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A NOTE ON FROBENIUS NORM PRESERVERS OF JORDAN PRODUCT

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Abstract. We classify maps on n–by–n complex matrices which preserve the Frobenius norm of Jordan product.

1. Introduction

Recently, preserver problems with respect to various algebraic operations on M_n, the algebra of all $n\times n$ complex matrices, attracted a lot of attention. In our recent work [4], we completely characterized surjective maps on M_n, $n\geqslant 3$, the algebra of $n\times n$ complex matrices, having the following property:

$$\|\phi(A)\phi(B)+\phi(B)\phi(A)\| = \|AB+BA\| \quad \text{for all } A,B\in M_n, \tag{1}$$

where $\|\cdot\|$ denotes the Frobenius norm,

$$\|(a_{ij})\| = \sqrt{\operatorname{trace}(A^*A)} = \left(\sum_{i,j=1}^{n} |a_{ij}|^2\right)^{1/2}.$$

As it is well known, the Frobenius norm is unitary invariant; i.e. $\|UAV\| = \|A\|$ for all unitary U,V and $A\in M_n$.

In this note we characterize maps on M_n, $n\geqslant 2$, having the property (1) without surjectivity assumption. We replace it by demand that ϕ is also norm preserving in a sense that $\|A\| = \|\phi(A)\|$ for all A.

REMARK. To counter the lack of surjectivity, we might have assumed unitality. However, we decided to assume that ϕ preserves the norm of every matrix. Namely, the assumption that $\phi(I) = \mu I$, for some unimodular complex number μ, immediately implies that $\|\phi(A)\| = \|A\|$ for every matrix A. Indeed,

$$\|A\| = \frac{1}{2}\|A\circ I\| = \frac{1}{2}\|\phi(A)\circ\phi(I)\| = \frac{1}{2}\|\phi(A)\circ\mu I\| = \|\phi(A)\|.$$

The converse statement, that property (1) together with norm preserving property imply that all unimodular scalar multiples of the identity are preserved, is not that obvious.

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The following four *standard* bijective maps on M_n will be used:

$$X \mapsto X \quad \text{identity map}, \qquad X \mapsto \overline{X} \quad \text{complex conjugation},$$
$$X \mapsto X^{\mathrm{tr}} \ \text{transposition}, \qquad X \mapsto X^{*} \ \text{conjugate transposition}.$$

By the map $\# : M_n \to M_n, \quad A \mapsto A^{\#}$, any of the above standard maps will be referred to.

Denote by $\mathbb{C}$ and $\mathsf{T} \subset \mathbb{C}$ the complex field and the unit circle, respectively. By projections we mean Hermitian idempotents, i.e. matrices P satisfying $P^2 = P = P^*$. As usual, $\mathbb{C}^n$ is the vector space of complex column vectors of length n and $\mathbf{e}_1, \dots, \mathbf{e}_n$ is its standard orthonormal basis. Let $E_{ij} = \mathbf{e}_i \mathbf{e}_j^*$, $1 \leqslant i, j \leqslant n$, be the standard basis for M_n.

In the sequel, we will often, possibly without referencing, use the folowing elementary fact on complex numbers [4, Lemma 3.2].

LEMMA 1.1. *Let $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$, $n \geqslant 2$, be complex numbers such that*

$$|a_i| = |b_i|, \quad i = 1, 2, \dots, n,$$
$$\left|a_i + a_j\right| = \left|b_i + b_j\right|, \quad j \neq i, \ i, j = 1, 2, \dots, n.$$

Then there exists a $\mu \in \mathsf{T}$ such that at least one of the following two possibilities holds:
(1) $(a_1, a_2, \dots, a_n) = \mu\,(b_1, b_2, \dots, b_n)$; (2) $(a_1, a_2, \dots, a_n) = \mu\left(\overline{b_1}, \overline{b_2}, \dots, \overline{b_n}\right)$.

2. Main result and proofs

Our aim is to prove the following Theorem.

THEOREM 2.1. *Let $\|\cdot\|$ be the Frobenius norm. A map $\phi : M_n \to M_n$, $n \geqslant 2$, satisfies*

$$\|\phi(A) \circ \phi(B)\| = \|A \circ B\|, \quad A, B \in M_n, \tag{2}$$
$$\|\phi(A)\| = \|A\|, \quad A \in M_n, \tag{3}$$

if and only if there exist:
(1) a unitary matrix W;
(2) a map $\gamma : M_n \to \mathsf{T}$;
(3) a standard map $X \mapsto X^{\#}$;
(4) a subset $\mathcal{N}_0$, possibly empty, of $\mathcal{N}_n$, the set of $n \times n$ normal matrices, such that

$$\phi(X) = \begin{cases} \gamma(X) W X^{\#} W^* & \text{if} \quad X \in M_n \backslash \mathcal{N}_0, \\ \gamma(X) W \left(X^{\#}\right)^* W^* & \text{if} \quad X \in \mathcal{N}_0. \end{cases} \tag{4}$$

Before presenting the proof we need some Lemmas. The first one is a characterization of multiples of rank-one projections via equality of Frobenius norms.

LEMMA 2.2. *Let $\|\cdot\|$ be the Frobenius norm. A matrix B is a scalar multiple of a rank-one projection if and only if $\|B^2\| = \|B\|^2$.*

Proof. Let $B = \lambda P$, $\lambda \in \mathbb{C}$, $P^2 = P$, $P = P^*$ and $\operatorname{rank} P = 1$. Then there exists a unitary matrix U such that $B = \lambda U^* E_{11} U$. Since $\|U^* E_{11} U\| = 1$,

$$\|B^2\| = |\lambda^2| \, \|U^* E_{11} U\| = |\lambda^2| = \|\lambda U^* E_{11} U\|^2 = \|B\|^2.$$

Assume now that $\|B^2\| = \|B\|^2$. Then, a singular value decomposition gives $B = UDV$ for some unitary U, V and some diagonal $D = \operatorname{diag}(s_1, \ldots, s_n)$, with $s_1 \geqslant s_2 \geqslant \ldots \geqslant s_n \geqslant 0$. Therefore, by the unitary invariance,

$$\|B^2\| = \|UDVUDV\| = \|DWD\|; \qquad (W = (w_{ij}) := VU).$$

We claim that $\operatorname{rank} B \leqslant 1$. From $DWD = (s_i s_j w_{ij})$ we deduce that

$$\|B^2\| = \left(\sum_{i,j=1}^{n} s_i^2 s_j^2 |w_{ij}|^2 \right)^{1/2}.$$

As $|w_{ij}| \leqslant 1$, for all i, j, we have

$$\|B^2\| = \left(\sum_{i,j=1}^{n} s_i^2 s_j^2 |w_{ij}|^2 \right)^{1/2} \leqslant \left(\sum_{i,j=1}^{n} s_i^2 s_j^2 \right)^{1/2} = \sum_{i,j=1}^{n} s_i^2 = \|B\|^2.$$

Squaring both sides reveals that the equality $\|B^2\| = \|B\|^2$ holds if and only if we have $s_i^2 s_j^2 \left(1 - |w_{ij}|^2\right) = 0$ for all i, j. Assume for distinct indices i, j we have that s_i and s_j are both nonzero. Then $|w_{ii}| = 1 = |w_{jj}| = |w_{ij}|$, which contradicts the fact that W is unitary. Hence, at most one singular value of B can be nonzero and so $\operatorname{rank} B \leqslant 1$.

If $s_1 = 0$, then $B = 0$. Else, $s_1 > 0 = s_2 = \ldots = s_n$. It is easy to see that $|w_{11}| = 1$, wherefrom $W = w_{11} \oplus W'$ because W is unitary. Using $W = VU$ we get that $V = (w_{11} \oplus W')U^*$, so that $B = UDV = U s_1 E_{11} V = s_1 U (w_{11} E_{11}) U^*$ must be a scalar multiple of a rank-one projection. $\square$

LEMMA 2.3. *Let $\|\cdot\|$ be the Frobenius norm. Suppose that for matrices $A = (a_{ij})$, $B = (b_{ij}) \in M_n$ we have $\|A\| = \|B\|$ and $\|A \circ E_{ii}\| = \|B \circ E_{ii}\|$, $i = 1, 2, \ldots, n$. Then*

$$\sum_{\substack{1 \leqslant i,j \leqslant n \\ i \neq j}} |a_{ij}|^2 = \sum_{\substack{1 \leqslant i,j \leqslant n \\ i \neq j}} |b_{ij}|^2 \tag{5}$$

and

$$\sum_{i=1}^{n} |a_{ii}|^2 = \sum_{i=1}^{n} |b_{ii}|^2.$$

Moreover, if $n = 2$, we have $|a_{ii}| = |b_{ii}|$, $i = 1, 2$.

Hence, the matrix A is diagonal if and only if B is diagonal and in that case, we have also $|a_{ii}| = |b_{ii}|$, $i = 1, 2, \ldots n$.

Proof. Equality of norms of A and B implies that

$$\sum_{i,j=1}^{n} |a_{ij}|^2 = \sum_{i,j=1}^{n} |b_{ij}|^2. \tag{6}$$

From $\|A \circ E_{ii}\| = \|B \circ E_{ii}\|$ it follows that

$$\|A \circ E_{ii}\|^2 = 4|a_{ii}|^2 + \sum_{i<j} \left(|a_{ij}|^2 + |a_{ji}|^2\right) \tag{7}$$

$$\sum_{i=1}^{n} \|A \circ E_{ii}\|^2 = 4\sum_{i=1}^{n} |a_{ii}|^2 + \sum_{i=1}^{n}\sum_{i<j} \left(|a_{ij}|^2 + |a_{ji}|^2\right)$$
$$= 2\sum_{i=1}^{n} |a_{ii}|^2 + 2\|A\|^2$$

$$\sum_{i=1}^{n} \|B \circ E_{ii}\|^2 = 4\sum_{i=1}^{n} |b_{ii}|^2 + \sum_{i=1}^{n}\sum_{i<j} \left(|b_{ij}|^2 + |b_{ji}|^2\right)$$
$$= 2\sum_{i=1}^{n} |b_{ii}|^2 + 2\|B\|^2$$

wherefrom it follows that $\sum_{i=1}^{n} |a_{ii}|^2 = \sum_{i=1}^{n} |b_{ii}|^2$. Equality (5) then follows from the equality of norms of A and B.

Clearly, A is diagonal if and only if $\sum_{i\neq j} |a_{ij}|^2 = 0 = \sum_{i\neq j} |b_{ij}|^2$ which is equivalent to the diagonality of B. That $|a_{ii}| = |b_{ii}|$, $i = 1,2,\ldots,n$, in this case, follows from (7).

Let now $n = 2$. Then $|a_{11}|^2 + |a_{22}|^2 = |b_{11}|^2 + |b_{22}|^2$ and also, as

$$\|A \circ E_{11}\|^2 - \|A \circ E_{22}\|^2 = 4|a_{11}|^2 - 4|a_{22}|^2$$
$$\|B \circ E_{11}\|^2 - \|B \circ E_{22}\|^2 = 4|b_{11}|^2 - 4|b_{22}|^2,$$

$|a_{11}|^2 - |a_{22}|^2 = |b_{11}|^2 - |b_{22}|^2$, the desired conclusion follows. □

LEMMA 2.4. *Let $\phi : M_2 \to M_2$ have the properties* (2) *and* (3) *from Theorem* 2.1 *and let $\phi(E_{ii}) = \mu_{ii}E_{ii}$, $|\mu_{ii}| = 1$, $i = 1,2$. Then there exist functions μ_{12}, $\mu_{21} : \mathbb{C} \to \mathbb{C}$, such that $|\mu_{ij}(x)| = |x|$ for every $x \in \mathbb{C}$ and*

$$\phi(xE_{12}) = \mu_{12}(x)E_{12} \text{ and } \phi(xE_{21}) = \mu_{21}(x)E_{21},$$

or,

$$\phi(xE_{12}) = \mu_{12}(x)E_{21} \text{ and } \phi(xE_{21}) = \mu_{21}(x)E_{12}.$$

Proof. Let $x \neq 0$ and let $B = \phi(xE_{12}) = (b_{ij})$. By Lemma 2.3, $b_{11} = b_{22} = 0$ and $|b_{12}|^2 + |b_{21}|^2 = |x|^2$. Since xE_{12} is a square-zero nilpotent, $\|(xE_{12}) \circ (xE_{12})\| = 0$, therefore, $\frac{1}{2}\|B\circ B\| = \|B^2\| = \|b_{12}b_{21}I\| = 0$. So, either $B = b_{12}E_{12}$ or $B = b_{21}E_{21}$. Consider $\phi(yE_{21}) = C = (c_{ij})$, $y \neq 0$. In the same way as above we get $C = c_{12}E_{12}$ or $C = c_{21}E_{21}$. But it is impossible that $B = b_{12}E_{12}$ and $C = c_{12}E_{12}$ since $xE_{12}\circ yE_{21} = xyE_{11}$ but $B\circ C = 0$. Also $B = b_{21}E_{21}$ and $C = c_{21}E_{21}$ cannot hold true simultaneously. So, either $B = b_{12}E_{12}$, $C = c_{21}E_{21}$ or, $B = b_{21}E_{21}$, $C = c_{12}E_{12}$. Clearly, b_{ij} and c_{ij} are dependent on x so the equality $|\mu_{ij}(x)| = |x|$ follows from the equality of norms. □

LEMMA 2.5. *Let $\phi : M_2 \to M_2$ have the properties (2) and (3) from Theorem 2.1. Assume further that it maps rank-one projections to scalar multiples of rank-one projections and that $\phi(E_{ij}) = \mu_{ij}E_{ij}$, $\mu_{ij} \in \mathsf{T}$, $i,j = 1,2$. Then there exists a diagonal unitary matrix U such that either*

$$\phi(P) = \mu_P UPU^*, \quad \mu_P \in \mathsf{T},$$

for every rank-one projection P, or,

$$\phi(P) = \mu_P U\overline{P}U^*, \quad \mu_P \in \mathsf{T},$$

for every rank-one projection P.

Proof. We will first show that there exists a diagonal unitary matrix U such that for every $x \in \mathbb{C}$ and every $i,j,k = 1,2$, $i \neq j$, $\phi(E_{kk} + xE_{ij}) = \alpha_x U(E_{kk} + xE_{ij})U^*$ simultaneously or, $\phi(E_{kk} + xE_{ij}) = \alpha_x U(E_{kk} + \overline{x}E_{ij})U^*$ simultaneously, where $\alpha_x \in \mathsf{T}$ is also dependent on i,j,k. Let us start with $(i,j) = (1,2)$, and set $\phi(E_{11} + xE_{12}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Since $\phi(E_{ii}) = \mu_{ii}E_{ii}$, by Lemma 2.3 it follows that $|a| = 1$, $|d| = 0$, $|b|^2 + |c|^2 = |x|^2$. By equating the norms of matrices

$$(E_{11} + xE_{12})\circ E_{12} = E_{12}$$

and

$$\phi(E_{11} + xE_{12})\circ\phi(E_{12}) = \phi(E_{11} + xE_{12})\circ\mu_{12}E_{12} = \mu_{12}\begin{pmatrix} c & a \\ 0 & c \end{pmatrix}$$

it follows $2|c|^2 + |a|^2 = 1$, so $c = 0$. Moreover, $|b| = |x|$, so

$$\phi(E_{11} + xE_{12}) = \alpha_x(E_{11} + b_xE_{12})$$

for some $\alpha_x \in \mathsf{T}$, $|b_x| = |x|$. Similarly we get that $\phi(E_{22} - xE_{12}) = \beta_{-x}(E_{22} + b'_xE_{12})$ for some $\beta_{-x} \in \mathsf{T}$, $|b'_x| = |x|$. Since $(E_{11} + xE_{12})\circ(E_{22} - xE_{12}) = 2E_{11}$ we get $b'_x = -b_x$. Let $\phi(E_{11} + E_{12}) = \alpha_1(E_{11} + b_1E_{12})$. Then by the equality of norms of matrices

$$(E_{11} + E_{12})\circ(E_{11} + xE_{12}) = 2E_{11} + (1+x)E_{12}$$

$$\alpha_1(E_{11} + b_1E_{12})\circ\alpha_x(E_{11} + b_xE_{12}) = \alpha_1\alpha_x(2E_{11} + (b_1 + b_x)E_{12})$$

we obtain $|b_1 + b_x| = |1 + x|$ whence it follows $(b_1, b_x) = \mu(1, x)$ or $(b_1, b_x) = \mu(1, \overline{x})$ for some $\mu \in \mathsf{T}$. Then $\mu = b_1$ and $b_x = b_1x$ or $b_x = b_1\overline{x}$. Replacing ϕ by $X \mapsto$

$B\phi(X)B^*$, $B=\operatorname{diag}(1,b_1)$, we may assume $b_1=1$. Next we show that $b_x=x$ for all x, or, $b_x=\bar{x}$ for all $x\in\mathbb{C}$. Assume that $x\neq\bar{x}$ and $y\neq\bar{y}$ and that $b_x=x$ and $b_y=\bar{y}$. Comparing the norms of

$$(E_{11}+xE_{12})\circ(E_{11}+yE_{12})=2E_{11}+(x+y)E_{12}$$
$$\alpha_x(E_{11}+xE_{12})\circ\alpha_y(E_{11}+\bar{y}E_{12})=\alpha_x\alpha_y(2E_{11}+(x+\bar{y})E_{12})$$

we see that $|x+\bar{y}|=|x+y|$, so $(x,y)=\mu'(x,\bar{y})$ or $(x,y)=\mu'(\bar{x},y)$ for some $\mu'\in\mathsf{T}$. Since $x\neq\bar{x}$ and $y\neq\bar{y}$, both cases lead to a contradiction. Therefore, we conclude that

$$\phi(E_{11}+xE_{12})=\alpha_x(E_{11}+xE_{12}),\quad x\in\mathbb{C},$$

or,

$$\phi(E_{11}+xE_{12})=\alpha_x(E_{11}+\bar{x}E_{12}),\quad x\in\mathbb{C}.$$

In the second case we compose ϕ with conjugation to achieve that for all $x\in\mathbb{C}$

$$\phi(E_{11}+xE_{12})=\alpha_x(E_{11}+xE_{12}).$$

Note that $A=E_{22}-xE_{12}$ is the only matrix, up to scalar multiplication, with $(E_{11}+xE_{12})\circ A=0$ which further implies that

$$\phi(E_{22}-xE_{12})=\beta_{-x}(E_{22}-xE_{12}).$$

In the same way as above, we get that $\phi(E_{11}+xE_{21})=\gamma_x(E_{11}+c_xE_{21})$, and $\phi(E_{22}-xE_{21})=\delta_{-x}(E_{22}-c_xE_{21})$, where $|c_x|=|x|$, $\gamma_x,\ \delta_{-x}\in\mathsf{T}$. Then

$$(E_{11}+xE_{12})\circ(E_{11}+yE_{21})=\begin{pmatrix}2+xy & x\\ y & xy\end{pmatrix}$$

implies that

$$\alpha_x(E_{11}+xE_{12})\circ\delta_y(E_{11}+c_yE_{21})=\alpha_x\delta_y\begin{pmatrix}2+xc_y & x\\ y & xc_y\end{pmatrix},$$

and by equating the norms we get $|2+xy|=|2+xc_y|$ for every $x\in\mathbb{C}$. It follows that either $c_y=y$ or $c_y=\frac{\overline{xy}}{x}$. The later case wrongly implies c_y is dependent on x, wherefrom $c_y=y$.

In order to finish the proof of the Lemma let

$$P=\frac{1}{1+|x|^2}\begin{pmatrix}1 & x\\ \bar{x} & |x|^2\end{pmatrix}\quad\text{and}\quad Q=\phi(P)=\frac{\mu_P}{1+|y|^2}\begin{pmatrix}1 & y\\ \bar{y} & |y|^2\end{pmatrix},$$

for some $\mu_P\in\mathsf{T}$ and $y\in\mathbb{C}$. Note that if $P=E_{11}$ or $P=E_{22}$ $(=\lim_{|x|\to\infty}P)$ there is nothing to do. So assume $x\neq 0$. We will show first that $|x|=|y|$. Computing

$$P\circ E_{11}=\frac{1}{1+|x|^2}\begin{pmatrix}2 & x\\ \bar{x} & 0\end{pmatrix},\quad Q\circ E_{11}=\frac{\mu_P}{1+|y|^2}\begin{pmatrix}2 & y\\ \bar{y} & 0\end{pmatrix},$$

and comparing the norms, we get $\frac{4+2|x|^2}{(1+|x|^2)^2} = \frac{4+2|y|^2}{(1+|y|^2)^2}$, wherefrom $|x| = |y|$ easily follows. It remains to show that $x = y$. Compare the norms of

$$\frac{1}{1+|x|^2}\begin{pmatrix} 1 & x \\ \overline{x} & |x|^2 \end{pmatrix} \circ \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \frac{1}{1+|x|^2}\begin{pmatrix} \overline{x} & 1+|x|^2+x \\ \overline{x} & 2|x|^2+\overline{x} \end{pmatrix}$$
$$\frac{\mu_P}{1+|y|^2}\begin{pmatrix} 1 & y \\ \overline{y} & |y|^2 \end{pmatrix} \circ \beta_1 \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \frac{\mu_P \beta_1}{1+|y|^2}\begin{pmatrix} \overline{y} & 1+|y|^2+y \\ \overline{y} & 2|y|^2+\overline{y} \end{pmatrix}$$

and

$$\frac{1}{1+|x|^2}\begin{pmatrix} 1 & x \\ \overline{x} & |x|^2 \end{pmatrix} \circ \begin{pmatrix} 0 & i \\ 0 & 1 \end{pmatrix} = \frac{1}{1+|x|^2}\begin{pmatrix} i\overline{x} & \left(1+|x|^2\right)i+x \\ \overline{x} & 2|x|^2+i\overline{x} \end{pmatrix}$$
$$\frac{\mu_P}{1+|y|^2}\begin{pmatrix} 1 & y \\ \overline{y} & |y|^2 \end{pmatrix} \circ \beta_i \begin{pmatrix} 0 & i \\ 0 & 1 \end{pmatrix} = \frac{\mu_P \beta_i}{1+|y|^2}\begin{pmatrix} i\overline{y} & \left(1+|y|^2\right)i+y \\ \overline{y} & 2|y|^2+i\overline{y} \end{pmatrix},$$

and use $|x| = |y|$ to obtain that $\operatorname{Re} x = \operatorname{Re} y$ and $\operatorname{Re} ix = \operatorname{Re} iy$. So, $y = x$ and $Q = \mu_P P$. $\square$

In our subsequent Lemmas 2.6 and 2.7 we assume that $\phi : M_n \to M_n$ is a map with the properties (2) and (3) from Theorem 2.1.

LEMMA 2.6. *Assume $\phi(E_{ii}) = \mu_{ii} E_{ii}$, for all i. Then either $\phi(E_{ij}) = \mu_{ij} E_{ij}$, $i, j = 1, 2, \ldots, n$, or, $\phi(E_{ij}) = \mu_{ij} E_{ji}$, $i, j = 1, 2, \ldots, n$. If $\phi(E_{12}) = \mu_{12} E_{12}$, $\mu_{12} \in \mathsf{T}$, then $\phi(E_{ij}) = \mu_{ij} E_{ij}$ for all $i \neq j$.*

Proof. Given indices $i < j$ let ϕ_{ij} be the restriction of ϕ to the space $\mathscr{W}_{ij} := \operatorname{span}\{E_{ii}, E_{ij}, E_{ji}, E_{jj}\}$. Since for every matrix $A \in \mathscr{W}_{ij}$ it holds that $A \circ E_{kk} = 0$ if $k \neq i, j$, then $\phi(A) \circ E_{kk} = 0$ for all $k \neq i, j$, as well. Therefore, $\phi(A) \in W_{ij}$. Mapping $\phi_{ij} : \mathscr{W}_{ij} \to \mathscr{W}_{ij}$ satisfies hypotheses of Lemma 2.4, therefore, $\phi_{ij}(E_{ij}) = \mu_{ij} E_{ij}$ and $\phi_{ij}(E_{ji}) = \mu_{ji} E_{ji}$, or, $\phi_{ij}(E_{ij}) = \mu_{ij} E_{ji}$ and $\phi_{ij}(E_{ji}) = \mu_{ji} E_{ij}$. So, for any $i < j$, $\phi(E_{ij}) = \mu_{ij} E_{ij}$ and $\phi(E_{ji}) = \mu_{ji} E_{ji}$, or, $\phi(E_{ij}) = \mu_{ij} E_{ji}$ and $\phi(E_{ji}) = \mu_{ji} E_{ij}$. Hence, by composing ϕ with transposition, if necessary, we assume that $\phi(E_{12}) = \mu_{12} E_{12}$. Then it follows $\phi(E_{1k}) = \mu_{1k} E_{1k}$, $k = 3, \ldots, n$, because otherwise $\phi(E_{1k}) = \mu_{1k} E_{k1}$, for some k, would imply $E_{12} \circ E_{1k} = 0$, while $\phi(E_{12}) \circ \phi(E_{1k}) = \mu_{12}\mu_{1k}(E_{12} \circ E_{k1}) = \mu_{12}\mu_{1k} E_{k2} \neq 0$, a contradiction. With a similar argument we then show $\phi(E_{ik}) = \mu_{ik} E_{ik}$, $k = 1, 2, \ldots, n$, $i \neq k$. $\square$

LEMMA 2.7. *If $\phi(E_{ij}) = \mu_{ij} E_{ij}$, $i, j = 1, 2, \ldots, n$, or if $\phi(E_{ij}) = \mu_{ij} E_{ji}$, $i, j = 1, 2, \ldots, n$, then for every diagonal matrix D, there exists a $\mu_D \in \mathsf{T}$ such that $\phi(D) = \mu_D D$ or $\phi(D) = \mu_D \overline{D}$.*

Proof. Let $D = \operatorname{diag}(d_1, \ldots, d_n)$. By Lemma 2.3, $B = \phi(D) = \operatorname{diag}(b_1, \ldots, b_n)$ and $|b_i| = |d_i|$, $i = 1, 2, \ldots, n$. Then

$$\left\|D \circ E_{ij}\right\|^2 = \left\|(d_i + d_j) E_{ij}\right\|^2 = \left|d_i + d_j\right|^2$$

and

$$\left\|B\circ E_{ij}\right\|^2 = \left\|(b_i+b_j)E_{ij}\right\|^2 = \left|b_i+b_j\right|^2.$$

This implies that $\left|d_i+d_j\right| = \left|b_i+b_j\right|$, $i,j=1,2,\ldots,n$, $i\neq j$. From Lemma 1.1 the desired conclusion follows. □

LEMMA 2.8. *Let A be an upper or lower triangular $n\times n$ matrix and assume that $\left\|A\circ E_{ij}\right\| = \left\|A^*\circ E_{ij}\right\|$ for all $i,j=1,2,\ldots,n$. Then A is diagonal.*

Proof. Using adjoints, it suffices to consider only upper triangular matrices. Assume, to reach a contradiction, that A is nondiagonal, and let i-th row be the first row of A with nonzero off-diagonal entry. Then, $A\circ E_{in} = (\alpha_{ii}+\alpha_{nn})E_{in}$ while $A^*\circ E_{in} = (\overline{\alpha_{ii}}+\overline{\alpha_{nn}})E_{in} + \sum_{k>i}\overline{\alpha_{ik}}E_{kn} + \sum_{i\leqslant k<n}\overline{\alpha_{kn}}E_{ik}$. Comparing the Frobenius norms reveals that $\alpha_{ik}=0$, for $k=i+1,\ldots,n$ which contradicts the fact that the i-th row of upper-triangular A contains nonzero off-diagonal entry. Recall that a unitary U is generalized permutation matrix, corresponding to a permutation π on the set $\{1,2,\ldots,n\}$ if $E_{ii}U = UE_{\pi(i),\pi(i)}$ for $i=1,\ldots,n$. Equivalently, if each row of U contains only one nonzero entry. □

LEMMA 2.9. *Let $\mathscr{T}_n$ be the subspace of all upper-triangular matrices. Then, for every unitary U, either the intersection $\mathscr{T}_n\cap(U\mathscr{T}_nU^*)$ contains a nondiagonal matrix or, U is a generalized permutation matrix, corresponding to the permutation π defined by $\pi(i)=n+1-i$, $i=1,2,\ldots,n$. In the latter case, $U\mathscr{T}_nU^*$ is the set of all lower triangular matrices.*

Proof. Note that $\operatorname{codim}(\mathscr{T}_n\cap(U\mathscr{T}_nU^*)) \leqslant \operatorname{codim}\mathscr{T}_n + \operatorname{codim}(U\mathscr{T}_nU^*) = 2\frac{n(n-1)}{2}$. Wherefrom, $\dim(\mathscr{T}_n\cap(U\mathscr{T}_nU^*))\geqslant n$. Therefore, if $\mathscr{T}_n\cap(U\mathscr{T}_nU^*)$ contains only diagonal matrices, then its dimension implies that it is equal to the space of diagonal matrices. In which case we conclude that there exists a permutation π on the set $\{1,2,\ldots,n\}$ such that for every i we have $E_{ii} = UE_{\pi(i),\pi(i)}U^*$. So, U is a generalized permutation matrix, corresponding to the permutation π.

Writing π as product of cycles we find that either there exist indices $i<j$, such that $\pi(i)<\pi(j)$, or, π is strictly decreasing, i.e. for every $i<j$, we have $\pi(i)>\pi(j)$. In the first case, $UE_{ij}U^* = u_{ij}E_{\pi(i)\pi(j)}$, $|u_{ij}|=1$, is the desired nondiagonal matrix in the intersection while in the second case, $\pi(i)=n+1-i$, $i=1,2,\ldots,n$. □

Proof of Theorem 2.1. Let us begin with a simple fact that $\left\|A^2\right\| = \frac{1}{2}\|A\circ A\| = \frac{1}{2}\|\phi(A)\circ\phi(A)\| = \left\|\phi(A)^2\right\|$ for every $A\in M_n$. Next, we observe that ϕ maps the set of nonzero scalar multiples of rank-one projections into itself. Indeed, let $A=\lambda P$, P being a rank-one projection, $\lambda\in\mathbb{C}\setminus\{0\}$, and denote $B=\phi(A)$. In view of Lemma 2.2

$$\|B\|^2 = \|\phi(A)\|^2 = \|A\|^2 = \left\|A^2\right\| = \left\|\phi(A)^2\right\| = \left\|B^2\right\|$$

imply that $B=\delta Q$, where Q is a rank-one projection and $\delta\in\mathbb{C}$. Furthermore, $\|B\| = \|A\|$ gives that $|\delta| = |\lambda|\neq 0$. Clearly, $A=0$ if and only if $\phi(A)=0$.

Moreover, if P_1,P_2 are mutually orthogonal rank-one projections, then we claim that $\phi(P_1)\phi(P_2)=\phi(P_2)\phi(P_1)=0$. Namely, there exist unimodular numbers μ_1,μ_2 and rank-one projections Q_1,Q_2, such that $\phi(P_i)=\mu_iQ_i,\ \mu_i\in\mathsf{T},\ i=1,2$. Note that for every rank-one projections $Q_1,Q_2,\ Q_1\perp Q_2$ if and only if $Q_1\circ Q_2=0$. Thus it suffices to show that $\|Q_1\circ Q_2\|=0$ which follows from

$$\|Q_1\circ Q_2\|=\|\mu_1\mu_2Q_1\circ Q_2\|=\|\phi(P_1)\circ\phi(P_2)\|=\|P_1\circ P_2\|=0.$$

Since each rank-one projection equals $\mathbf{x}\mathbf{x}^*$ for some unit vector $\mathbf{x}\in\mathbb{C}^n$, it follows that for every $\mathbf{x}\in\mathbb{C}^n,\ \|\mathbf{x}\|=1$, there exist a unit vector $\mathbf{y}\in\mathbb{C}^n$ and $\mu\in\mathsf{T}$ such that $\phi(\mathbf{x}\mathbf{x}^*)=\mu\mathbf{y}\mathbf{y}^*$.

If $n=2$, there exists a unitary matrix V such that $V^*\phi(E_{ii})V=\mu_{ii}E_{ii},\ i=1,2$, $\mu_{ii}\in\mathsf{T}$. Then by Lemma 2.4, $V^*\phi(E_{ij})V=\mu_{ij}E_{ij},\ i,j=1,2$, or, $V^*\phi(E_{ij})V=\mu_{ij}E_{ji}$, $i,j=1,2$. Replacing ϕ by $X\mapsto V^*\phi(X)V$ or $X\mapsto(V^*\phi(X)V)^*$, if necessary, we may assume that $\phi(E_{ij})=\mu_{ij}E_{ij},\ i,j=1,2,\ \mu_{ij}\in\mathsf{T}$. By Lemma 2.5 there is a diagonal unitary matrix U such that $\phi(P)=\mu_PUPU^*,\ \mu_P\in\mathsf{T}$, for every rank-one projection P, or $\phi(P)=\mu_PU\overline{P}U^*,\ \mu_P\in\mathsf{T}$, for every rank-one projection P. Replacing ϕ by $X\mapsto U^*\phi(X)U$ or, by $X\mapsto\overline{U^*\phi(X)U}$, if neccessary, enables us to further assume that $\phi(P)=\mu_PP,\ \mu_P\in\mathsf{T}$, for every rank-one projection P; the new map still satisfies assumptions (2) and (3) of Theorem 2.1. □

If $n\geqslant 3$, then ϕ induces a map from the projective space

$$\mathbb{P}(\mathbb{C}^n)=\{[\mathbf{x}]=\mathbb{C}\mathbf{x};\ \mathbf{x}\in\mathbb{C}^n\setminus\{\mathbf{0}\}\}$$

into itself which preserves orthogonality. We can now use the following Lemma from [1].

LEMMA 2.10. *Let $n\geqslant 3$. Suppose a map $\varphi:\mathbb{P}(\mathbb{C}^n)\to\mathbb{P}(\mathbb{C}^n)$ preserves orthogonality. Then there exists a unitary matrix V such that $\varphi([\mathbf{x}])=[V\mathbf{x}],\ \mathbf{x}\in\mathbb{C}^n$, or $\varphi([\mathbf{x}])=[V\overline{\mathbf{x}}],\ \mathbf{x}\in\mathbb{C}^n$.*

It follows that $\phi(\mathbf{x}\mathbf{x}^*)=\mu_{\mathbf{x}}V\mathbf{x}(V\mathbf{x})^*=\mu_{\mathbf{x}}V\mathbf{x}\mathbf{x}^*V^*$ for all unit vectors $\mathbf{x}$, or, $\phi(\mathbf{x}\mathbf{x}^*)$ $=\mu_{\mathbf{x}}V\overline{\mathbf{x}\mathbf{x}^*}V^*$ for all unit $\mathbf{x}$. We may replace ϕ by the map $X\mapsto V^*\phi(X)V$ or by $X\mapsto V^*\overline{\phi(X)}V$, to achieve $\phi(P)=\mu_PP,\ \mu_P\in\mathsf{T}$, for every rank-one projection P.

Let now $n\geqslant 2$. We will next show that for every normal matrix A there exists $\mu_A\in\mathsf{T}$ such that $\phi(A)=\mu_AA$ or $\phi(A)=\mu_AA^*$. In order to do it let us choose a unitary matrix U such that $U^*AU=D$ is a diagonal matrix. Let $\phi_1(X):=U^*\phi(UXU^*)U$. Then we have $\phi_1(E_{ii})=\mu_{ii}E_{ii},\ \mu_{ii}\in\mathsf{T}$, for every $i=1,2,\ldots,n$. By Lemma 2.6, we have either $\phi_1(E_{ij})=\mu_{ij}E_{ij}$ for all i,j or, $\phi_1(E_{ij})=\mu_{ij}E_{ji}$ for all i,j. Then Lemma 2.7 assures that there exists a $\mu_D\in\mathsf{T}$ such that $\phi_1(D)=\mu_DD$ or $\phi_1(D)=\mu_D\overline{D}$. This now implies that $\phi(A)=\mu_AA$ or $\phi(A)=\mu_AA^*$ for some unimodular μ_A.

Replacing ϕ by $X\mapsto\phi(X)^*$, if necessary, we can now assume that $\phi(E_{ij})=\mu_{ij}E_{ij}$ for all i,j. Note that in the Frobenius norm, $\|\mu_AA^*\circ\phi(B)\|=\|A\circ\phi(B)\|$, for every normal matrix A [4, Lemma 6.4]. Therefore, we can further adjust the map ϕ on normal matrices so that ϕ fixes every normal matrix.

Pick now any nonnormal matrix A and let $B=\phi(A)$. Then

$$\|A\circ X\| = \|\phi(A)\circ\phi(X)\| = \|B\circ X\|$$

for every normal X. By [4, Lemma 6.6] there exists $\gamma\in\mathsf{T}$ such that either $\operatorname{diagv}(A) = \gamma\operatorname{diagv}(B)$, or $\operatorname{diagv}(A)=\gamma\overline{\operatorname{diagv}(B)}$ where $\operatorname{diagv}((c_{ij}))=(c_{11},c_{22},\ldots,c_{nn})^{\mathrm{tr}}\in\mathbb{C}^n$. We remark that if $n=2$, the same result can be more directly obtained by applying Lemma 2.3 to get $|a_{ii}|=|b_{ii}|$, $i=1,2$, and $|a_{12}|^2+|a_{21}|^2=|b_{12}|^2+|b_{21}|^2$. Then, by summing up the equalities

$$|a_{11}+a_{22}|^2+2|a_{21}|^2 = \|A\circ E_{12}\| = \|\phi(A)\circ\mu_{12}E_{12}\| = |b_{11}+b_{22}|^2+2|b_{21}|^2$$
$$|a_{11}+a_{22}|^2+2|a_{12}|^2 = \|A\circ E_{21}\| = \|\phi(A)\circ\mu_{21}E_{21}\| = |b_{11}+b_{22}|^2+2|b_{12}|^2$$

and using Lemma 1.1 it follows that $\operatorname{diagv}(A)=\gamma\operatorname{diagv}(B)$, or $\operatorname{diagv}(A)=\gamma\overline{\operatorname{diagv}(B)}$ for some $\gamma\in\mathsf{T}$. Then [3, Theorem 3.2] implies $B=\gamma_A A$ or $B=\gamma_A A^*$, $\gamma_A\in\mathsf{T}$.

To bring the proof to the end, it suffices to show that the latter is impossible. Since A is not normal, there exists a unitary matrix U such that $U^*AU=T_0$ is non-diagonal upper triangular matrix. Recall that $\phi(E_{ij})=\mu_{ij}E_{ij}$ for all i,j, so, the Lemma 2.8 provides that $\phi(T)=\gamma_T T$ for every upper or lower triangular matrix T. Since $\phi(UE_{ii}U^*)=\gamma_{ii}UE_{ii}U^*$, $\gamma_{ii}\in\mathsf{T}$, for every i, passing to $X\mapsto U^*\phi(UXU^*)U$ and applying Lemma 2.6, we get either $\phi(UE_{ij}U^*)=\gamma_{ij}UE_{ij}U^*$, for all i,j, or, $\phi(UE_{ij}U^*)=\gamma_{ij}UE_{ji}U^*$ for all i,j.

Lemma 2.8 shows that the latter case would imply that $U^*\phi(UTU^*)U=\gamma_T T^*$ or, equivalently, $\phi(UTU^*)=\gamma_T UT^*U^*$ for every upper-triangular T. However, by Lemma 2.10, either U is a generalized permutation matrix, corresponding to permutation π, $\pi(i)=n+1-i$, $i=1,2,\ldots,n$, or, there exists a nondiagonal $T\in\mathscr{T}_n\cap(U\mathscr{T}_nU^*)$. In the first case, $A\in U\mathscr{T}_nU^*$ is in fact lower triangular, so, $\phi(A)=\gamma_A A$. But on the other hand, $\phi(A)=\phi(UT_0U^*)=\gamma' UT_0^*U^*=\gamma' A^*$, which is impossible since A is not diagonal. In the second case, $T\in\mathscr{T}_n$ implies $\phi(T)=\gamma_1 T$, while $T=UT_1U^*\in(U\mathscr{T}_nU^*)$ implies $\phi(T)=\phi(UT_1U^*)=\gamma_2 UT_1^*U^*=\gamma_2 T^*$, a contradiction.

That any map of the form (4) satisfies (2) and (3), is easy to check and was done in [4].

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