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ONE–DIMENSIONAL SCHRÖDINGER OPERATORS WITH δ'–INTERACTIONS ON A SET OF LEBESGUE MEASURE ZERO

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Abstract. Let Γ be a compact subset of $\mathbb{R}$ of Lebesgue measure zero. The notion 'Schrödinger operator defining a δ'-interaction on Γ' is introduced. The dimension of the range of the spectral projection $\chi_{(-\infty,0)}(A)$ of a Schrödinger operator A defining a δ'-interaction on Γ is not less than the number of isolated points of Γ where the intensity of the δ'-interaction is negative. In the case that the set Γ is endowed with a Radon measure a method how to construct a large class of such operators is presented and for the operators from this class it is shown that their absolutely continuous spectra and their essential spectra are equal to the nonnegative real half-axis. Constructive examples of such operators with infinitely many negative eigenvalues are given.

1. Introduction

Let Γ be a compact subset of $\mathbb{R}$ of Lebesgue measure zero. One says that A is a Schrödinger operator defining an interaction on Γ, if A is a self-adjoint operator in $L_2(\mathbb{R})$, its domain $\mathfrak{D}(A)$ contains the space $C_0^\infty(\mathbb{R}\setminus\Gamma)$ of smooth functions with compact support in $\mathbb{R}\setminus\Gamma$ and $A\psi = -\psi''$ for every $\psi \in C_0^\infty(\mathbb{R}\setminus\Gamma)$. Since the pioneering work of Berezin and Faddeev [8] such singular Schrödinger operators have been studied in numerous publications, cf., e.g., [1, 4, 5, 7, 9–12, 14–16, 19–23].

One is strongly interested in point interactions, i.e. interactions on a discrete set, because in this case one gets solvable models in quantum mechanics [2, 3]. The starting point for this paper is a certain kind of point interactions, the so called δ'-interactions. Usually δ'-interactions on a finite set are described with the aid of certain boundary conditions, cf. (2.10). We solve the problem how to define δ'-interactions on an arbitrary compact subset Γ of $\mathbb{R}$ of Lebesgue measure zero in the following way. We give certain properties $P_1(\Gamma)$, $P_2(\Gamma)$, $P_3(\Gamma)$ for operators A in $L_2(\mathbb{R})$ and prove that an operator A in $L_2(\mathbb{R})$ defines a δ'-interaction on the finite set X if, and only if, it has the properties $P_1(X)$, $P_2(X)$, $P_3(X)$ (Proposition 2.3). Our new characterization of δ'-interactions on finite sets X is then used in order to introduce δ'-interactions on Γ. By definition the operator A in $L_2(\mathbb{R})$ defines a δ'-interaction on Γ if, and only if, it has the properties $P_1(\Gamma)$, $P_2(\Gamma)$, $P_3(\Gamma)$ (Definition 2.3).

For operators A defining a δ'-interaction on a compact null set Γ we show that the dimension of the range of the spectral projection $\chi_{(-\infty,0)}(A)$ is not less than the number

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of isolated points of Γ where the intensity of the δ'-interaction is negative (Theorem 3.1). If A has a pure point spectrum below zero than this dimension is equal to the number (counting multiplicities) of negative eigenvalues of A.

We present a method how to construct a large class of operators defining a δ'-interaction with the aid of Radon measures supported by Γ and certain boundary conditions on Γ (Theorem 5.2, 1°). For the operators from this special class we prove that their absolutely continuous spectra and their essential spectra are equal to the nonnegative real half-axis, and zero is not an accumulation point of the set of their negative eigenvalues (Theorem 5.2, 2°, 3°). Moreover we give a condition that is sufficient in order that such operators have infinitely many negative eigenvalues (Theorem 5.3).

2. Local interactions on a set of measure zero

Let Γ be a compact subset of $\mathbb{R}$ of Lebesgue measure zero. The minimal operator $L_{\min,\Gamma}$ in the space $L_2(\mathbb{R})$ is defined as follows:

$$\mathfrak{D}(L_{\min,\Gamma}) = C_0^\infty(\mathbb{R}\setminus\Gamma), \quad L_{\min,\Gamma}\varphi(x) = -\varphi''(x).$$

The maximal operator $L_{\max,\Gamma}$ in $L^2(\mathbb{R})$ is the adjoint of the minimal operator:

$$\mathfrak{D}(L_{\max,\Gamma}) = W_2^2(\mathbb{R}\setminus\Gamma), \quad L_{\max,\Gamma}\psi(x) = -\psi''(x), \quad x\notin\Gamma,$$

where the Sobolev space $W_2^2(\mathbb{R}\setminus\Gamma)$ consists of the functions ψ, such that ψ and ψ' are absolutely continuous on $\mathbb{R}\setminus\Gamma$ and ψ,ψ',ψ'' are square integrable. Note that for $\psi\in W_2^2(\mathbb{R}\setminus\Gamma)$ the limits $\psi(a+0),\psi'(a+0),\psi(b-0),\psi'(b-0)$ exist, if the open interval (a,b) is contained in $\mathbb{R}\setminus\Gamma$.

DEFINITION 2.1. An operator A in $L_2(\mathbb{R})$ defines a local interaction on Γ if A is a Schrödinger operator defining an interaction on Γ, and $\psi\in\mathfrak{D}(A)$ implies that $\chi\psi\in\mathfrak{D}(A)$ for every function $\chi\in C^\infty(\mathbb{R})$ such that $\chi'\in C_0^\infty(\mathbb{R}\setminus\Gamma)$.

REMARK 2.1. We neither exclude the trivial case $\Gamma=\emptyset$ nor 'the trivial interaction', i.e. the kinetic energy Hamiltonian $-\Delta$ in $L_2(\mathbb{R})$, viz.

$$-\Delta\psi = -\psi'', \quad \psi\in\mathfrak{D}(-\Delta) = W_2^2(\mathbb{R}),$$

defines an interaction on Γ for every compact subset Γ of $\mathbb{R}$ of Lebesgue measure zero.

LEMMA 2.1. *Let A be a self-adjoint operator in the space $L_2(\mathbb{R})$ defining a local interaction on Γ. Let $x_0\in\mathbb{R}\setminus\Gamma$ or let x_0 be an isolated point of the set Γ. Then, for all functions $\varphi,\psi\in\mathfrak{D}(A)$*

$$\int_{-\infty}^{x_0} [(A\varphi)(x)\overline{\psi(x)} - \varphi(x)\overline{(A\psi)(x)}]\,dx = \varphi(x_0-0)\overline{\psi'(x_0-0)} - \varphi'(x_0-0)\overline{\psi(x_0-0)}, \tag{2.1}$$

$$\int_{x_0}^{+\infty}[(A\varphi)(x)\overline{\psi(x)}-\varphi(x)\overline{(A\psi)(x)}]\,dx=\varphi'(x_0+0)\overline{\psi(x_0+0)}-\varphi(x_0+0)\overline{\psi'(x_0+0)}. \tag{2.2}$$

Proof. (2.2) follows from (2.1). In fact, since for any a

$$\int_a^{\infty}[(A\varphi)(x)\overline{\psi(x)}-\varphi(x)\overline{(A\psi)(x)}]\,dx=-\int_{-\infty}^{a}[(A\varphi)(x)\overline{\psi(x)}-\varphi(x)\overline{(A\psi)(x)}]\,dx,$$

we obtain the equality (2.2) if we set $a=x_0+\varepsilon\notin\Gamma$, apply (2.1) and pass to the limit $\varepsilon\to+0$.

In order to prove (2.1) first consider the case that $x_0\notin\Gamma$. Let $b>x_0$ be such that the interval (x_0,b) does not contain any point of Γ. We may assume that $\varphi(x)=0$ and $\psi(x)=0$ for every $x\geqslant b$. This may be achieved by multiplying φ and ψ by a function $\chi\in C^\infty(\mathbb{R})$ such that $\chi(x)=1$ for $x\leqslant x_0$ and $\chi(x)=0$ for $x\geqslant b$. The functions $\chi\varphi$ and $\chi\psi$ belong to the domain of the operator A since the operator A defines a local interaction on Γ. Therefore

$$\int_{-\infty}^{x_0}[(A\varphi)(x)\overline{\psi(x)}-\varphi(x)\overline{(A\psi)(x)}]\,dx=(A\varphi,\psi)-(\varphi,A\psi)-\int_{x_0}^{b}[\varphi\cdot\overline{\psi''}-\varphi''\overline{\varphi}]\,dx.$$

This leads to equality (2.1) since A is self–adjoint and $\varphi(b)=\varphi'(b)=\psi(b)=\psi'(b)=0$.

Finally let $x_0\in\Gamma$ be an isolated point of the set Γ. Then $x_0-\varepsilon\notin\Gamma$ for small $\varepsilon>0$. Applying equality (2.1) for $x_0-\varepsilon$ and passing to the limit $\varepsilon\to+0$, we receive that (2.1) holds. $\square$

PROPOSITION 2.1. *Let x_0 be an isolated point of Γ or let $x_0\in\mathbb{R}\setminus\Gamma$. Let A be an operator in $L_2(\mathbb{R})$ defining a local interaction on Γ and let $A_\circ$ be the restriction of A on the set*

$$\mathfrak{D}(A_\circ):=\{\psi\in\mathfrak{D}(A):\ \psi(x_0\pm0)=0=\psi'(x_0\pm0)\}.$$

Then the following holds:

1° *$A_\circ$ has deficiency indices $(2,2)$.*

2°

$$\mathfrak{D}(A_\circ^*)=\{\psi_1+\psi_2:\ \psi_1\in\mathfrak{D}(A_\circ),\ \psi_2\in W_2^2(\mathbb{R}\setminus\{x_0\}),\ supp(\psi_2)\cap\Gamma\subset\{x_0\}\}. \tag{2.3}$$

3° *$\tilde{A}$ is a self-adjoint extension of $A_\circ$ if, and only if, there exists a Lagrangian plane L for the form*

$$\omega(v,w):=v_3\overline{w_1}-v_1\overline{w_3}-v_4\overline{w_2}+v_2\overline{w_4},\quad v,w\in\mathbb{C}^4, \tag{2.4}$$

such that $\tilde{A}$ is the restriction of $A_\circ^*$ on the space

$$\mathfrak{D}(\tilde{A}) = \{\psi \in \mathfrak{D}(A_\circ^*) : J\psi \in L\}, \tag{2.5}$$

where

$$J\psi := \begin{bmatrix} \psi(x_0+0) \\ \psi(x_0-0) \\ \psi'(x_0+0) \\ \psi'(x_0-0) \end{bmatrix}, \quad \psi \in W_2^2(\mathbb{R}\setminus\{x_0\}). \tag{2.6}$$

(Recall that a Lagrangrian plane L for ω is a maximal subspace of $\mathbb{C}^4$ such that $\omega(v,v) = 0$ for every $v \in L$).

4° *Every self-adjoint extension of $A_\circ$ defines a local interaction on $\Gamma \cup \{x_0\}$.*

5° *The restriction of $L_{\max,\Gamma}$ on the set*

$$\{\psi_1 + \psi_2 : \psi_1 \in \mathfrak{D}(A_\circ), \psi_2 \in W_2^2(\mathbb{R}), supp(\psi_2) \cap \Gamma \subset \{x_0\}\}$$

defines a local interaction on $\Gamma \setminus \{x_0\}$.

6° *If $\tilde{A}_1$ and $\tilde{A}_2$ are self-adjoint extensions of $A_\circ$, then the difference $(\tilde{A}_1 - z)^{-1} - (\tilde{A}_2 - z)^{-1}$ is a finite-rank-operator for every z that belongs both to the resolvent set of $\tilde{A}_1$ and to the resolvent set of $\tilde{A}_2$.*

Proof. 1° Put $\hat{\Gamma} := \Gamma \cup \{x_0\}$ and

$$\mathfrak{D}_0 := \{\psi \in W_2^2(\mathbb{R}\setminus\{x_0\}) : \operatorname{supp}(\psi) \cap \Gamma \subset \{x_0\}\}.$$

Integrating by parts we obtain that

$$(L_{\max,\hat{\Gamma}}\psi, \varphi) - (\psi, L_{\max,\hat{\Gamma}}\varphi) = \omega(J\psi, J\varphi), \quad \psi \in \mathfrak{D}_0,\ \varphi \in W_2^2(\mathbb{R}\setminus\hat{\Gamma}). \tag{2.7}$$

Since A defines a local interaction, its domain is contained in

$$\mathfrak{D}_1 := \{\psi_1 + \psi_2 : \psi_1 \in \mathfrak{D}(A_\circ), \psi_2 \in \mathfrak{D}_0\}.$$

By (2.1), (2.2), and (2.7),

$$(L_{\max,\hat{\Gamma}}\psi, \varphi) - (\psi, L_{\max,\hat{\Gamma}}\varphi) = \omega(J\psi, J\varphi), \quad \psi, \varphi \in \mathfrak{D}_1. \tag{2.8}$$

Since A is a restriction of $L_{\max,\hat{\Gamma}}$ this implies that $\omega(J\psi, J\psi) = 0$ for every $\psi \in \mathfrak{D}(A)$. Choose a Lagrangian plane L for ω such that $\{J\psi : \psi \in \mathfrak{D}(A)\} \subset L$. By (2.8), the restriction of $L_{\max,\hat{\Gamma}}$ on the space $\{\psi \in \mathfrak{D}_1 : J\psi \in L\}$ is a symmetric extension of A. Since A is self-adjoint, it is, in particular, maximal symmetric, i.e. it is symmetric and does not possess any proper symmetric extension. Thus

$$\mathfrak{D}(A) = \{\psi \in \mathfrak{D}_1 : J\psi \in L\}.$$

Since the mapping $J : \mathfrak{D}_1 \longrightarrow \mathbb{C}^4$ is surjective, $\{J\psi : \psi \in \mathfrak{D}(A)\} = L$. Choose $\psi_1, \psi_2 \in \mathfrak{D}(A)$ such that $J\psi_1, J\psi_2$ is a basis of L. Then for every $\psi \in \mathfrak{D}(A)$ there exist $c_1, c_2 \in \mathbb{C}$ such that $J(\psi - c_1\psi_1 - c_2\psi_2) = 0$ and hence $\psi - c_1\psi_1 - c_2\psi_2 \in \mathfrak{D}(A_\circ)$. Thus the dimension of the quotient space $\mathfrak{D}(A)/\mathfrak{D}(A_\circ)$ is equal to 2. This implies that $A_\circ$ has deficiency indices $(2,2)$.

2°: By 1°, the dimension of the quotient space $\mathfrak{D}(A_\circ^*)/\mathfrak{D}(A_\circ)$ is equal to 4. Since $\dim(\mathfrak{D}_1/\mathfrak{D}(A_\circ)) = 4$ and, by (2.8), $\mathfrak{D}_1 \subset \mathfrak{D}(A_\circ^*)$, it follows that

$$\mathfrak{D}(A_\circ^*) = \mathfrak{D}_1.$$

3° By 1°, an extension $\tilde{A}$ of $A_\circ$ is self-adjoint if, and only if, it is a maximal symmetric restriction of $A_\circ^*$. Since A^* is a restriction of $L_{\max,\hat{\Gamma}}$, it follows from (2.3) and (2.8), that

$$(A_\circ^*\psi, \varphi) - (\psi, A_\circ^*\varphi) = \omega(J\psi, J\varphi), \quad \psi, \varphi \in \mathfrak{D}(A_\circ^*),$$

and the mapping $J : \mathfrak{D}(A_\circ^*) \longrightarrow \mathbb{C}^4$ is surjective. Thus a restriction $\tilde{A}$ of $A_\circ^*$ is maximal symmetric if, and only if, there exists a Lagrangian plane L for ω such that

$$\mathfrak{D}(\tilde{A}) = \{\psi \in \mathfrak{D}(A_\circ^*) : J\psi \in L\}.$$

4° Let $\tilde{A}$ be a self-adjoint extension of $A_\circ$ and L the Lagrangian plane for ω such that (2.5) holds. Let $\chi \in C^\infty(\mathbb{R})$ and $\chi' \in C_0^\infty(\mathbb{R} \setminus \hat{\Gamma})$. Then $\chi\psi_1 \in \mathfrak{D}(A)$ for every $\psi_1 \in \mathfrak{D}(A)$, since A defines a local interaction on Γ. Moreover $J(\chi\psi) = \chi(x_0)J\psi$ for every $\psi \in W_2^2(\mathbb{R} \setminus \{x_0\})$, since χ is constant on a neighborhood of x_0. Thus $\chi\psi_1 \in \mathfrak{D}(A_\circ)$ and $\chi\psi_2 \in \mathfrak{D}_0$ for every $\psi_1 \in \mathfrak{D}(A_\circ)$ and every $\psi_2 \in \mathfrak{D}_0$, respectively. It follows now from (2.3), that $\chi\psi \in \mathfrak{D}(A_\circ^*)$ for every $\psi \in \mathfrak{D}(A_\circ^*)$. Since $J(\chi\psi) = \chi(x_0)J\psi \in L$ for every $\psi \in \mathfrak{D}(\tilde{A})$, this implies that $\chi\psi \in \mathfrak{D}(\tilde{A})$ for every $\psi \in \mathfrak{D}(\tilde{A})$ and $\tilde{A}$ defines a local interaction on $\hat{\Gamma}$.

5° $L_0 := \{v \in \mathbb{C}^4 : v_1 = v_2, v_3 = v_4\}$ is a Lagrangian plane for ω, and $A_\circ^*$ is a restriction of $L_{\max,\hat{\Gamma}}$. By 3°, this implies that the restriction $\tilde{A}_0$ of $L_{\max,\hat{\Gamma}}$ on the space $\{\psi \in \mathfrak{D}(A^*) : J\psi \in L_0\}$ is a self-adjoint extension of $A_\circ$ and hence it defines a local interaction on Γ. Let $\psi_2 \in W_2^2(\mathbb{R} \setminus \{x_0\})$. Then $\psi_2 \in W_2^2(\mathbb{R})$ if, and only if, $J\psi_2 \in L_0$, and it follows from 1° that

$$\mathfrak{D}(\tilde{A}_0) = \{\psi_1 + \psi_2 : \psi_1 \in \mathfrak{D}(A_\circ), \psi_2 \in W_2^2(\mathbb{R}), \operatorname{supp}(\psi_2) \cap \Gamma \subset \{x_0\}\}.$$

Thus the space $C_0^\infty(\mathbb{R} \setminus (\Gamma \setminus \{x_0\}))$ is contained in the domain of $\tilde{A}_0$ and $\tilde{A}_0$ defines even a local interaction on $\Gamma \setminus \{x_0\}$.

6° follows immediately from 1° and Krein's resolvent formula. $\square$

REMARK 2.2. 1° Since the form ω in the preceeding proposition satisfies

$$\omega(v,w) = \left(\begin{bmatrix} v_3 \\ -v_4 \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \right)_{\mathbb{C}^2} - \left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \begin{bmatrix} w_3 \\ -w_4 \end{bmatrix} \right)_{\mathbb{C}^2}, \quad v, w \in \mathbb{C}^4,$$

L is a Lagrangian plane for ω if, and only if, there exists a $U \in U(2)$ such that

$$L = \{v \in \mathbb{C}^4 : \begin{bmatrix} v_3 + iv_1 \\ -v_4 + iv_2 \end{bmatrix} = U \cdot \begin{bmatrix} v_3 - iv_1 \\ -v_4 - iv_2 \end{bmatrix}\}.$$

By assertion 3° in the preceeding proposition this implies that $\tilde{A}$ is a self-adjoint extension of $A_\circ$ if, and only if, there exists a $U \in U(2)$ such that $\tilde{A}$ is the restriction of $A_\circ^*$ on the space of all ψ from the domain of $A_\circ^*$ satisfying the following boundary condition:

$$\begin{bmatrix} \psi'(x_0+0) + i\psi(x_0+0) \\ -\psi'(x_0-0) + i\psi(x_0-0) \end{bmatrix} = U \cdot \begin{bmatrix} \psi'(x_0+0) - i\psi(x_0+0) \\ -\psi'(x_0-0) - i\psi(x_0-0) \end{bmatrix}.$$

2° In the special case that $\Gamma = \emptyset$ the operator A is the kinetic energy Hamiltonian $-\Delta$ and every self-adjoint extension of $A_\circ$ defines a so called point-interaction at the point x_0.

We are interested in a special kind of local interaction, the so called δ'-interaction. For an isolated point x_0 of the set Γ a δ'-interaction at x_0 is defined as follows:

DEFINITION 2.2. Let A be a self-adjoint operator in $L_2(\mathbb{R})$ defining a local interaction on Γ and let x_0 be an isolated point of Γ. The operator A defines a δ'-interaction at the point x_0, if there exists a real number β such that

$$\psi'(x_0-0) = \psi'(x_0+0) =: \psi_r'(x_0), \quad \psi(x_0+0) - \psi(x_0-0) = \beta\psi_r'(x_0), \quad \psi \in \mathfrak{D}(A). \tag{2.9}$$

The real number β is called the intensity of the δ'-interaction at the point x_0.

For a Borel subset B of $\mathbb{R}$ we denote the characteristic function of B by χ_B, i.e. $\chi_B(x) = 1$ for $x \in B$ and $\chi_B(x) = 0$ for $x \in \mathbb{R} \setminus B$, and we put $L_2(B) := \{\chi_B\psi : \psi \in L_2(\mathbb{R})\}$. The following simple observation will play a key role in our discussion of δ'-interactions at isolated points.

PROPOSITION 2.2. *Let A be a self-adjoint operator in $L_2(\mathbb{R})$ defining a local interaction on Γ and let x_0 be an isolated point of Γ. Suppose that there exists a differentiable function $\chi \in \mathfrak{D}(A)$ such that $\chi(x_0) \neq 0$ and $\chi'(x_0) = 0$. Then*

either *there exists a real number β such that the operator A defines a δ'-interaction at the point x_0 with intensity β and every $\psi \in W_2^2(\mathbb{R} \setminus \{x_0\})$ such that supp$(\psi) \cap \Gamma \subset \{x_0\}$ and ψ satisfies the boundary condition (2.9) belongs to the domain of A,*

or *every $\psi \in \mathfrak{D}(A)$ satisfies Neumann boundary conditions at x_0, i.e*

$$\psi'(x_0-0) = \psi'(x_0+0) = 0, \quad \psi \in \mathfrak{D}(A),$$

and there exists an operator A_1 in $L_2(-\infty, x_0)$ and an operator A_2 in $L_2(x_0, \infty)$ such that $A = A_1 \oplus A_2$.

Proof. We may assume that $\chi(x_0) = 1$. As in the previous proposition let $A_\circ$ be the restriction of A on the space of all $\psi \in \mathfrak{D}(A)$ such that $\psi(x_0 \pm 0) = \psi'(x_0 \pm 0) = 0$ and let L be the Lagrangian plane for the form

$$\omega(v,w) = v_3\bar{w}_1 - v_1\bar{w}_3 - v_4\bar{w}_2 + v_2\bar{w}_4, \quad v,w \in \mathbb{C}^4,$$

satisfying $\mathfrak{D}(A) = \{\psi \in \mathfrak{D}(A_\circ^*) : J\psi \in L\}$. Since $\omega(J\chi, v) = \bar{v}_4 - \bar{v}_3 = 0$ for every $v \in L$, we have that $v_3 = v_4$ for every $v \in L$.

If there exists a $v \in L$ such that $v_3 \neq 0$, then there exist $v \in L$ such that $v_3 = v_4 = 1$. For $v,w \in L$ such that $v_3 = v_4 = w_3 = w_4 = 1$ we have

$$\omega(v,w) = \bar{w}_1 - v_1 - \bar{w}_2 + v_2 = 0;$$

the special case that $v = w$ provides that $v_1 - v_2$ is real and then it follows that there exists a real number β such that $v_1 - v_2 = \beta$ for every $v \in L$ satisfying $v_3 = v_4 = 1$. Since L is a linear space it follows that $v_1 - v_2 = \beta v_3$, if $v \in L$ and $v_3 \neq 0$. If $v \in L$ and $v_3 = 0$, then $v_4 = 0$ and for every $w \in L$ such that $w_3 = 1 = w_4$ we have $\omega(v,w) = -v_1 + v_2 = 0$. Thus $v_3 = v_4$ and $v_1 - v_2 = \beta v_3$ for every $v \in L$. It follows that $\psi \in W_2^2(\mathbb{R} \setminus \{x_0\})$ satisfies $J\psi \in L$ if, and only if, ψ satisfies the boundary condition (2.9). Thus A defines a δ'-interaction with intensity β at the point x_0 and every $\psi \in W_2^2(\mathbb{R} \setminus \{x_0\})$ such that $\operatorname{supp}(\psi) \cap \Gamma \subset \{x_0\}$ and ψ satisfies the boundary condition (2.9) belongs to the domain of A.

In the case that $v_3 = v_4 = 0$ for every $v \in L$ the functions ψ from the domain of A satisfy Neumann boundary conditions at the point x_0, the Lagrangian plane L is equal to the space of all $v \in \mathbb{C}^4$ such that $v_3 = v_4 = 0$, and A does not define a δ'-interaction at the point x_0. It only remains to prove that $\chi_{(-\infty,x_0)}\psi$ belongs to the domain of A for every ψ from the domain of A. Let $\psi \in \mathfrak{D}(A)$. Then $J(\chi_{(-\infty,x_0)}\psi) \in L$ and, by (2.1), $\chi_{(-\infty,x_0)}\psi \in \mathfrak{D}(A_\circ^*)$, and hence $\chi_{(-\infty,x_0)}\psi \in \mathfrak{D}(A)$. $\square$

Definition 2.2 is motivated by the well known work on δ'-point interactions. Let $X = \{x_k\}_{k=1}^n$ be a finite subset of $\mathbb{R}$ with n points and let $\beta = \{\beta_k\}_{k=1}^n$ be a family of real numbers. Denote by $L_{X,\beta}$ the restriction of $L_{\max,X}$ on the space of all $\psi \in W_2^2(\mathbb{R} \setminus X)$ satisfying the following boundary condition:

$$\psi'(x_k - 0) = \psi'(x_k + 0) =: \psi_r'(x_k), \quad \psi(x_k + 0) - \psi(x_k - 0) = \beta_k \psi_r'(x_k), \quad 1 \leqslant k \leqslant n. \tag{2.10}$$

One says that $L_{X,\beta}$ defines a δ'-interaction on X with intensity β. Note that the trivial interaction, i.e. the case $\beta_k = 0$, is not excluded. With the aid of Proposition 2.2, we can derive another characterization of operators defining a δ'-interaction on a finite set X:

PROPOSITION 2.3. *Let X be a finite subset of $\mathbb{R}$. The operator A in $L_2(\mathbb{R})$ defines a δ'-interaction on X if, and only if, it has the following three properties:*

$P_1(X)$ *A defines a local interaction on X.*

$P_2(X)$ *Every function χ from $C_0^\infty(\mathbb{R})$ such that $\chi' \in C_0^\infty(\mathbb{R} \setminus X)$ belongs to the domain of A.*

$P_3(X)$ *There does not exist a point $a \in X$ such that $A = A_1 \oplus A_2$ for any operators A_1 and A_2 in $L_2(-\infty, a)$ and $L_2(a, \infty)$, respectively.*

Proof. Necessity. Let A be an operator in $L_2(\mathbb{R})$ that defines a δ'-interaction on X.

$P_1(X)$ A is a self-adjoint extension of the minimal operator $L_{\min,X}$ ([2], p. 155). Let $\chi \in C^\infty(\mathbb{R})$ and $\chi' \in C_0^\infty(\mathbb{R} \setminus X)$. Then for every point a of X the function χ is constant on a neighborhood of a. Thus $\chi\psi$ satisfies the boundary condition (2.10), if ψ satisfies (2.10), and hence $\chi\psi \in \mathfrak{D}(A)$ for every $\psi \in \mathfrak{D}(A)$. Thus A has the property $P_1(X)$.

$P_2(X)$ Every $\chi \in C_0^\infty(\mathbb{R})$ such that $\chi' \in C_0^\infty(\mathbb{R} \setminus X)$ satisfies the boundary condition (2.10) for every choice of the real numbers β_k and hence it belongs to the domain of A.

$P_3(X)$ Choose any $\chi \in C_0^\infty(\mathbb{R})$ that is equal to 1 on a neighborhood of X. Let $a \in X$. Then $\chi \in \mathfrak{D}(A)$ and $\chi_{(-\infty,a)}\chi \notin \mathfrak{D}(A)$. Thus A cannot be decomposed as $A = A_1 \oplus A_2$ for an operator A_1 and A_2 in $L_2(-\infty, a)$ and $L_2(a, \infty)$, respectively.

Sufficiency. Suppose that the operator A in $L_2(\mathbb{R})$ has the properties $P_1(X)$, $P_2(X)$ and $P_3(X)$ and X consists of the n points $x_1, x_2, \ldots, x_n$. Let $1 \leqslant k \leqslant n$. Since there exists a differentiable function χ from the domain of A such that $\chi'(x_k) = 0$ and $\chi(x_k) \neq 0$, it follows from Proposition 2.2, that there exists a real number β_k such that A defines a δ'-interaction at the point x_k with intensity β_k. Thus A is a restriction of the operator $L_{X,\{\beta_k\}_{k=1}^n}$. Since A and $L_{X,\{\beta_k\}_{k=1}^n}$ are self-adjoint, this implies that $A = L_{X,\{\beta_k\}_{k=1}^n}$. $\square$

By the preceeding proposition, the following definition generalizes the definition of δ'-interactions on finite sets.

DEFINITION 2.3. Let Γ be a compact subset of $\mathbb{R}$ of Lebesgue measure zero. The operator A in $L_2(\mathbb{R})$ defines a δ'-interaction on Γ if, and only if, it has the following three properties:

$P_1(\Gamma)$ A defines a local interaction on Γ.

$P_2(\Gamma)$ Every function χ from $C_0^\infty(\mathbb{R})$ such that $\chi' \in C_0^\infty(\mathbb{R} \setminus \Gamma)$ belongs to the domain of A.

$P_3(\Gamma)$ There does not exist a point $a \in \Gamma$ such that $A = A_1 \oplus A_2$ for any operators A_1 and A_2 in $L_2(-\infty, a)$ and $L_2(a, \infty)$, respectively.

3. Number of negative eigenvalues for δ'-interactions

It is known [13, 17, 18] that for an operator A that defines a δ'-interaction on a discrete set the number (counting multiplicities) of negative eigenvalues of A and hence the dimension of the range of the spectral projection $\chi_{(-\infty,0)}(A)$ is equal to the number of points where the intensity of the δ'-interaction is negative. An analogous result is true for operators defining a δ'-interaction on any compact set Γ of Lebesgue measure zero, cf. Theorem 3.1 and Remark 3.3 below.

For the proof of the theorem we shall use special test functions: Let Γ be a compact subset of $\mathbb{R}$ of Lebesgue measure zero. Let x_0 be an isolated point of Γ. Let A be a self-adjoint operator in $L_2(\mathbb{R})$ that defines a δ'-interaction on Γ. Then, in particular, the functions $\psi \in \mathfrak{D}(A)$ satisfy the boundary condition (2.9) at the point x_0 where β is the intensity of the δ'-interaction at the point x_0. We construct a function that belongs to $\mathfrak{D}(A)$, has compact support, satisfies condition (2.9) at the point x_0, and consists piecewise of parabolas and constants.

DEFINITION 3.4. Consider the following test function that depends on 4 parameters ε, β, l, r:

$$t(x,\varepsilon,\beta,l,r) = \begin{cases} 0, & x \leqslant -\varepsilon, \\ \dfrac{1}{2\varepsilon}(x+\varepsilon)^2, & -\varepsilon \leqslant x < 0, \\ \beta+\varepsilon-\frac{1}{2\varepsilon}(x-\varepsilon)^2, & 0 < x \leqslant \varepsilon, \\ \beta+\varepsilon, & \varepsilon \leqslant x \leqslant l, \\ \beta+\varepsilon-\frac{\beta+\varepsilon}{2r^2}(l-x)^2, & l \leqslant x \leqslant l+r, \\ \dfrac{\beta+\varepsilon}{2r^2}(l+2r-x)^2, & l+r \leqslant x \leqslant l+2r, \\ 0, & l+2r \leqslant x. \end{cases}$$

It is straightforward to prove the following proposition:

PROPOSITION 3.4. *Let $\varepsilon, l.r > 0$ and $x_0, \beta \in \mathbb{R}$. Put*

$$\tilde{t}(x) := t(x_0 - x, \varepsilon, \beta, l.r), \quad x \in \mathbb{R}.$$

Then the following holds:

1^0 *The function $\tilde{t}$ belongs to the space $W_2^2(\mathbb{R} \setminus \{x_0\})$ and has compact support.*

2^0 *$\tilde{t}$ satisfies the boundary condition (2.9).*

3^0 *If $0 < \varepsilon \leqslant \varepsilon_0$ is such that the $2\varepsilon_0$-neighborhood of the point x_0 does not contain any point of Γ different from x_0 and the value of l is larger than the diameter of Γ, then $\tilde{t}$ belongs to the domain of every self-adjoint operator A defining a δ'-interaction on Γ and the δ'-interaction with intensity β at the point x_0.*

4^0 *If 3^0 holds, then $(A\tilde{t},\tilde{t}) = \beta + \dfrac{2}{3}\varepsilon + \dfrac{2}{3r}(\beta+\varepsilon)^2$ for every self-adjoint operator A defining a δ'-interaction on Γ and the δ'-interaction with intensity β at the point x_0.*

THEOREM 3.1. *Let A be an operator in $L_2(\mathbb{R})$ that defines a δ'-interaction on a compact set Γ of Lebesgue measure zero. Then the dimension of the range of the spectral projection $\chi_{(-\infty,0)}(A)$ is not less than the number of isolated points of Γ where the intensity of the δ'-interaction is negative.*

Proof. Let $x_1,\dots,x_n$ be isolated points of the set Γ such that the intensity of the δ'-interaction at the point x_k, $k=1,\dots,n$, is equal to the negative number β_k. Let $\varepsilon_0>0$ be such that for $k=1,\dots,n$ the $2\varepsilon_0$-neighborhood of x_k does not contain any point of the set Γ different from x_k.

Let $\mathscr{L}_n$ be the n-dimensional subspace of $\mathfrak{D}(A)$ spanned by the test-functions $t_k(x)=t(x-x_k;\beta_k,\varepsilon_k,l_k,r_k)$, $k=1,\dots,n$. Here the numbers ε_k,r_k and l_k are chosen such that $0<\varepsilon_k\leqslant\varepsilon_0$,

$$\beta_k+\frac{2}{3}\varepsilon_k+\frac{2}{3r_k}(\beta_k+\varepsilon_k)^2=\frac{1}{2}\beta_k<0 \tag{3.1}$$

and l_k is larger than the diameter of Γ for $k=1,\dots,n$ and the intervals $I_k=(l_k,l_k+2r_k)$, $k=1,\dots,n$, are pairwise disjoint.

Every function $\psi\in\mathscr{L}_n$ can be represented as

$$\psi(x)=\sum_{k=1}^{n}a_kt_k(x), \tag{3.2}$$

where a_k are complex constants. By Proposition 3.4 and (3.1) it is easy to see that the quadratic form $(A\psi,\psi)$ is negative definite on the n-dimensional subspace $\mathscr{L}_n$, i.e. for $\psi\in\mathscr{L}_n\setminus\{0\}$ we have

$$(A\psi,\psi)=\sum_{k=1}^{n}|a_k|^2(At_k,t_k)=\frac{1}{2}\sum_{k=1}^{n}\beta_k|a_k|^2<0. \tag{3.3}$$

Let $\mathscr{L}_-$ be the range of the spectral projection $\chi_{(-\infty,0)}(A)$. If the dimension of $\mathscr{L}_-$ would be smaller than n, then we could choose $\psi\in\mathscr{L}_n\setminus\{0\}$ that are orthogonal to $\mathscr{L}_-$ and hence satisfy $(A\psi,\psi)\geqslant 0$. This contradicts (3.3). $\square$

REMARK 3.3. If, in addition, the operator A in the previous theorem has a pure point spectrum below 0, then it follows that the number (counting multiplicities) of negative eigenvalues of A is not less than the number of isolated points of Γ where the intensity of the δ'-interaction is negative. In the following section we shall introduce a large class of Schrödinger operators defining a δ'-interaction on Γ such that their negative spectra are even discrete.

4. Boundary conditions for δ'-interactions

In this section we shall show how to construct a large class of operators defining a δ'-interaction on Γ. For this construction we shall use derivatives with respect to a measure. Let μ be Radon measure and $\operatorname{supp}(\mu)=\Gamma$. Let $\psi\in W_2^2(\mathbb{R}\setminus\Gamma)$ be a function such that ψ and its derivative ψ' have the following representations for $x, s \in \mathbb{R}\setminus\Gamma$ with $s < x$:

$$\begin{aligned} \psi(x) &= \psi(s) + \int_s^x \psi'(\xi)\,d\xi + \int_{(s,x)} f(\xi)\mu(d\xi), \\ \psi'(x) &= \psi'(s) + \int_s^x \psi''(\xi)\,d\xi + \int_{(s,x)} g(\xi)\mu(d\xi), \end{aligned} \tag{4.1}$$

where f and g are defined on Γ and absolutely integrable with respect to the measure μ. The functions f and g are called derivatives of the functions ψ and ψ' with respect to the measure μ, respectively, and they are denoted by $f = \dfrac{d\psi}{d\mu}$, $g = \dfrac{d\psi'}{d\mu}$. It follows from (4.1) that the functions $\psi_r(x) = \dfrac{1}{2}[\psi(x+0) + \psi(x-0)]$ and $\psi_r'(x) = \dfrac{1}{2}[\psi'(x+0) + \psi'(x-0)]$ on Γ are essentially bounded on Γ, i.e., belong to the space $L_\infty(\Gamma, d\mu)$. The set of all functions in the space $W_2^2(\mathbb{R}\setminus\Gamma)$ that admit a representation of the form (4.1) will be denoted by $W_2^2(\mathbb{R}\setminus\Gamma;\Gamma,\mu)$. The functions ψ_r, ψ_r', $\dfrac{d\psi}{d\mu}$, $\dfrac{d\psi'}{d\mu}$ are called boundary values of $\psi \in W_2^2(\mathbb{R}\setminus\Gamma;\Gamma,\mu)$ on Γ. By construction $W_2^2(\mathbb{R}) \subset W_2^2(\mathbb{R}\setminus\Gamma;\Gamma,\mu) \subset W_2^2(\mathbb{R}\setminus\Gamma)$.

Let $\chi \in C^\infty(\mathbb{R})$ and $\chi' \in C_0^\infty(\mathbb{R}\setminus\Gamma)$. Then there exists finitely many pairwise disjoint open intervals I_k, $k = 1, 2, \ldots, n$, such that χ is constant on I_k for $k = 1, 2, \ldots, n$ and $\Gamma \subset \bigcup_{k=1}^n I_k$. It follows that

$$\frac{d(\chi\psi)}{d\mu} = \chi\frac{d\psi}{d\mu}, \quad \frac{d(\chi\psi)'}{d\mu} = \chi\frac{d\psi'}{d\mu}. \tag{4.2}$$

For functions $\psi, \varphi \in W_2^2(\mathbb{R}\setminus\Gamma;\Gamma,\mu)$, it was proved in [7], Theorem 4.3, that Green's first and second formulas hold with boundary values of ψ and φ on Γ.

Green's first formula is

$$(-\psi'', \varphi)_{L_2(\mathbb{R})} = (\psi', \varphi')_{L_2(\mathbb{R})} + \int_\Gamma \Big[\frac{d\psi'}{d\mu}\overline{\varphi_r} + \psi_r'\overline{\frac{d\varphi}{d\mu}}\Big]d\mu. \tag{4.3}$$

Green's second formula is

$$(-\psi'', \varphi)_{L_2(\mathbb{R})} - (\psi, -\varphi'')_{L_2(\mathbb{R})} = \int_\Gamma \Big[\frac{d\psi'}{d\mu}\overline{\varphi}_r + \psi_r'\overline{\frac{d\varphi}{d\mu}} - \psi_r\overline{\frac{d\varphi'}{d\mu}} - \frac{d\psi}{d\mu}\overline{\varphi_r'}\Big]d\mu \tag{4.4}$$

Green's second formula allows to consider different self-adjoint boundary conditions that are similar to well known boundary conditions for point interactions. In particular, we get δ'-interactions by the following definition, as we shall show below (Theorem 5.2).

DEFINITION 4.5. Let β be a real-valued function defined on Γ that is absolutely integrable with respect to the measure μ. The operator $L_{\Gamma,\beta}$ in $L_2(\mathbb{R})$ is the restriction of $L_{\max,\Gamma}$ on the space of all $\psi \in W_2^2(\mathbb{R}\setminus\Gamma;\Gamma,\mu)$ satisfying following boundary condition:

$$\frac{d\psi'(x)}{d\mu} = 0,\ \frac{d\psi(x)}{d\mu} = \beta(x)\psi_r'(x),\ x \in \Gamma. \tag{4.5}$$

β is the intensity of the δ'-interaction.

5. Spectral properties of Schrödinger operators with δ'-interaction

We shall show that the operator $L_{\Gamma,\beta}$ defined via the boundary conditions (4.5) defines a δ'-interaction on Γ and its negative spectrum is discrete. We prepare the proof by the following two lemmata.

LEMMA 5.2. *Let Γ be a non-empty compact subset of $\mathbb{R}$ of Lebesgue measure zero. Let μ be a Radon measure such that supp$(\mu) = \Gamma$ and let a and b real numbers such that $\Gamma \subset (a,b)$. Let β be a real-valued function defined on Γ that is absolutely integrable with respect to the measure μ. Define the operator $L_{\Gamma,\beta}^{(a,b)}$ in $L_2(a,b)$ as follows:*

$$L_{\Gamma,\beta}^{(a,b)}\psi(x) := -\psi''(x), \quad x \in \mathbb{R} \setminus (\Gamma \cup \{a,b\}),$$
$$\mathfrak{D}(L_{\Gamma,\beta}^{(a,b)}) := \{\chi_{(a,b)}\psi : \psi \in \mathfrak{D}(L_{\Gamma,\beta}), \psi(a) = 0 = \psi'(b)\}.$$

Then $L_{\Gamma,\beta}^{(a,b)}$ is an invertible self-adjoint operator in $L_2(a,b)$ and its inverse $(L_{\Gamma,\beta}^{(a,b)})^{-1}$ is compact.

Proof. Green's second formula (4.4) and an integration by parts yields that the operator $L_{\Gamma,\beta}^{(a,b)}$ is symmetric. Let $h \in L_2(a,b)$. Put $\psi'(x) := \int_x^b h(s)ds$ for every $x \in (a,b)$. Then $\psi'(x) = \psi_r'(x)$ for every $x \in (a,b)$. Put

$$\psi(x) := \int_a^x \psi'(s)\,ds + \int_{(a,x)} \beta(s)\psi'(s)\,\mu(ds), x \in (a,b)\setminus\Gamma, \quad \psi(x) := 0, x \in \mathbb{R}\setminus(a,b). \tag{5.1}$$

Then $\psi(a+0) = \psi'(b-0) = 0$, $\psi \in \mathfrak{D}(L_{\Gamma,\beta}^{(a,b)}) \subset L_2(a,b)$ and $\psi(\cdot) = \int_{-\infty}^{\infty} \mathscr{G}(\cdot,s)h(s)\,ds$, where $\mathscr{G}(x,s) = \chi_{(a,b)\times(a,b)}(x,s)\,(\min(x,s) - a + \int_{(a,\min(x,s))} \beta(\xi)\mu(d\xi))$. The integral operator in $L_2(\mathbb{R})$ with kernel $\mathscr{G}(x,s)$ is compact and self-adjoint and $L_2(a,b)$ is an invariant subspace for this integral operator. Thus the restriction $\mathscr{G}$ of this integral operator on $L_2(a,b)$ is a compact and self-adjoint operator in $L_2(a,b)$. By formula (5.1), $L_{\Gamma,\beta}^{(a,b)}\mathscr{G}h(x) = -\psi''(x) = h(x)$ for $x \in \mathbb{R}\setminus(\Gamma\cup\{a,b\})$ and $h \in L_2(a,b)$. Hence the operator $\mathscr{G}$ is invertible and $L_{\Gamma,\beta}^{(a,b)}$ is an extension of its inverse $\mathscr{G}^{-1}$. Since the inverse of a self-adjoint operator is self-adjoint and a self-adjoint operator does not possess any proper symmetric extension, it follows that $L_{\Gamma,\beta}^{(a,b)} = \mathscr{G}^{-1}$. □

LEMMA 5.3. *Let a,b and $L_{\Gamma,\beta}^{(a,b)}$ be as in the previous lemma. Let L_D be the Dirichlet-Laplacian in $L_2(-\infty,a)$ and let L_N be the Neumann-Laplacian in $L_2(b,\infty)$. Then the orthogonal sum $A := L_D \oplus L_{\Gamma,\beta}^{(a,b)} \oplus L_N$ defines a local interaction on $\hat{\Gamma} := \Gamma \cup \{a,b\}$, the essential spectrum $\sigma_{ess}(A)$ of A and the absolutely continuous spectrum $\sigma_{ac}(A)$ of A are equal to the nonnegative real half-axis, and their exists an $\alpha < 0$ such that the interval $(\alpha,0)$ is contained in the resolvent set of A.*

Proof. A is self-adjoint since it is the orthogonal sum of self-adjoint operators. Obviously it is an extension of $L_{\min,\hat{\Gamma}}$.

$\psi \in \mathfrak{D}(A)$ if, and only if, it can be represented as $\psi = \psi_1 + \chi_{(a,b)}\psi_2 + \psi_3$, where $\psi_1 \in W_2^2(\mathbb{R}\setminus\{a\})$, $\psi_1(a-0) = 0$ and $\psi_1 = 0$ in (a,∞), $\psi_2 \in \mathfrak{D}(L_{\Gamma,\beta})$ and $\psi_2(a) = 0 = \psi_2'(b)$, and $\psi_3 \in W_2^2(\mathbb{R}\setminus\{b\})$, $\psi_3'(b+0) = 0$ and $\psi_3 = 0$ in $(-\infty,b)$. Let $\chi \in C^\infty(\mathbb{R})$ and $\chi' \in C_0^\infty(\mathbb{R}\setminus\hat{\Gamma})$. By (4.2), $\chi\psi_2 \in \mathfrak{D}(L_{\Gamma,\beta})$ for every $\psi_2 \in \mathfrak{D}(L_{\Gamma,\beta})$. Moreover χ is constant on a neighborhood of a and on a neighborhood of b. It follows now that $\chi\psi \in \mathfrak{D}(A)$ for every $\psi \in \mathfrak{D}(A)$. Hence A defines a local interaction on $\hat{\Gamma}$.

Since A is the orthogonal sum of the self-adjoint operators L_D, $L_{\Gamma,\beta}^{(a,b)}$, and L_N,

$$\begin{aligned}\sigma(A) &= \sigma(L_D) \cup \sigma(L_{\Gamma,\beta}^{(a,b)}) \cup \sigma(L_N), \\ \sigma_x(A) &= \sigma_x(L_D) \cup \sigma_x(L_{\Gamma,\beta}^{(a,b)}) \cup \sigma_x(L_N), \quad x \in \{ess, ac\}\end{aligned} \tag{5.2}$$

($\sigma(\cdot)$ denotes the spectrum). It is well known that

$$\sigma(L_D) = \sigma_x(L_D) = \sigma(L_N) = \sigma_x(L_N) = [0,\infty), \quad x \in \{ess, ac\}. \tag{5.3}$$

Since $L_{\Gamma,\beta}^{(a,b)}$ is the inverse of a compact self-adjoint operator, its essential spectrum and its absolutely continuous spectrum are empty and its resolvent set contains a neighborhood of zero. In conjunction with (5.2) and (5.3), this proves the assertions about the spectral properties of A. $\square$

THEOREM 5.2. *Let Γ be a compact subset of the real line of Lebesgue measure zero. Let μ be a Radon measure and $supp(\mu) = \Gamma$, and let β be a real-valued function on Γ that is absolutely integrable with respect to μ. Let $L_{\Gamma,\beta}$ be the operator in $L_2(\mathbb{R})$ given by Definition 4.5. Then the following holds:*

1^0 *$L_{\Gamma,\beta}$ is self-adjoint and it defines a δ'-interaction on Γ.*

2^0 *The essential spectrum $\sigma_{ess}(L_{\Gamma,\beta})$ of $L_{\Gamma,\beta}$ and the absolutely continuous spectrum $\sigma_{ac}(L_{\Gamma,\beta})$ of $L_{\Gamma,\beta}$ are equal to the nonnegative real half-axis.*

3^0 *There exists an $\alpha_0 < 0$ such that the interval $(\alpha_0,0)$ is contained in the resolvent set of $L_{\Gamma,\beta}$.*

4^0 *The number (counting multiplicities) of negative eigenvalues of $L_{\Gamma,\beta}$ is not less than the number of isolated points of Γ where the intensity of the δ'-interaction is negative.*

Proof. 1° We have to show that $L_{\Gamma,\beta}$ has the properties $P_1(\Gamma)$-$P_3(\Gamma)$ in Definition 2.3.

$P_1(\Gamma)$ Let $A = L_D \oplus L_{\Gamma,\beta}^{(a,b)} \oplus L_N$ be the operator from the previous lemma. Define the operator $\tilde{A}_{00}$ as the restriction of $L_{\max,\Gamma}$ on the space of all $\psi \in W_2^2(\mathbb{R}\setminus\Gamma)$ that can be represented as $\psi = \psi_1 + \psi_2$, where $\psi_1 \in \mathfrak{D}(A), \psi_1(a\pm 0) = \psi_1'(a\pm 0) = \psi_1(b\pm 0) = \psi_1'(b\pm 0) = 0$, and $\psi_2 \in W_2^2(\mathbb{R})$ and $\operatorname{supp}(\psi_2)\cap\Gamma = \emptyset$. Applying Proposition 2.1 5° twice we obtain that $\tilde{A}_{00}$ is self-adjoint and it defines a local interaction on Γ. Obviously $\tilde{A}_{00} = L_{\Gamma,\beta}$. Thus $L_{\Gamma,\beta}$ has property $P_1(\Gamma)$.

$P_2(\Gamma)$ Every function $\chi \in C_0^\infty(\mathbb{R})$ such that $\chi' \in C_0^\infty(\mathbb{R}\setminus\Gamma)$ has trivial boundary data $\dfrac{d\chi}{d\mu} = 0$, $\dfrac{d\chi'}{d\mu} = 0$, $\chi_r' = 0$, $\chi_r = \chi$. Thus it satisfies the boundary conditions (4.5), and hence $\chi \in D(L_{\Gamma,\beta})$.

$P_3(\Gamma)$ We prove by contradiction that $L_{\beta,\Gamma}$ has the property $P_3(\Gamma)$. Suppose that there exists a point $x_0 \in \Gamma$ such that $L_{\Gamma,\beta} = A_1 \oplus A_2$, where A_1 and A_2 is an operator in the space $L_2(-\infty, x_0)$ and $L_2(x_0, +\infty)$, respectively. Choose any function $\chi_0 \in C_0^\infty(\mathbb{R})$, that is equal to 1 in the interval $(a,b) \supset \Gamma$. Then both χ_0 and $\psi_+ := \chi_{(x_0,\infty)}\chi_0$ belong to $\mathfrak{D}(L_{\Gamma,\beta})$. It follows from representation (4.1) that $x_0 \in \Gamma$, $\mu(\{x_0\}) > 0$, $\frac{d\psi_+(x)}{d\mu}|_{x=x_0} = [\mu(\{x_0\})]^{-1} \neq 0$. This contradicts the boundary conditions (4.5), since $\chi_{0,r}' = 0$.

$2^\circ, 3^\circ$ Applying Proposition 2.1, 1° and 6°, twice we obtain that A and $L_{\Gamma,\beta}$ have a common restriction with finite deficiency indices and for every z that belongs both to the resolvent set of A and the resolvent set of $L_{\Gamma,\beta}$ the difference $(A-z)^{-1} - (L_{\Gamma,\beta} - z)^{-1}$ is a finite rank operator. In conjunction with the previous lemma that proves the assertions 2° and 3°.

4° follows from 2° and Remark 3.3. □

THEOREM 5.3. *Let Γ be a compact subset of $\mathbb{R}$ of Lebesgue measure zero, μ a Radon measure and supp$(\mu) = \Gamma$. Let β be a real-valued function defined on Γ. Suppose that β is absolutely integrable with respect to μ and assumes negative mean values on an infinite number of closed pairwise disjoint nonintersecting subsets Γ_k of Γ. Then the Schrödinger operator $L_{\Gamma,\beta}$ with δ'-interaction on Γ, having intensity β, is a self-adjoint operator in the space $L_2(\mathbb{R})$, it has infinitely many negative eigenvalues and the set of its negative eigenvalues is not lower bounded.*

Proof. The proof is similar to the proof of Theorem 3.1. By Theorem 5.2, the operator $L_{\Gamma,\beta}$ is self-adjoint in $L_2(\mathbb{R})$, its negative spectrum is discrete and there exists an $\alpha_0 < 0$ such that the interval $(\alpha_0, 0)$ does not contain any point of the spectrum of $L_{\Gamma,\beta}$. Thus it is sufficient to show that the range of the spectral projection $\chi_{(-\infty,0)}(L_{\Gamma,\beta})$ is infinite-dimensional. To this end, it is sufficient to show that for every positive integer N there exists an N–dimensional subspace $\mathscr{L}_N$ of the domain of $L_{\Gamma,\beta}$ such that $(L_{\Gamma,\beta}u, u) < 0$, for every $u \in \mathscr{L}_N \setminus \{0\}$.

Fix N. By the conditions of the theorem there exists an $\varepsilon > 0$ and N pairwise disjoint closed subsets Γ_k of Γ, such that $\mu(\Gamma_k) > 0$ and $\int\limits_{\Gamma_k} \beta(x)\mu(dx) \leqslant -\varepsilon\mu(\Gamma_k)$

for $k = 1,...,N$. Consider analogues of the test functions of Section 3. Since the subsets Γ_k are compact and pairwise disjoint and the number of these subsets is finite, there exists a $\delta > 0$ such that the δ-neighborhoods $\mathscr{U}_\delta(\Gamma_k) = \{y : |y-x| < \delta, x \in \Gamma_k\}$ of the sets Γ_k are also pairwise disjoint. Let us construct a test function for each set Γ_k as follows. Consider the function $\chi_k \in C_0^\infty(\mathbb{R})$ that is equal to 1 on Γ_k, takes values between 0 and 1, and is equal to zero outside of $\mathscr{U}_\delta(\Gamma_k)$. As a candidate for the test function, we take

$$\hat{t}_k(x;\beta,\Gamma_k,\delta) = \int_a^x \chi_k(\xi)\,d\xi + \int_{(a,x)} \beta(\xi)\chi_k(\xi)\,d\mu(\xi), \tag{5.4}$$

where the number a is chosen so that all bounded sets $\mathscr{U}_\delta(\Gamma_k)$, $k = 1,...,N$, lie to the left of the set Γ. For x that lie on the right of the set Γ, this function takes the constant value c_k. While the function t_k does not belong to the space $L_2(\mathbb{R})$, we can turn it into a function with compact support using two parabolas on the interval $[l, l+2r]$ that lies to the right of Γ. We thus get the test function

$$t_k(x;\beta,\Gamma_k,\delta,l,r) = \begin{cases} \hat{t}_k(x), & x \leqslant l, \\ -\dfrac{c_k}{2r^2}(l-x)^2 + c_k, & l \leqslant x \leqslant l+r, \\ \dfrac{c_k}{2r^2}(l+2r-x)^2, & l+r \leqslant x \leqslant l+2r, \\ 0, & l+2r < x. \end{cases} \tag{5.5}$$

Here, the parameters l and r may depend on k.

PROPOSITION 5.5. *The test functions t_k, defined by (5.5), have following properties:*

1^0 $t_k \in \mathscr{D}(L_{\Gamma,\beta})$.

2^0 *By choosing δ sufficiently small and r sufficiently large, we have*

$$(L_{\Gamma,\beta}t_k, t_k) \leqslant -\frac{1}{8}\varepsilon\mu(\Gamma_k), \tag{5.6}$$

that is, the quadratic form takes negative values.

3^0 *The quadratic form of the linear combination $t = \sum\limits_{k=1}^{N} a_k \cdot t_k$ of test functions that satisfy the condition 1^0, if l_k and r_k are chosen so that the intervals $[l_k, l_k+2r_k]$ are pairwise disjoint, takes negative values,*

$$(L_{\Gamma,\beta}t, t) = \sum_{k=1}^{N} |a_k|^2 (L_{\Gamma,\beta}t_k, t_k) \leqslant -\frac{1}{8}\varepsilon \min_k \mu(\Gamma_k) \sum_{k=1}^{N} |a_k|^2 < 0. \tag{5.7}$$

If these three conditions are satisfied, then we can complete the proof in the same way as in the proof of Theorem 3.1.

Let us now show that test functions satisfy properties 1^0—3^0. The first property is clearly satisfied by the construction of t_k and $\hat{t}_k$ in (5.4) and (5.5) and the definition of the operator $L_{\Gamma,\beta}$. The second property is most important. Since the function β is absolutely integrable on Γ with respect to the Radon measure μ and $0 \leqslant \chi_k \leqslant 1$, we see that there exists a $\delta > 0$ such that

$$\Big| \int\limits_{\mathscr{U}_\delta(\Gamma_k)\cap\Gamma} \beta(\xi)\chi_k(\xi)\,d\mu(\xi) - \int\limits_{\Gamma_k} \beta(\xi)\,d\mu(\xi)\Big| < \frac{1}{2}\varepsilon\mu(\Gamma_k). \tag{5.8}$$

Moreover, since the set Γ has Lebesgue measure zero, there exists a $\delta > 0$ such that the following estimate holds for the Lebesgue measure of the set $\mathscr{U}_\delta(\Gamma_k)$:

$$|\mathscr{U}_\delta(\Gamma_k)| \leqslant \frac{1}{4}\varepsilon\mu(\Gamma_k). \tag{5.9}$$

If inequalities (5.8) and (5.9) hold, then the constant c_k, which is equal to the value of the function $\hat{t}$ for large x, satisfies the estimate

$$|c_k| \leqslant (\frac{3}{4}\varepsilon + ||\beta||_{L_1(\Gamma,d\mu)})\mu(\Gamma_k). \tag{5.10}$$

By choosing r_k sufficiently large, we get

$$\int\limits_{l_k}^{l_k+2r_k} |t'(x)|^2\,dx \leqslant \frac{1}{8}\varepsilon\mu(\Gamma_k). \tag{5.11}$$

In virtue of Green's first formula (4.3), since the function t_k satisfies the boundary conditions (4.5) and because $t_k'(x) = \chi_k(x)$ for $x \leqslant l_k$, we have

$$(L_{\Gamma,\beta}t_k,t_k) = \int\limits_a^{l_k} |\chi_k(x)|^2\,dx + \int\limits_{l_k}^{l_k+2r_k} |t_k'(x)|^2\,dx + \int\limits_\Gamma \beta(\xi)|\chi_k(\xi)|^2\,d\mu(\xi). \tag{5.12}$$

The first integral $\mathscr{I}_1$ in (5.12) can be estimated in terms of the Lebesgue measure of $\mathscr{U}_\delta(\Gamma_k)$, since the values of the function χ_k belong to the interval $[0,1]$. The second integral $\mathscr{I}_2 = \frac{2}{3}c_k^2\cdot r_k^{-1}$ can be explicitly calculated, since the function t_k' on the interval $[l_k, l_k+2r_k]$ consists of two parabolas by (5.5). The third integral $\mathscr{I}_3$ in (5.12) can be estimated as follows:

$$\mathscr{I}_3 \leqslant \int\limits_{\Gamma_k} \beta(\xi)\,d\mu(\xi) + \Big| \int\limits_{\mathscr{U}_\delta(\Gamma_k)} \beta(\xi)\chi_k^2(\xi)\,d\mu(\xi) - \int\limits_{\Gamma_k} \beta(\xi)\,d\mu(\xi)\Big|.$$

Since, by choosing sufficiently small δ and sufficiently large r_k we can satisfy estimates (5.8)–(5.11), we see that the quadratic form $(L_{\Gamma,\beta}t_k,t_k)$ is negative, i.e., inequality (5.6) is satisfied.

Consider now property 3^0. Since the intervals $(l_k, l_k + 2r_k)$ and the regions $\mathscr{U}_\delta(\Gamma_k)$ are mutually disjoint, we have that $(L_{\Gamma,\beta} t_k, t_j) = 0$ for $k \neq j$. This leads to property (5.7). □

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