Im} B') \leqslant 0$, $\beta = \max \sigma(\mathrm{Im} B') \geqslant 0$ *and* $B' = I_k'^* B \oplus [0]$.

Proof. Let $\widetilde{A} = A \oplus [0]$, that is,

$$\widetilde{A} = \begin{bmatrix} 0 & I_k'' + B_1 \\ -I_k'' + B_1^* & 0 \end{bmatrix} \in M_{2k},$$

where $I_k'' = \begin{bmatrix} I_k' & 0 \end{bmatrix} = I_{k-1} \oplus [0] \in M_k$ and $B_1 = \begin{bmatrix} B & 0 \end{bmatrix} \in M_k$. Notice that

$$\mathrm{Re}\tilde{A} = \begin{bmatrix} 0 & B_1 \\ B_1^* & 0 \end{bmatrix} \quad \text{and} \quad \mathrm{Im}\tilde{A} = \begin{bmatrix} 0 & -iI_k'' \\ iI_k'' & 0 \end{bmatrix}.$$

Let $f(\theta) = \max \sigma(\mathrm{Re}\,(e^{-i\theta}\tilde{A}))$ for $\theta \in [0, 2\pi)$. Observe that

$$\mathrm{Re}\,(e^{-i\theta}\tilde{A}) = \cos\theta\,\mathrm{Re}\tilde{A} + \sin\theta\,\mathrm{Im}\tilde{A} = \begin{bmatrix} 0 & T_\theta \\ T_\theta^* & 0 \end{bmatrix},$$

where $T_\theta = \cos\theta \cdot B_1 - i\sin\theta \cdot I_k''$. Note that $\begin{bmatrix} 0 & T_\theta \\ T_\theta^* & 0 \end{bmatrix}$ is unitarily equivalent to $\begin{bmatrix} 0 & -T_\theta \\ -T_\theta^* & 0 \end{bmatrix}$. This gives $\sigma(\mathrm{Re}\,(e^{-i\theta}\tilde{A})) = \sigma(-\mathrm{Re}\,(e^{-i\theta}\tilde{A}))$. Hence

$$f(\theta) = \|\mathrm{Re}\,(e^{-i\theta}\tilde{A})\| = \left\| \begin{bmatrix} 0 & T_\theta \\ T_\theta^* & 0 \end{bmatrix} \begin{bmatrix} 0 & T_\theta \\ T_\theta^* & 0 \end{bmatrix} \right\|^{1/2} = \left\| \begin{bmatrix} T_\theta T_\theta^* & 0 \\ 0 & T_\theta^* T_\theta \end{bmatrix} \right\|^{1/2}.$$

Since $T_\theta^* T_\theta = I_k'' - \sin 2\theta \cdot \mathrm{Im} B'$ and $\|T_\theta^* T_\theta\| = \|T_\theta T_\theta^*\|$, we have

$$\begin{aligned} f(\theta) &= \|I_k'' - \sin 2\theta \cdot \mathrm{Im} B'\|^{1/2} \\ &= \begin{cases} \sqrt{1 - \alpha \sin 2\theta}, & \text{if } 0 \leqslant \theta \leqslant \dfrac{\pi}{2} \quad \text{or} \quad \pi \leqslant \theta \leqslant \dfrac{3\pi}{2}; \\ \sqrt{1 - \beta \sin 2\theta}, & \text{if } \dfrac{\pi}{2} \leqslant \theta \leqslant \pi \quad \text{or} \quad \dfrac{3\pi}{2} \leqslant \theta \leqslant 2\pi. \end{cases} \end{aligned}$$

Next, let

$$C = \begin{bmatrix} \sqrt{\beta}(-1+i) & 2\sqrt{1-\beta} \\ 0 & \sqrt{\beta}(1-i) \end{bmatrix}$$

and $g(\theta) = \max \sigma(\mathrm{Re}\,(e^{-i\theta}C))$ for $\theta \in [0, 2\pi)$. An easy computation shows that

$$\mathrm{Re}\,(e^{-i\theta}C) = \begin{bmatrix} -\sqrt{\beta}(\cos\theta - \sin\theta) & \sqrt{1-\beta}(\cos\theta - i\sin\theta) \\ \sqrt{1-\beta}(\cos\theta + i\sin\theta) & \sqrt{\beta}(\cos\theta - \sin\theta) \end{bmatrix}$$

and $g(\theta) = \sqrt{1-\beta\sin 2\theta}$ for $\theta \in [\pi/2, \pi] \cup [3\pi/2, 2\pi]$. Similarly, let

$$D = \begin{bmatrix} \sqrt{(-\alpha)}(1+i) & 2\sqrt{1+\alpha} \\ 0 & \sqrt{(-\alpha)}(-1-i) \end{bmatrix}$$

and $h(\theta) = \max \sigma(\mathrm{Re}\,(e^{-i\theta}D))$ for all $\theta \in [0, 2\pi)$. Since

$$\mathrm{Re}\,(e^{-i\theta}D) = \begin{bmatrix} \sqrt{-\alpha}(\cos\theta + \sin\theta) & \sqrt{1+\alpha}(\cos\theta - i\sin\theta) \\ \sqrt{1+\alpha}(\cos\theta + i\sin\theta) & -\sqrt{-\alpha}(\cos\theta + \sin\theta) \end{bmatrix},$$

it is easy to check that $h(\theta) = \sqrt{1-\alpha\sin 2\theta}$ for $\theta \in [0, \pi/2] \cup [\pi, 3\pi/2]$. Hence we conclude that

$$f(\theta) = \begin{cases} h(\theta), \text{ if } 0 \leqslant \theta \leqslant \dfrac{\pi}{2} \quad \text{or} \quad \pi \leqslant \theta \leqslant \dfrac{3\pi}{2}; \\ g(\theta), \text{ if } \dfrac{\pi}{2} \leqslant \theta \leqslant \pi \quad \text{or} \quad \dfrac{3\pi}{2} \leqslant \theta \leqslant 2\pi, \end{cases}$$

or, $W(\tilde{A}) = W(A) = W(C \oplus D)$, thus completing the proof. □

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