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UNIVERSAL INEQUALITIES FOR EIGENVALUES OF THE LAMÉ SYSTEM

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Abstract. In this paper, we investigate the Dirichlet eigenvalue problem of the Lamé system: $\Delta\mathbf{u}+\alpha\mathrm{grad}(\mathrm{div}\mathbf{u})=-\sigma\mathbf{u}$ on a bounded domain Ω in an n-dimensional Euclidean space $\mathbb{R}^n$, where α is a nonnegative constant and $\mathbf{u}$ is a vector-valued function on Ω. We establish a Levitin-Parnovski-type inequality for its eigenvalues, which gives an estimate for the upper bounds of $\sum_{i=1}^{n}\sigma_{i+j}$ for any positive integer j. Moreover, we obtain some other universal inequalities for eigenvalues of this problem.

1. Introduction

Let Ω be a bounded domain in an n-dimensional Euclidean space $\mathbb{R}^n$. Let $\mathbf{u}=(u_1,\cdots,u_l,\cdots,u_n)$ be a vector-valued function on $\overline{\Omega}$. Denote by div the divergence operator and grad the gradient operator. The Dirichlet eigenvalue problem of the Lamé system is described by

$$\begin{cases}\Delta\mathbf{u}+\alpha\mathrm{grad}(\mathrm{div}\mathbf{u})=-\sigma\mathbf{u}, & \text{in } \Omega,\\ \mathbf{u}|_{\partial\Omega}=0,\end{cases} \tag{1.1}$$

where α is a nonnegative constant and Δ is the Laplacian in $\mathbb{R}^n$. This problem has definite physical background. When $n=3$, it describes the behavior of an elastic medium. Its eigenvectors describe the deformation of vibrating elastic bodies with fixed boundaries (cf. [16, 12]). This problem has a real discrete spectrum

$$0<\sigma_1\leqslant\sigma_2\leqslant\cdots\leqslant\sigma_l\leqslant\cdots\to\infty, \tag{1.2}$$

where each eigenvalue is repeated according to its multiplicity.

Eigenvalues of problem (1.1) have been studied from different angles (see [7, 9, 10, 14]). In particular, some universal inequalities for its eigenvalues have been established. In 1990, Hook [6] proved

$$\sum_{i=1}^{k}\frac{\sigma_i}{\sigma_{k+1}-\sigma_i}\geqslant\frac{n^2k}{4(n+\alpha)}. \tag{1.3}$$

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In 2002, Levitin and Parnovski [11] derived

$$\sigma_{k+1}-\sigma_k \leqslant \frac{\max\{4+\alpha^2;(n+2)\alpha+8\}}{n+\alpha}\frac{1}{k}\sum_{i=1}^{k}\sigma_i, \tag{1.4}$$

which gives an estimate for the gap of $\sigma_{k+1}-\sigma_k$ in terms of the first k eigenvalues. In 2009, Cheng and Yang [5] obtained

$$\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i) \leqslant \frac{2\sqrt{n+\alpha}}{n}\left[\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^{\frac{1}{2}}\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^{\frac{1}{2}}\sigma_i\right]^{\frac{1}{2}}. \tag{1.5}$$

It implies

$$\sigma_{k+1} \leqslant \left[1+\frac{4(n+\alpha)}{n^2}\right]\frac{1}{k}\sum_{i=1}^{n}\sigma_i, \tag{1.6}$$

which gives an estimate for the upper bound of σ_{k+1} in terms of the first k eigenvalues. In 2012, Chen, Cheng, Wang and Xia [4] further strengthened (1.5) to

$$\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^2 \leqslant B(n,\alpha)\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)\sigma_i,$$

where $B(n,\alpha)$ is a constant depended on n and α. Cheng and Yang [5] also gave the following estimate for the upper bound of the sum of consecutive eigenvalues:

$$\sum_{i=1}^{n}\sigma_{i+1} \leqslant (n+4+4\alpha)\sigma_1. \tag{1.7}$$

It is interesting to relate problem (1.1) with the fixed membrane problem which is described by

$$\begin{cases}\Delta u=-\lambda u, & \text{in } \Omega,\\ u|_{\partial\Omega}=0,\end{cases} \tag{1.8}$$

where Ω is a bounded domain in $\mathbb{R}^n$. When $n=2$ (namely for $\Omega\subset\mathbb{R}^2$), Payne, Pólya and Weinberger [13] proved

$$\lambda_2+\lambda_3 \leqslant 6\lambda_1. \tag{1.9}$$

It lead us to the famous Payne, Pólya and Weinberger conjecture (cf. [1]). In 1993, Ashbaugh and Benguria [2] derived

$$\sum_{i=1}^{n}\lambda_{i+1} \leqslant (n+4)\lambda_1 \tag{1.10}$$

for $\Omega\subset\mathbb{R}^n$. On the one hand, (1.10) have been extended to bounded domains in some other Riemannian manifolds. In 2008, Sun, Cheng and Yang [15] obtained

$$\sum_{i=1}^{n}\lambda_{i+1} \leqslant n^2+(n+4)\lambda_1. \tag{1.11}$$

on a bounded domain in the unite sphere $S^n(1)$. It is optimal for the unite sphere since it becomes an equality when $\Omega = S^n(1)$. Chen and Cheng [3] proved that (1.10) also holds on bounded domains in complete Riemannian manifolds. On the other hand, Levitin and Parnovski [11] generalized (1.10) to

$$\sum_{i=1}^{n} \lambda_{i+j} \leqslant (n+4)\lambda_j, \tag{1.12}$$

where j is any positive integer. A remarkable point of (1.12) is that it gives some estimates for the upper bounds of $\lambda_{j+1} + \cdots + \lambda_{j+n}$ in terms of λ_j. Moreover, it covers (1.10) when $j = 1$. This inequality will be referred to henceforth as the Levitin-Parnovski inequality. Observe that (1.7) also becomes the same as (1.10) when $\alpha = 0$. It is natural to consider the following question: Whether can one obtain a Levitin-Parnovski-type inequality for problem (1.1)?

The purpose of this paper is to establish a Levitin-Parnovski-type inequality and some other universal inequalities for problem (1.1). In this paper, we obtain the following result:

THEOREM 1. *Let Ω be a bounded domain in $\mathbb{R}^n$. Denote by σ_i the i-th eigenvalue of problem (1.1). For any positive integer j, we have*

$$\sum_{i=1}^{n} \sigma_{i+j} \leqslant \big(n + C(n,\alpha)\big)\sigma_j - \alpha(\sigma_{j+1} - \sigma_j), \tag{1.13}$$

where the constant

$$C(n,\alpha) = \begin{cases} (n+2)\alpha + 8, & \text{when } 0 \leqslant \alpha \leqslant \dfrac{\sqrt{(n+2)^2+16}+n+2}{2}; \\ 4+\alpha^2, & \text{when } \alpha \geqslant \dfrac{\sqrt{(n+2)^2+16}+n+2}{2}. \end{cases}$$

Hence, we answer the preceding question. Observe that (1.13) becomes

$$\sum_{i=1}^{n} \sigma_{i+j} \leqslant (n+8)\sigma_j,$$

when $\alpha = 0$. Of course, it is also interesting to consider whether it is possible to establish a sharper inequality which becomes the same as (1.12) when $\alpha = 0$.

Furthermore, we derive some other universal inequalities for problem (1.1).

THEOREM 2. *Let Ω be a bounded domain in $\mathbb{R}^n$. Denote by σ_i the i-th eigenvalue of problem (1.1). Then we have*

$$\sum_{i=1}^{k} (\sigma_{k+1} - \sigma_i)^2 \leqslant \frac{2\sqrt{n+\alpha}}{n} \left[\sum_{i=1}^{k} \sigma_i \sum_{i=1}^{k} (\sigma_{k+1} - \sigma_i)^3 \right]^{\frac{1}{2}} \tag{1.14}$$

and

$$\sum_{i=1}^{k} (\sigma_{k+1} - \sigma_i)^{\frac{3}{2}} \leqslant \frac{4(n+\alpha)}{n^2} \sum_{i=1}^{k} (\sigma_{k+1} - \sigma_i)^{\frac{1}{2}} \sigma_i. \tag{1.15}$$

REMARK 1. In the proof of Theorem 2, we obtain inequality (2.32) by making use of an abstract inequality attributed to Ilias and Makhoul [8]. Besides (1.14) and (1.15), we can also get (1.5) and (1.6) of Cheng and Yang [5] by using (2.32). In fact, taking $f(\sigma_i) = \sigma_{k+1} - \sigma_i$ and $g(\sigma_i) = (\sigma_{k+1} - \sigma_i)^{\frac{1}{2}}$ in (2.32), we can derive (1.5). Taking $f(\sigma_i) = g(\sigma_i) = \sigma_{k+1} - \sigma_i$ in (2.32), we can get (1.6).

2. Proofs of the main results

In this section, we give the proofs of Theorems 1 and 2. The proof of Theorem 1 is based on the observation that estimates in the proof of Corollary 2.7 of [11] can be sharpened. In the proof of Theorem 1, we need the following abstract formula established by Levitin and Parnovski [11].

LEMMA 1. *Let $\mathscr{H}$ be a complex Hilbert space with a given inner product $\langle , \rangle$. Let $H : \mathscr{D} \subset \mathscr{H} \longrightarrow \mathscr{H}$ be a self-adjoint operator defined on a dense domain $\mathscr{D}$ which is semibounded beblow and has a discrete spectrum $\mu_1 \leqslant \mu_2 \leqslant \mu_3 \leqslant \cdots$. Let $\{G_l : H(\mathscr{D}) \longrightarrow \mathscr{H}\}_{l=1}^{N}$ be a collection of symmetric operators which leave $\mathscr{D}$ invariant. Denote by $\{u_i\}_{i=1}^{\infty}$ the normalized eigenvectors of H and u_i corresponding to the i-th eigenvalue μ_i. Moreover, this family of eigenvectors is further assumed to be an orthonormal basis for $\mathscr{H}$. For any positive integer j, we have*

$$\sum_{k=1}^{\infty} \frac{|\langle [H, G_l] u_j, u_k \rangle|^2}{\mu_k - \mu_j} = -\frac{1}{2} \langle [[H, G_l], G_l] u_j, u_j \rangle, \tag{2.1}$$

where $[H, G_l] := HG_l - G_l H$ is the commutator of H and G_l.

Now we give the proof of Theorem 1.

Proof of Thereom 1. Denote by $\mathbf{e}_1 = (1, 0, \cdots, 0)$, $\cdots$, $\mathbf{e}_n = (0, \cdots, 1)$ the unit vectors in $\mathbb{R}^n$. Then we have $u_l = \mathbf{u} \cdot \mathbf{e}_l$ for a vector-valued function $\mathbf{u} = (u_1, \cdots, u_l, \cdots, u_n)$ on Ω. For the sake of convenience, we denote by

$$L\mathbf{u} = -\Delta \mathbf{u} + \alpha M \mathbf{u},$$

where $M\mathbf{u} = -\mathrm{grad}(\mathrm{div}\mathbf{u})$. Let $\mathbf{u}_i$ be the orthonormal eigenvectors corresponding to the i-th eigenvalues σ_i of problem (1.1). That is to say, $\mathbf{u}_i$ satisfies

$$\begin{cases} L\mathbf{u}_i = \sigma_i \mathbf{u}_i, & \text{in } \Omega, \\ \mathbf{u}_i|_{\partial\Omega} = 0, & \\ \displaystyle\int_{\Omega} \mathbf{u}_i \cdot \mathbf{u}_j = \delta_{ij}. & \end{cases} \tag{2.2}$$

We claim that we can choose the functions $x_1, \cdots, x_n$ as the standard coordinates functions of $\mathbb{R}^n$ such that

$$\langle [L, x_l] \mathbf{u}_j, \mathbf{u}_{j+k} \rangle = 0, \qquad \text{for } 1 \leqslant k < l \leqslant n. \tag{2.3}$$

In fact, let $y_1,\cdots,y_n$ be the standard coordinate functions of $\mathbb{R}^n$. Consider an $n\times n$ matrix B defined by

$$B := \begin{pmatrix} \langle [L,y_1]\mathbf{u}_j,\mathbf{u}_{j+1}\rangle & \langle [L,y_1]\mathbf{u}_j,\mathbf{u}_{j+2}\rangle & \cdots & \langle [L,y_1]\mathbf{u}_j,\mathbf{u}_{j+n}\rangle \\ \langle [L,y_2]\mathbf{u}_j,\mathbf{u}_{j+1}\rangle & \langle [L,y_2]\mathbf{u}_j,\mathbf{u}_{j+2}\rangle & \cdots & \langle [L,y_2]\mathbf{u}_j,\mathbf{u}_{j+n}\rangle \\ \cdots & \cdots & \cdots & \cdots \\ \langle [L,y_n]\mathbf{u}_j,\mathbf{u}_{j+1}\rangle & \langle [L,y_n]\mathbf{u}_j,\mathbf{u}_{j+2}\rangle & \cdots & \langle [L,y_n]\mathbf{u}_j,\mathbf{u}_{j+n}\rangle \end{pmatrix}.$$

According to the QR-factorization theorem, we know that there is an orthogonal $n\times n$ matrix $Q=(q_{lr})_{n\times n}$ such that $A=QB$ is an upper triangle matrix. Namely, it holds

$$\sum_{r=1}^{n} q_{lr}\langle [L,y_r]\mathbf{u}_j,\mathbf{u}_{j+k}\rangle = 0, \qquad \text{for } 1\leqslant k<l\leqslant n.$$

Putting $x_l=\sum_{r=1}^{n} q_{lr}y_r$, we know that our claim is true. Therefore, according to (2.3), we find that it holds

$$\sum_{k=1}^{l-1}\frac{|\langle [L,x_l]\mathbf{u}_j,\mathbf{u}_{j+k}\rangle|^2}{\sigma_{j+k}-\sigma_j}=0. \tag{2.4}$$

Taking $H=L$ and $G_l=x_l$ in (2.1), we have

$$\sum_{k=1}^{\infty}\frac{|\langle [L,x_l]\mathbf{u}_j,\mathbf{u}_k\rangle|^2}{\sigma_k-\sigma_j}=-\frac{1}{2}\langle [[L,x_l],x_l]\mathbf{u}_j,\mathbf{u}_j\rangle. \tag{2.5}$$

Utilizing (2.4), we can get an inequality. In fact, rewriting the summation index, one can deduce

$$\begin{aligned}
&\sum_{k=1}^{\infty}\frac{|\langle [L,x_l]\mathbf{u}_j,\mathbf{u}_k\rangle|^2}{\sigma_k-\sigma_j}\\
=&\sum_{k=1}^{j-1}\frac{|\langle [L,x_l]\mathbf{u}_j,\mathbf{u}_k\rangle|^2}{\sigma_k-\sigma_j}+\sum_{k=j+1}^{j+l-1}\frac{|\langle [L,x_l]\mathbf{u}_j,\mathbf{u}_k\rangle|^2}{\sigma_k-\sigma_j}+\sum_{k=j+l}^{\infty}\frac{|\langle [L,x_l]\mathbf{u}_j,\mathbf{u}_k\rangle|^2}{\sigma_k-\sigma_j}\\
=&\sum_{k=1}^{j-1}\frac{|\langle [L,x_l]\mathbf{u}_j,\mathbf{u}_k\rangle|^2}{\sigma_k-\sigma_j}+\sum_{k=1}^{l-1}\frac{|\langle [L,x_l]\mathbf{u}_j,\mathbf{u}_{j+k}\rangle|^2}{\sigma_{j+k}-\sigma_j}+\sum_{k=j+l}^{\infty}\frac{|\langle [L,x_l]\mathbf{u}_j,\mathbf{u}_k\rangle|^2}{\sigma_k-\sigma_j}.
\end{aligned} \tag{2.6}$$

Moreover, it follows from (1.2) that

$$\sum_{k=1}^{j-1}\frac{|\langle [L,x_l]\mathbf{u}_j,\mathbf{u}_k\rangle|^2}{\sigma_k-\sigma_j}\leqslant 0. \tag{2.7}$$

Combining (2.4), (2.6) and (2.7), we have

$$\begin{aligned}
\sum_{k=1}^{\infty}\frac{|\langle [L,x_l]\mathbf{u}_j,\mathbf{u}_k\rangle|^2}{\sigma_k-\sigma_j}\leqslant&\sum_{k=j+l}^{\infty}\frac{|\langle [L,x_l]\mathbf{u}_j,\mathbf{u}_k\rangle|^2}{\sigma_k-\sigma_j}\\
\leqslant&\frac{1}{\sigma_{j+l}-\sigma_j}\sum_{k=1}^{\infty}|\langle [L,x_l]\mathbf{u}_j,\mathbf{u}_k\rangle|^2.
\end{aligned} \tag{2.8}$$

Furthermore, Parseval's identity implies

$$\sum_{k=1}^{\infty} |\langle [L,x_l]\mathbf{u}_j, \mathbf{u}_k\rangle|^2 = \|[L,x_l]\mathbf{u}_j\|^2. \tag{2.9}$$

Combining (2.8) and (2.9), we obtain

$$\sum_{k=1}^{\infty} \frac{|\langle [L,x_l]\mathbf{u}_j, \mathbf{u}_k\rangle|^2}{\sigma_k - \sigma_j} \leqslant \frac{1}{\sigma_{j+l} - \sigma_j}\|[L,x_l]\mathbf{u}_j\|^2. \tag{2.10}$$

Substituting (2.10) into (2.5) and taking sum on l from 1 to n, we derive

$$-\frac{1}{2}\sum_{l=1}^{n}(\sigma_{j+l} - \sigma_j)\langle [[L,x_l],x_l]\mathbf{u}_j, \mathbf{u}_j\rangle \leqslant \sum_{l=1}^{n}\|[L,x_l]\mathbf{u}_j\|^2. \tag{2.11}$$

Now we calculate the terms in the both sides of (2.11). On the one hand, according to

$$\operatorname{div}(x_l\mathbf{u}) = \operatorname{grad}x_l \cdot \mathbf{u} + x_l \operatorname{div}\mathbf{u},$$

it yields (cf. Lemma 5 of [6])

$$[-\Delta, x_l]\mathbf{u} = -2\frac{\partial \mathbf{u}}{\partial x_l} \tag{2.12}$$

and

$$[M, x_l]\mathbf{u} = -R_l\mathbf{u}, \tag{2.13}$$

where $R_l\mathbf{u} = (\operatorname{div}\mathbf{u})\operatorname{grad}x_l + \operatorname{grad}(\mathbf{u}\cdot\mathbf{e}_l)$. Hence, making use of (2.12) and (2.13), we deduce

$$\begin{aligned}
-\frac{1}{2}\langle [[L,x_l],x_l]\mathbf{u}_j, \mathbf{u}_j\rangle =& \frac{1}{2}\langle [[\Delta,x_l],x_l]\mathbf{u}_j, \mathbf{u}_j\rangle - \frac{1}{2}\alpha\langle [[M,x_l],x_l]\mathbf{u}_j, \mathbf{u}_j\rangle \\
=& \Big\langle \Big[\frac{\partial}{\partial x_l}, x_l\Big]\mathbf{u}_j, \mathbf{u}_j\Big\rangle + \frac{1}{2}\alpha\langle [R_l, x_l]\mathbf{u}_j, \mathbf{u}_j\rangle \\
=& \int_\Omega \mathbf{u}_j \cdot \Big[\frac{\partial}{\partial x_l}(x_l\mathbf{u}_j) - x_l\frac{\partial \mathbf{u}_j}{\partial x_l}\Big] \\
&+ \frac{1}{2}\alpha\int_\Omega \mathbf{u}_j\cdot[(\operatorname{grad}x_l\cdot\mathbf{u})\operatorname{grad}x_l + (\mathbf{u}\cdot\mathbf{e}_l)\operatorname{grad}x_l] \\
=& \|\mathbf{u}_j\|^2 + \alpha\int_\Omega (\mathbf{u}_j\cdot\mathbf{e}_l)\operatorname{grad}x_l\cdot\mathbf{u}_j \\
=& 1 + \alpha\int_\Omega (\mathbf{u}_j\cdot\mathbf{e}_l)^2.
\end{aligned} \tag{2.14}$$

On the other hand, it follows from (2.12) and (2.13) that

$$\sum_{l=1}^{n} \| [L,x_l]\mathbf{u}_j \|^2 = \sum_{l=1}^{n}\Big(4\Big\|\frac{\partial}{\partial x_l}\mathbf{u}_j\Big\|^2 + \alpha^2\|R_l\mathbf{u}_j\|^2 + 4\alpha\Big\langle\frac{\partial \mathbf{u}_j}{\partial x_l}, R_l\mathbf{u}_j\Big\rangle\Big). \tag{2.15}$$

According to Lemma 4.5 of [11], it holds

$$\sum_{l=1}^{n} \|R_l \mathbf{u}_j\|^2 = (n+2)\int_\Omega (\mathrm{div}\mathbf{u}_j)^2 - \int_\Omega \mathbf{u}_j \cdot \Delta \mathbf{u}_j \tag{2.16}$$

and

$$\sum_{l=1}^{n} \left\langle \frac{\partial \mathbf{u}_j}{\partial x_l}, R_l \mathbf{u}_j \right\rangle = -2\int_\Omega \mathbf{u}_j \cdot \mathrm{grad}(\mathrm{div}\mathbf{u}_j). \tag{2.17}$$

Substituting (2.16), (2.17) and

$$\sum_{l=1}^{n} \left\| \frac{\partial \mathbf{u}_j}{\partial x_l} \right\|^2 = -\int_\Omega \mathbf{u}_j \cdot \Delta \mathbf{u}_j$$

into (2.15), we obtain

$$\sum_{l=1}^{n} \| \, [L, x_l]\mathbf{u}_j \, \|^2 = w_j, \tag{2.18}$$

where

$$w_j = -(4+\alpha^2)\int_\Omega \mathbf{u}_j \cdot \Delta \mathbf{u}_j - [(n+2)\alpha^2 + 8\alpha]\int_\Omega \mathbf{u}_j \cdot \mathrm{grad}(\mathrm{div}\mathbf{u}_j).$$

When $\alpha \geqslant \frac{1}{2}[n+2+\sqrt{(n+2)^2+16}]$, it yields $\alpha^2-(n+2)\alpha-4 \geqslant 0$. In this case, we have

$$w_j = (4+\alpha^2)\sigma_j - [\alpha^2-(n+2)\alpha-4]\int_\Omega (\mathrm{div}\mathbf{u}_j)^2 \leqslant (4+\alpha^2)\sigma_j. \tag{2.19}$$

When $0 \leqslant \alpha \leqslant \frac{1}{2}[n+2+\sqrt{(n+2)^2+16}]$, it yields $\alpha^2-(n+2)\alpha-4 \leqslant 0$. In this case, since

$$-\int_\Omega \mathbf{u}_j \cdot \Delta \mathbf{u}_j \geqslant 0,$$

we get

$$\begin{aligned} w_j =& [(n+2)\alpha+8]\sigma_j - [\alpha^2-(n+2)\alpha-4]\int_\Omega \mathbf{u}_j \cdot \Delta \mathbf{u}_j \\ \leqslant& [(n+2)\alpha+8]\sigma_j. \end{aligned} \tag{2.20}$$

It follows from (2.18), (2.19) and (2.20) that

$$\sum_{l=1}^{n} \| \, [L, x_l]\mathbf{u}_j \, \|^2 \leqslant C(n,\alpha)\sigma_j. \tag{2.21}$$

Substituting (2.14) and (2.21) into (2.11), we get

$$\sum_{l=1}^{n} (\sigma_{j+l}-\sigma_j)\left[1+\alpha\int_\Omega (\mathbf{u}_j\cdot\mathbf{e}_l)^2\right] \leqslant C(n,\alpha)\sigma_j. \tag{2.22}$$

Since

$$\sum_{l=1}^{n} \int_\Omega (\mathbf{u}_j\cdot\mathbf{e}_l)^2 = \|\mathbf{u}_j\|^2 = 1,$$

we deduce

$$\begin{aligned}\sum_{l=1}^{n}(\sigma_{j+l}-\sigma_j)\left[1+\alpha\int_\Omega(\mathbf{u}_j\cdot\mathbf{e}_l)^2\right] &\geqslant \sum_{l=1}^{n}(\sigma_{j+l}-\sigma_j)+\alpha(\sigma_{j+1}-\sigma_j)\sum_{l=1}^{n}\int_\Omega(\mathbf{u}_j\cdot\mathbf{e}_l)^2\\ &=\sum_{l=1}^{n}(\sigma_{j+l}-\sigma_j)+\alpha(\sigma_{j+1}-\sigma_j).\end{aligned} \tag{2.23}$$

Finally, combining (2.22) and (2.23), we obtain

$$\sum_{l=1}^{n}(\sigma_{j+l}-\sigma_j)+\alpha(\sigma_{j+1}-\sigma_j)\leqslant C(n,\alpha)\sigma_j. \tag{2.24}$$

It yields (1.13). This completes the proof of Theorem 1. □

In the proof of Theorem 2, we use the following lemma of Ilias and Makhoul [8].

LEMMA 2. *Let $\mathscr{H}$ be a complex Hilbert space with a given inner product $\langle,\rangle$. The notations of H, G_l, μ_i and u_i denote the same meanings as Lemma 1. Let $\{T_l : \mathscr{D}\longrightarrow\mathscr{H}\}_{l=1}^{N}$ be a collection of skew-symmetric operators which leave $\mathscr{D}$ invariant. A couple (f,g) of functions defined on $]0,\mu[$ belongs to $\mathfrak{J}_\mu$ provided that f and g are positive functions which satisfy*

$$\left[\frac{f(x)-f(y)}{x-y}\right]^2+\frac{g(x)-g(y)}{x-y}\left[\frac{f^2(x)}{g(x)(\mu-x)}+\frac{f^2(y)}{g(y)(\mu-y)}\right]\leqslant 0,$$

for any $x,y\in]0,\mu[$ and $x\neq y$. Then we have

$$\begin{aligned}&\left[\sum_{i=1}^{k}\sum_{l=1}^{n}f(\mu_i)\langle[T_l,G_l]u_i,u_i\rangle\right]^2\\ \leqslant&4\left[\sum_{i=1}^{k}\sum_{l=1}^{n}g(\mu_i)\langle[H,G_l]u_i,G_lu_i\rangle\right]\left[\sum_{i=1}^{k}\sum_{l=1}^{n}\frac{f^2(\mu_i)}{g(\mu_i)(\mu_{k+1}-\mu_i)}\|T_lu_i\|^2\right],\end{aligned} \tag{2.25}$$

where $\|T_lu_i\|$ denotes the norm of T_lu_i.

Now we give the proof of Theorem 2.

Proof of Thereom 2. Taking $H=L$, $G_l=x_l$ and $T_l=\frac{\partial}{\partial x_l}$ in (2.25), we have

$$\begin{aligned}&\left[\sum_{i=1}^{k}\sum_{l=1}^{n}f(\sigma_i)\left\langle\left[\frac{\partial}{\partial x_l},x_l\right]\mathbf{u}_i,\mathbf{u}_i\right\rangle\right]^2\\ \leqslant&4\left[\sum_{i=1}^{k}\sum_{l=1}^{n}g(\sigma_i)\langle[L,x_l]\mathbf{u}_i,x_l\mathbf{u}_i\rangle\right]\left[\sum_{i=1}^{k}\sum_{l=1}^{n}\frac{f^2(\sigma_i)}{g(\sigma_i)(\sigma_{k+1}-\sigma_i)}\|\frac{\partial}{\partial x_l}\mathbf{u}_i\|^2\right].\end{aligned} \tag{2.26}$$

Now we need to calculate the terms in the both side of (2.26). Since

$$\left\langle\frac{\partial\mathbf{u}_i}{\partial x_l},x_l\mathbf{u}_i\right\rangle=\int_\Omega x_l\mathbf{u}_i\cdot\frac{\partial\mathbf{u}_i}{\partial x_l}=-\int_\Omega\mathbf{u}_i^2-\int_\Omega x_l\mathbf{u}_i\cdot\frac{\partial\mathbf{u}_i}{\partial x_l},$$

we get

$$\Big\langle \frac{\partial \mathbf{u}_i}{\partial x_l}, x_l\mathbf{u}_i \Big\rangle = -\frac{1}{2}\int_\Omega \mathbf{u}_i\cdot\mathbf{u}_i = -\frac{1}{2}. \tag{2.27}$$

Moreover, we have

$$\begin{aligned}\langle R_l\mathbf{u}_i, x_l\mathbf{u}_i\rangle &= \int_\Omega x_l\mathbf{u}_i\cdot[(\mathrm{div}\mathbf{u}_i)\mathrm{grad}x_l + \mathrm{grad}(\mathbf{u}_i\cdot\mathbf{e}_l)]\\ &= \int_\Omega x_l(\mathrm{div}\mathbf{u}_i)\mathbf{u}_i\cdot\mathbf{e}_l - \int_\Omega \mathbf{u}_i\cdot\mathbf{e}_l[x_l(\mathrm{div}\mathbf{u}_i)+\mathbf{u}_i\cdot\mathbf{e}_l]\\ &= -\int_\Omega(\mathbf{u}_i\cdot\mathbf{e}_l)^2.\end{aligned} \tag{2.28}$$

Hence, it follows from (2.12), (2.13), (2.27) and (2.28) that

$$\begin{aligned}\sum_{l=1}^n\langle[L,x_l]\mathbf{u}_i, x_l\mathbf{u}_i\rangle &= \sum_{l=1}^n\langle(([-\Delta,x_l]+\alpha[M,x_l])\mathbf{u}_i, x_l\mathbf{u}_i\rangle\\ &= -2\sum_{l=1}^n\Big\langle\frac{\partial\mathbf{u}_i}{\partial x_l}, x_l\mathbf{u}_i\Big\rangle - \alpha\sum_{l=1}^n\langle R_l\mathbf{u}_i, x_l\mathbf{u}_i\rangle\\ &= \sum_{l=1}^n\Big[1+\alpha\int_\Omega(\mathbf{u}_i\cdot\mathbf{e}_l)^2\Big]\\ &= n+\alpha.\end{aligned} \tag{2.29}$$

At the same time, we derive

$$\Big\langle\Big[\frac{\partial}{\partial x_l}, x_l\Big]\mathbf{u}_i, \mathbf{u}_i\Big\rangle = \int_\Omega \mathbf{u}_i\cdot\Big[\frac{\partial}{\partial x_l}(x_l\mathbf{u}_i) - x_l\frac{\partial\mathbf{u}_i}{\partial x_l}\Big] = \|\mathbf{u}_i\|^2 = 1. \tag{2.30}$$

Since $\alpha \geqslant 0$, it holds

$$\sum_{l=1}^n\|\frac{\partial}{\partial x_l}\mathbf{u}_i\|^2 = -\int_\Omega\mathbf{u}_i\cdot\Delta\mathbf{u}_i = \sigma_i - \alpha\int_\Omega(\mathrm{div}\mathbf{u}_i)^2 \leqslant \sigma_i. \tag{2.31}$$

Substituting (2.29), (2.30) and (2.31) into (2.26), we can deduce

$$\Big[\sum_{i=1}^k f(\sigma_i)\Big]^2 \leqslant \frac{4(n+\alpha)}{n^2}\sum_{i=1}^k g(\sigma_i)\sum_{i=1}^k\frac{f^2(\sigma_i)}{g(\sigma_i)(\sigma_{k+1}-\sigma_i)}\sigma_i. \tag{2.32}$$

Taking $f(\sigma_i) = (\sigma_{k+1}-\sigma_i)^2$ and $g(\sigma_i) = (\sigma_{k+1}-\sigma_i)^3$ in (2.32), we obtain (1.14). Taking $f(\sigma_i) = g(\sigma_i) = (\sigma_{k+1}-\sigma_i)^{\frac{3}{2}}$ in (2.32), we get (1.15). This concludes the proof of Theorem 2. □

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