Operators
and
Matrices
Volume 7, Number 4 (2013), 927–946 doi:10.7153/oam-07-52

ESSENTIAL SPECTRA OF QUASI–PARABOLIC COMPOSITION OPERATORS ON HARDY SPACES OF THE POLY–DISC

UĞUR GÜL

(*Communicated by Raúl Curto*)

Abstract. In this paper we study the essential spectra of a class of composition operators on the Hilbert-Hardy space of the bi-disc which is called ”quasi-parabolic” and whose one variable analogue was studied in [2]. As in [2], quasi-parabolic composition operators on the Hilbert-Hardy space of the bi-disc are written as a linear combination of Toeplitz operators and Fourier multipliers. The C*-algebra generated by Toeplitz operators and Fourier multipliers on the Hilbert-Hardy space of the bi-disc is written as the tensor product of the similar C*-algebra in one variable with itself. As a result we find a nontrivial set consisting of spiral curves lying inside the essential spectra of quasi-parabolic composition operators.

Introduction

Quasi-parabolic composition operators is a generalization of the composition operators induced by parabolic linear fractional non-automorphisms of the unit disc that fix a point ξ on the boundary. These linear fractional transformations for $\xi = 1$ take the form

$$\varphi_a(z) = \frac{2iz + a(1-z)}{2i + a(1-z)}$$

with $\Im(a) > 0$. Quasi-parabolic composition operators on $H^2(\mathbb{D})$ are composition operators induced by the symbols where 'a' is replaced by a bounded analytic function 'ψ' for which $\Im(\psi(z)) > \delta > 0 \quad \forall z \in \mathbb{D}$. We recall that the set of cluster points $\mathscr{C}_\xi(\psi)$ of $\psi \in H^\infty(\mathbb{D})$ at $\xi \in \mathbb{T}$ is defined to be the set of points $z \in \mathbb{C}$ for which there is a sequence $\{z_n\} \subset \mathbb{D}$ so that $z_n \to \xi$ and $\psi(z_n) \to z$. Similarly for the bi-disc, $\mathscr{C}_{(\xi_1,\xi_2)}(\psi)$ of $\psi \in H^\infty(\mathbb{D}^2)$ at $(\xi_1,\xi_2) \in \mathbb{T}^2$ is defined to be the set of points $z \in \mathbb{C}$ for which there is a sequence $\{z_n\} \subset \mathbb{D}^2$ so that $z_n \to (\xi_1,\xi_2)$ and $\psi(z_n) \to z$. In [2] we showed that if $\psi \in QC(\mathbb{T})$ then these composition operators are essentially normal and their essential spectra are given as

$$\sigma_e(C_\varphi) = \{e^{izt} : t \in [0,\infty], z \in \mathscr{C}_1(\psi)\} \cup \{0\}$$

where $\mathscr{C}_1(\psi)$ is the set of cluster points of ψ at 1.

Mathematics subject classification (2010): 32A45.
Keywords and phrases: Composition operators, Hardy spaces of the poly-disc, essential spectra.

Paper OaM-07-52

In this work we investigate this phenomenon in the bi-disc setting. We look at the composition operators on the Hardy space of the bi-disc induced by symbols of the form

$$\varphi(z_1,z_2)=\left(\frac{2iz_1+\psi_1(z_1,z_2)(1-z_1)}{2i+\psi_1(z_1,z_2)(1-z_1)},\frac{2iz_2+\psi_2(z_1,z_2)(1-z_2)}{2i+\psi_2(z_1,z_2)(1-z_2)}\right)$$

where $\psi_1,\psi_2\in H^\infty(\mathbb{D}^2)$ such that $\Im(\psi_j(z_1,z_2))>\delta>0 \quad \forall(z_1,z_2)\in\mathbb{D}^2,\ j=1,2.$ These symbols are carried over via the Cayley transform to the symbols of the form

$$\tilde{\varphi}(w_1,w_2)=(w_1+\psi_1\circ\mathfrak{C}_2(w_1,w_2),w_2+\psi_2\circ\mathfrak{C}_2(w_1,w_2))$$

on the two dimensional upper half-plane $\mathbb{H}^2$,i.e.

$$\mathfrak{C}_2^{-1}\circ\varphi\circ\mathfrak{C}_2=\tilde{\varphi}$$

where

$$\mathfrak{C}_2(z_1,z_2)=\left(\frac{z_1-i}{z_1+i},\frac{z_2-i}{z_2+i}\right)$$

is the Cayley transform. In particular we prove the following result:

MAIN THEOREM 2. *Let $\varphi:\mathbb{D}^2\to\mathbb{D}^2$ be an analytic self-map of $\mathbb{D}^2$ such that*

$$\varphi(z_1,z_2)=\left(\frac{2iz_1+\psi_1(z_1,z_2)(1-z_1)}{2i+\psi_1(z_1,z_2)(1-z_1)},\frac{2iz_2+\psi_2(z_1,z_2)(1-z_2)}{2i+\psi_2(z_1,z_2)(1-z_2)}\right)$$

where $\psi_j\in H^\infty(\mathbb{D}^2)$ with $\Im(\psi_j(z_1,z_2))>\varepsilon>0$ for all $(z_1,z_2)\in\mathbb{D}^2,\ j=1,2$. Then $C_\varphi:\ H^2(\mathbb{D}^2)\to H^2(\mathbb{D}^2)$ is bounded. Moreover if $\psi_j\in(QC\otimes QC)\cap H^\infty(\mathbb{D}^2)$ then we have

$$\sigma_e(C_\varphi)\supseteq\{e^{i(z_1t_1+z_2t_2)}:t_1,t_2\in[0,\infty],z_1\in\mathscr{C}_{(1,1)}(\psi_1)\textit{and}\quad z_2\in\mathscr{C}_{(1,1)}(\psi_2)\}\cup\{0\},$$

where $\mathscr{C}_{(1,1)}(\psi)$ is the set of cluster points of ψ at $(1,1)\in\mathbb{T}^2$.

We work on the two dimensional upper half-plane $\mathbb{H}^2$ and use Banach algebra techniques to compute the essential spectra of operators that correspond to "quasi-parabolic" operators. As in [2] translation operators on $H^2(\mathbb{H}^2)$ can be considered as Fourier multipliers on $H^2(\mathbb{H}^2)$ where $\mathbb{H}^2$ is considered as a tubular domain (we refer the reader to [8] for the definition and properties of Fourier multipliers on Hardy and Bergman spaces of tubular domains in several complex variables). Throughout the present work, $H^2(\mathbb{H}^2)$ will be considered as a closed subspace of $L^2(\mathbb{R}^2)$ via the boundary values. With the help of Cauchy integral formula we prove an integral formula that gives composition operators as integral operators. Using this integral formula we show that operators that correspond to"quasi-parabolic" operators fall in a C*-algebra generated by Toeplitz operators and Fourier multipliers.

The remainder of this paper is organized as follows: In section 1 we give the basic definitions and preliminary material that we will use throughout. For the benefit of the reader we explicitly recall some facts about C*-algebras, tensor products of C*-algebras and nuclear C*-algebras. Using a version of Paley-Wiener theorem due to Bochner we also introduce the C*-algebra of Fourier multipliers acting on $H^2(\mathbb{H}^2)$. In Section 2 we first show that "quasi-parabolic" composition operators are bounded on $H^2(\mathbb{H}^2)$ and prove an integral representation formula for composition operators on $H^2(\mathbb{H}^2)$. Then we use this integral formula together with the boundedness result to prove that a "quasi-parabolic" composition operator is written as a series of Toeplitz operators and Fourier multipliers which converges in operator norm. In section 3 we analyze the C*-algebra generated by Toeplitz operators with $QC(\mathbb{R})\otimes QC(\mathbb{R})$ symbols and Fourier multipliers modulo compact operators. We write this C*-algebra as the tensor product of the C*-algebra Ψ in [2] with itself. In doing this we follow the approach taken by [1] for analyzing the Toeplitz C*-algebra of the bi-disc. We use this tensor product to identify the character space of the C*-algebra generated by Toeplitz operators with $QC(\mathbb{R})\otimes QC(\mathbb{R})$ symbols and Fourier multipliers modulo compact operators. In section 4, using the machinery developed in sections 2 and 3, we obtain some results about the essential spectra of "quasi-parabolic" composition operators.

1. Preliminaries

In this section we fix the notation that we will use throughout and recall some preliminary facts that will be used in the sequel.

Let S be a compact Hausdorff topological space. The space of all complex valued continuous functions on S will be denoted by $C(S)$. For any $f\in C(S)$, $\| f \|_\infty$ will denote the sup-norm of f, i.e.

$$\| f \|_\infty = \sup\{| f(s) | : s\in S\}.$$

For a Banach space X, $K(X)$ will denote the space of all compact operators on X and $B(X)$ will denote the space of all bounded linear operators on X. The open unit disc will be denoted by $\mathbb{D}$, the open upper half-plane will be denoted by $\mathbb{H}$, the real line will be denoted by $\mathbb{R}$ and the complex plane will be denoted by $\mathbb{C}$. The one point compactification of $\mathbb{R}$ will be denoted by $\dot{\mathbb{R}}$ which is homeomorphic to $\mathbb{T}$. For any $z\in \mathbb{C}$, $\Re(z)$ will denote the real part, and $\Im(z)$ will denote the imaginary part of z, respectively. For any subset $S\subset B(H)$, where H is a Hilbert space, the C*-algebra generated by S will be denoted by $C^*(S)$ and for any subset $S\subset A$ where A is a C*-algebra, the closed two-sided ideal generated by S will be denoted by $I^*(S)$.

The Hardy space of the bi-disc $H^2(\mathbb{D}^2)$ is identified as the tensor product of the two copies of the classical Hardy space of the unit disc $H^2(\mathbb{D})$, i.e. the closure of the linear span of the set of functions

$$\{h(z,w)=f(z)g(w) : f,g\in H^2(\mathbb{D})\}$$

with respect to the inner product

$$\langle h_1,h_2\rangle = \int_0^{2\pi}\int_0^{2\pi} h_1(e^{i\theta_1},e^{i\theta_2})\overline{h_2(e^{i\theta_1},e^{i\theta_2})}\,d\theta_1 d\theta_2.$$

In the same way the Hardy space of the two dimensional half-plane $H^2(\mathbb{H}^2)$ is identified as the tensor product of the two copies of the Hardy space of the upper half-plane

$H^2(\mathbb{H})$. Note that $H^2(\mathbb{D}^2)$ and $H^2(\mathbb{H}^2)$ are isometrically isomorphic. An isometric isomorphism $\Phi : H^2(\mathbb{D}^2) \to H^2(\mathbb{H}^2)$ is given by

$$(\Phi f)(z_1,z_2) = \left(\frac{1}{z_1+i}\right)\left(\frac{1}{z_2+i}\right) f\left(\frac{z_1-i}{z_1+i}, \frac{z_2-i}{z_2+i}\right)$$

Under this isometric isomorphism C_φ for an analytic self-map $\varphi : \mathbb{D}^2 \to \mathbb{D}^2$ is carried over to $(\frac{(\tilde{\varphi}_1(z_1,z_2)+i)(\tilde{\varphi}_2(z_1,z_2)+i)}{(z_1+i)(z_2+i)})C_{\tilde{\varphi}}$ on $H^2(\mathbb{H}^2)$ through Φ, where $\tilde{\varphi} = \mathfrak{C}_2^{-1} \circ \varphi \circ \mathfrak{C}_2$, i.e.we have

$$\Phi C_\varphi \Phi^{-1} = T_{(\frac{(\tilde{\varphi}_1(z_1,z_2)+i)(\tilde{\varphi}_2(z_1,z_2)+i)}{(z_1+i)(z_2+i)})} C_{\tilde{\varphi}} \tag{1}$$

A tubular domain $\Pi = X \oplus i\Lambda$ is a domain in $\mathbb{C}^n$ where $\Lambda \subseteq \mathbb{R}^n$ is a cone i.e. $x,y \in \Lambda \Rightarrow x+y \in \Lambda$ and $\forall t > 0$, $x \in \Lambda$, $tx \in \Lambda$. We observe that $\mathbb{H}^2 = \mathbb{R}^2 \oplus i(\mathbb{R}^+)^2$ is a tubular domain. We have the following Paley-Wiener type theorem due to Bochner (see [8], p. 93):

THEOREM 1. *Let $\Pi = X \oplus i\Lambda$ be a tubular domain where $\Lambda \subseteq \mathbb{R}^n$ is a cone and $X \cong \mathbb{R}^n$ then the Fourier transform*

$$\mathscr{F}(f)(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(t) e^{-ix.t} dt \tag{2}$$

maps $H^2(\Pi)$ isometrically onto $L^2(\Lambda^*)$ where $\Lambda^* = \{y \in \mathbb{R}^n : x.y > 0 \quad \forall x \in \Lambda\}$ is the dual cone of Λ.

Since $\mathbb{H}^2 = \mathbb{R}^2 \oplus i(\mathbb{R}^+)^2$ and $((\mathbb{R}^+)^2)^* = (\mathbb{R}^+)^2$, Bochner's theorem gives us that

$$\mathscr{F} : H^2(\mathbb{H}^2) \to L^2((\mathbb{R}^+)^2)$$

is an isometric isomorphism.

Using Bochner's theorem we define the following class of operators on $H^2(\mathbb{H}^2)$ which we call "Fourier Multipliers": let $\vartheta \in C_0((\mathbb{R}^+)^2)$ then D_ϑ defined in the following way

$$D_\vartheta = \mathscr{F}^{-1} M_\vartheta \mathscr{F}$$

maps $H^2(\mathbb{H}^2)$ into itself. Let

$$F_{C_0((\mathbb{R}^+)^2)} = \{D_\vartheta : \vartheta \in C_0((\mathbb{R}^+)^2)\}$$

then $F_{C_0((\mathbb{R}^+)^2)}$ is a commutative C*-algebra of operators on $H^2(\mathbb{H}^2)$ and

$$F_{C_0((\mathbb{R}^+)^2)} \cong C_0((\mathbb{R}^+)^2).$$

For any Banach algebra A let $M(A)$ denote the space of characters of A i.e.

$$M(A) = \{x \in A^* : x(ab) = x(a)x(b)\}.$$

where A^* is the dual space of A. If A has identity then $M(A)$ is a compact Hausdorff topological space with the weak* topology. If A is commutative then $M(A)$ coincides with the maximal ideal space of A. If A is a C*-algebra and I is a two-sided closed ideal of A, then the quotient algebra A/I is also a C*-algebra. For a Banach algebra A, we denote by $com(A)$ the two-sided closed ideal in A generated by the commutators $\{a_1a_2 - a_2a_1 : a_1, a_2 \in A\}$. It is not difficult to see that

$$M(A/I) = M(A) \tag{3}$$

for any closed two-sided ideal $I \subseteq com(A)$ since any character ϕ is zero on $com(A)$.For $a \in A$ the spectrum $\sigma_A(a)$ of a on A is defined as

$$\sigma_A(a) = \{\lambda \in \mathbb{C} : \lambda e - a \text{ is not invertible in } A\},$$

where e is the identity of A. We will use the spectral permanency property of C*-algebras (see [6], pp. 283); i.e. if A is a C*-algebra with identity and B is a closed *-subalgebra of A, then for any $b \in B$ we have

$$\sigma_B(b) = \sigma_A(b). \tag{4}$$

To compute essential spectra we employ the following important fact (see [6], pp. 268): If A is a commutative Banach algebra with identity then for any $a \in A$ we have

$$\sigma_A(a) = \{x(a) : x \in M(A)\}.$$

In general (for A not necessarily commutative), we have

$$\sigma_A(a) \supseteq \{x(a) : x \in M(A)\}. \tag{5}$$

Let H and K be two given Hilbert spaces. On the algebraic tensor product $H \otimes K$ of H and K, there is a unique inner product $\langle .,. \rangle$ satisfying the following equation

$$\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle = \langle x_1, x_2 \rangle_H \langle y_1, y_2 \rangle_K$$

$\forall x_1, x_2 \in H \quad y_1, y_2 \in K$ (See [5] pp. 185). For any $T \in B(H)$ and $S \in B(K)$ there is a unique operator $T \hat{\otimes} S \in B(H \otimes K)$ satisfying the following equation:

$$(T \hat{\otimes} S)(x \otimes y) = Tx \otimes Sy$$

Moreover $\| T \hat{\otimes} S \| = \| T \| \| S \|$ (See [5] pp. 187). For any two C*-algebras $A \subset B(H)$ and $B \subset B(K)$ the algebraic tensor product $A \odot B$ is defined to be the linear span of operators of the form $T \hat{\otimes} S$ i.e.

$$A \odot B = \{\sum_{j=1}^{n} T_j \hat{\otimes} S_j : T_j \in A, \quad S_j \in B\}$$

The algebraic tensor product $A \odot B$ becomes a *-algebra with multiplication

$$(x_1 \otimes y_1)(x_2 \otimes y_2) = x_1x_2 \otimes y_1y_2$$

and involution

$$(x\otimes y)^* = x^* \otimes y^*$$

However there might be more than one norm making the closure of $A\odot B$ into a C*-algebra. If γ is a pre C*-algebra norm on $A\odot B$ we denote by $A\otimes_\gamma B$ the closure of $A\odot B$ with respect to this pre C*-algebra norm γ. A C*-algebra A is called "nuclear" if for any C*-algebra B there is a unique pre C*-algebra norm on the algebraic tensor product $A\odot B$ of A and B. A well-known theorem of Takesaki asserts that any commutative C*-algebra is nuclear (see [5] p. 205). An extension of a C*-algebra by nuclear C*-algebras is nuclear, i.e. if A, B and C are C*-algebras s.t. the following sequence

$$0 \to A \xrightarrow{j} B \xrightarrow{\pi} C \to 0$$

is short exact and A and C are nuclear then B is also nuclear (see [5] p. 212). For any separable Hilbert space H the C*-algebra of all compact operators $K(H)$ on H is nuclear (see [5] pp. 183 and 196). For any separable Hilbert spaces H_1 and H_2 we have

$$K(H_1\otimes H_2) = K(H_1)\otimes K(H_2) \tag{6}$$

(See [1] pp. 207). We recall the following fact about tensor products of C*-algebras: If A and B are C*-algebras then we have

$$M(A\otimes B) \cong M(A)\times M(B) \tag{7}$$

that is the map $(\phi_1,\phi_2) \to \phi_1\hat{\otimes}\phi_2$ where

$$(\phi_1\hat{\otimes}\phi_2)(a\otimes b) = \phi_1(a)\phi_2(b) \tag{8}$$

is a homeomorphism of $M(A)\times M(B)$ onto $M(A\otimes B)$. See [5] pp. 189. The essential spectrum $\sigma_e(T)$ of an operator T acting on a Banach space X is the spectrum of the coset of T in the Calkin algebra $\mathscr{B}(X)/K(X)$, the algebra of bounded linear operators modulo compact operators. The well known Atkinson's theorem identifies the essential spectrum of T as the set of all $\lambda\in\mathbb{C}$ for which $\lambda I - T$ is not a Fredholm operator. The essential norm of T will be denoted by $\| T \|_e$ which is defined as

$$\| T \|_e = \inf\{\| T+K \| : K\in K(X)\}$$

The bracket $[\cdot]$ will denote the equivalence class modulo $K(X)$. Using the isometric isomorphism Φ, one may transfer Fatou's theorem in the bi-disc case to two dimensional upper half-plane and may embed $H^2(\mathbb{H}^2)$ in $L^2(\mathbb{R}^2)$ via $f \longrightarrow f^*$ where $f^*(x_1,x_2) = \lim_{y\to 0} f(x_1+iy, x_2+iy)$. This embedding is an isometry.

Throughout the paper, using Φ, we will go back and forth between $H^2(\mathbb{D}^2)$ and $H^2(\mathbb{H}^2)$. We use the property that Φ preserves spectra, compactness and essential spectra i.e. if $T\in B(H^2(\mathbb{D}^2))$ then

$$\sigma_{B(H^2(\mathbb{D}^2))}(T) = \sigma_{B(H^2(\mathbb{H}^2))}(\Phi\circ T\circ\Phi^{-1}),$$

$K\in K(H^2(\mathbb{D}^2))$ if and only if $\Phi\circ K\circ\Phi^{-1}\in K(H^2(\mathbb{H}^2))$ and hence we have

$$\sigma_e(T) = \sigma_e(\Phi\circ T\circ\Phi^{-1}). \tag{9}$$

The local essential range $\mathscr{R}_\infty(\psi)$ of $\psi \in L^\infty(\mathbb{R})$ at ∞ is defined as the set of points $z \in \mathbb{C}$ so that, for all $\varepsilon > 0$ and $n > 0$, we have

$$\lambda(\psi^{-1}(B(z,\varepsilon)) \cap (\mathbb{R} - [-n,n])) > 0,$$

where λ is the Lebesgue measure on $\mathbb{R}$. The following proposition from Hoffman's book (see [3] pp. 171) relates the local essential range to the values of a function $f \in A$ in a function algebra A on the fiber $M_\alpha(A)$ of the maximal ideal space of the function algebra:

PROPOSITION 2. *Let f be a function in $A \subseteq L^\infty(\mathbb{T})$ where A is a closed *-subalgebra of $L^\infty(\mathbb{T})$ which contains $C(\mathbb{T})$. The range of $\hat{f}$ on the fiber $M_\alpha(A)$ consists of all complex numbers ζ with this property: for each neighborhood N of α and each $\varepsilon > 0$, the set*

$$\{| f - \zeta | < \varepsilon\} \cap N$$

has positive Lebesgue measure.

Hoffman states and proves Proposition 2 for $A = L^\infty(\mathbb{T})$ but in fact his proof works for a general C*-subalgebra of $L^\infty(\mathbb{T})$ that contains $C(\mathbb{T})$. By Cayley transform Hoffman's proposition holds for $L^\infty(\mathbb{R})$ as well. Let us also recall the following fact from [2] and [7] that we will use in the last section:

LEMMA 3. *If $\psi \in QC(\mathbb{R}) \cap H^\infty(\mathbb{H})$ we have*

$$\mathscr{R}_\infty(\psi) = \mathscr{C}_\infty(\psi)$$

where $\mathscr{C}_\infty(\psi)$ is the cluster set of ψ at infinity which is defined as the set of points $z \in \mathbb{C}$ for which there is a sequence $\{z_n\} \subset \mathbb{H}$ so that $z_n \to \infty$ and $\psi(z_n) \to z$.

See [2] and [7] for a proof of this lemma.

We finish this section by recalling an elementary geometric lemma from [2] which we will use in the next section:

LEMMA 4. *Let $K \subset \mathbb{H}$ be a compact subset of $\mathbb{H}$. Then there is an $\alpha \in \mathbb{R}^+$ such that* $\sup\{| \frac{\alpha i - z}{\alpha} | : z \in K\} < \delta < 1$ *for some* $\delta \in (0,1)$.

See [2] for a proof of this lemma.

2. An approximation scheme for quasi-parabolic composition operators on Hardy spaces of the bi-disc

This section is a generalization of sec. 3 of [2] to bi-disc. As in [2] we devise an integral representation formula for composition operators and we develop an approximation scheme using this integral formula for composition operators induced by maps of the form $\varphi : \mathbb{H}^2 \to \mathbb{H}^2$

$$\varphi(z_1,z_2) = (p_1 z_1 + \psi_1(z_1,z_2), p_2 z_2 + \psi(z_1,z_2))$$

where $p_i > 0$, $i = 1,2$ and $\psi_i \in H^\infty(\mathbb{H}^2)$ such that $\Im(\psi_i(z_1,z_2)) > \varepsilon > 0$, $\forall (z_1,z_2) \in \mathbb{H}^2$. Boundedness of composition operators induced by such kind of mappings above is not trivial. In order to show that quasi-parabolic composition operators on $H^2(\mathbb{H}^2)$ are bounded we will use the following lemma due to Jafari (see [4] pp. 872):

LEMMA 5. *Suppose $\varphi : \mathbb{H}^n \to \mathbb{H}^n$ is holomorphic and C_φ is bounded(compact) on a dense subset of $H^p(\mathbb{H}^n)$ for $1 < p < \infty$. Then C_φ is bounded(compact)*

Although Jafari states and proves this lemma for the poly-disc $\mathbb{D}^n$, his proof carries over to our case in exactly the same manner as he does it for the poly-disc. Using lemma 5 we prove the following result:

PROPOSITION 6. *Let $\varphi(z_1,z_2) = (p_1 z_1 + \psi_1(z_1,z_2), p_2 z_2 + \psi_2(z_1,z_2))$ be an analytic self-map of $\mathbb{H}^2$ into itself such that $\psi_j \in H^\infty(\mathbb{H}^2)$, $p_j > 0$, and $\Im(\psi_j(z_1,z_2)) > \delta > 0$ for all $(z_1,z_2) \in \mathbb{H}^2$ where $j = 1,2$. Then C_φ is bounded on $H^2(\mathbb{H}^2)$.*

Proof. Without loss of generality we may take $p_1 = p_2 = 1$, for otherwise we consider the operator $C_{\tilde{\varphi}}$ instead of C_φ where

$$\tilde{\varphi}(z_1,z_2) = (z_1 + \psi_1(\frac{z_1}{p_1},\frac{z_2}{p_2}), z_2 + \psi_2(\frac{z_1}{p_1},\frac{z_2}{p_2})).$$

We observe that $V_{p_1,p_2} C_{\tilde{\varphi}} = C_\varphi$ where $V_{p_1,p_2}(f)(z_1,z_2) = f(p_1 z_1, p_2 z_2)$. Since V_{p_1,p_2} is invertible C_φ is bounded if and only if $C_{\tilde{\varphi}}$ is bounded.

The Hilbert space $H^2(\mathbb{H}^2)$ is a reproducing kernel Hilbert space with reproducing kernel functions

$$k_{w_1,w_2}(z_1,z_2) = \frac{1}{(2i)^2(\overline{w_1} - z_1)(\overline{w_2} - z_2)}$$

we observe that

$$C_\varphi^*(k_{w_1,w_2}) = k_{\varphi(w_1,w_2)}$$

and that

$$\| k_{w_1,w_2} \| = \frac{1}{\Im(w_1)\Im(w_2)} \tag{10}$$

where C_φ^* is the Hilbert space adjoint of C_φ. Let $E = \{\sum_{j=1}^n c_j k_{w_{1j},w_{2j}} : c_j \in \mathbb{C}\}$ then it is clear that E is dense in $H^2(\mathbb{H}^2)$. Observe that by equation (10) we have

$$\begin{aligned} \| C_\varphi^* k_{w_1,w_2} \| &= \frac{1}{\Im(\varphi_1(w_1,w_2))\Im(\varphi_2(w_1,w_2))} \\ &\leqslant \left(\frac{\Im(w_1)}{\Im(w_1)+\delta}\right)\left(\frac{\Im(w_2)}{\Im(w_2)+\delta}\right)\left(\frac{1}{\Im(w_1)\Im(w_2)}\right) \leqslant \| k_{w_1,w_2} \| \end{aligned}$$

for all $(w_1,w_2) \in \mathbb{H}^2$, since $\Im(\psi_1(w_1,w_2)) > \delta > 0$ where

$$\varphi(w_1,w_2) = (\varphi_1(w_1,w_2), \varphi_2(w_1,w_2)).$$

Hence C_φ^* is bounded on E and since

$$|\langle C_\varphi u, v\rangle| = |\langle u, C_\varphi^* v\rangle| = |\overline{\langle C_\varphi^* v, u\rangle}| = |\langle C_\varphi^* v, u\rangle| \leqslant C \| u \| \| v \|$$

for some $C > 0$ and for all $u, v \in E$, C_φ is also bounded on E. Since E is dense in $H^2(\mathbb{H}^2)$, by lemma 5 C_φ is bounded on $H^2(\mathbb{H}^2)$. $\square$

Like in one variable case, for any $f \in H^2(\mathbb{H}^2)$ we have the following Cauchy Integral formula

$$f(z_1, z_2) = \frac{1}{(2\pi i)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f^*(x_1, x_2) dx_1 dx_2}{(x_1 - z_1)(x_2 - z_2)} \tag{11}$$

Using this integral formula we prove the following proposition

PROPOSITION 7. *Let $\varphi : \mathbb{H}^2 \to \mathbb{H}^2$ be an analytic function such that for*

$$\varphi^*(\mathbf{x}) = \lim_{y \to 0} \varphi(\mathbf{x} + i\mathbf{y})$$

where $\mathbf{y} = (y, y) \in \mathbb{R}^2$ and $\varphi^*(x_1, x_2) = (\varphi_1(x_1, x_2), \varphi_2(x_1, x_2))$ we have $\Im(\varphi_i^*(x_1, x_2)) > 0$ for almost every $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$. Then the composition operator C_φ on $H^2(\mathbb{H}^2)$ is given by

$$\begin{aligned}(C_\varphi f)^*(x_1, x_2) &= \lim_{y \to 0} (C_\varphi f)(\mathbf{x} + i\mathbf{y}) \\ &= \frac{1}{(2\pi i)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f^*(t_1, t_2) dt_1 dt_2}{(t_1 - \varphi_1^*(\mathbf{x}))(t_2 - \varphi_2^*(\mathbf{x}))}.\end{aligned}$$

Proof. By the equation (11) above one has

$$(C_\varphi f)(\mathbf{x} + i\mathbf{y}) = \frac{1}{(2\pi i)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f^*(t_1, t_2) dt_1 dt_2}{(t_1 - \varphi_1(\mathbf{x} + i\mathbf{y}))(t_2 - \varphi_2(\mathbf{x} + i\mathbf{y}))}$$

where $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ and $\mathbf{y} = (y, y) \in \mathbb{R}^2$. Let $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ be such that $\lim_{y \to 0} \varphi(\mathbf{x} + i\mathbf{y}) = \varphi^*(\mathbf{x}) = (\varphi_1^*(\mathbf{x}), \varphi_2^*(\mathbf{x}))$ exists and $\Im(\varphi_j^*(\mathbf{x})) > 0$, $j = 1, 2$. We have

$$\begin{aligned}&\left| C_\varphi(f)(\mathbf{x} + i\mathbf{y}) - \frac{1}{(2\pi i)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f^*(t_1, t_2) dt_1 dt_2}{(t_1 - \varphi_1^*(\mathbf{x}))(t_2 - \varphi_2^*(\mathbf{x}))} \right| \qquad (12) \\ = &\left| \frac{1}{(2\pi i)^2} \left(\int_{\mathbb{R}^2} \frac{f^*(t_1, t_2) dt_1 dt_2}{\prod_{j=1}^2 (t_j - \varphi_j(\mathbf{x} + i\mathbf{y}))} - \int_{\mathbb{R}^2} \frac{f^*(t_1, t_2) dt_1 dt_2}{\prod_{j=1}^2 (t_j - \varphi_j^*(\mathbf{x}))} \right) \right| \\ = &\left| \frac{1}{(2\pi i)^2} \int_{\mathbb{R}^2} \left(\frac{1}{\prod_{j=1}^2 (t_j - \varphi_j(\mathbf{x} + i\mathbf{y}))} - \frac{1}{\prod_{j=1}^2 (t_j - \varphi_j^*(\mathbf{x}))} \right) f^*(\mathbf{t}) d\mathbf{t} \right|\end{aligned}$$

Consider

$$\begin{aligned}
&\frac{1}{(t_1-\varphi_1(\mathbf{x}+i\mathbf{y}))(t_2-\varphi_2(\mathbf{x}+i\mathbf{y}))}-\frac{1}{(t_1-\varphi_1^*(\mathbf{x}))(t_2-\varphi_2^*(\mathbf{x}))}\\
&=\frac{1}{(t_1-\varphi_1(\mathbf{x}+i\mathbf{y}))(t_2-\varphi_2(\mathbf{x}+i\mathbf{y}))}-\frac{1}{(t_1-\varphi_1^*(\mathbf{x}+i\mathbf{y}))(t_2-\varphi_2^*(\mathbf{x}))}\\
&\quad+\frac{1}{(t_1-\varphi_1(\mathbf{x}+i\mathbf{y}))(t_2-\varphi_2(\mathbf{x}))}-\frac{1}{(t_1-\varphi_1^*(\mathbf{x}))(t_2-\varphi_2^*(\mathbf{x}))}\\
&=\frac{\varphi_2(x_1+iy,x_2+iy)-\varphi_2^*(x_1,x_2)}{(t_1-\varphi_1(\mathbf{x}+i\mathbf{y}))(t_2-\varphi_2(\mathbf{x}))(t_2-\varphi_2(\mathbf{x}+i\mathbf{y}))}\\
&\quad+\frac{\varphi_1(x_1+iy,x_2+iy)-\varphi_1^*(x_1,x_2)}{(t_1-\varphi_1(\mathbf{x}+i\mathbf{y}))(t_2-\varphi_2(\mathbf{x}))(t_2-\varphi_2(\mathbf{x}+i\mathbf{y}))}
\end{aligned}$$

Inserting this into equation (12) above we obtain

$$\begin{aligned}
&\left|C_\varphi(f)(\mathbf{x}+i\mathbf{y})-\frac{1}{(2\pi i)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f^*(t_1,t_2)dt_1dt_2}{(t_1-\varphi_1^*(\mathbf{x}))(t_2-\varphi_2^*(\mathbf{x}))}\right|\\
&\leqslant\frac{|\ \varphi_2(\mathbf{x}+i\mathbf{y})-\varphi_2^*(\mathbf{x})\ |}{4\pi^2}\left|\int_{\mathbb{R}^2}\frac{f^*(t_1,t_2)dt_1dt_2}{(t_1-\varphi_1(\mathbf{x}+i\mathbf{y}))(t_2-\varphi_2(\mathbf{x}))(t_2-\varphi_2(\mathbf{x}+i\mathbf{y}))}\right|\\
&\quad+\frac{|\ \varphi_1(\mathbf{x}+i\mathbf{y})-\varphi_1^*(\mathbf{x})\ |}{4\pi^2}\left|\int_{\mathbb{R}^2}\frac{f^*(t_1,t_2)dt_1dt_2}{(t_1-\varphi_1(\mathbf{x}+i\mathbf{y}))(t_2-\varphi_2(\mathbf{x}))(t_2-\varphi_2(\mathbf{x}+i\mathbf{y}))}\right|\\
&\leqslant\|\ f\ \|_2\,\frac{|\ \varphi_2(\mathbf{x}+i\mathbf{y})-\varphi_2^*(\mathbf{x})\ |}{4\pi^2}\left(\int_{\mathbb{R}^2}\frac{dt_1dt_2}{|\ (t_2-\varphi_2(\mathbf{x}))\prod_{j=1}^2(t_j-\varphi_j(\mathbf{x}+i\mathbf{y}))\ |^2}\right)^{\frac{1}{2}}\\
&\quad+\|\ f\ \|_2\,\frac{|\ \varphi_1(\mathbf{x}+i\mathbf{y})-\varphi_1^*(\mathbf{x})\ |}{4\pi^2}\left(\int_{\mathbb{R}^2}\frac{dt_1dt_2}{|\ (t_2-\varphi_2(\mathbf{x}))\prod_{j=1}^2(t_j-\varphi_j(\mathbf{x}+i\mathbf{y}))\ |^2}\right)^{\frac{1}{2}}
\end{aligned}$$

by Cauchy-Schwarz inequality. Now consider

$$\begin{aligned}
&\int_{\mathbb{R}^2}\frac{dt_1dt_2}{|\ (t_1-\varphi_1(\mathbf{x}+i\mathbf{y}))(t_2-\varphi_2(\mathbf{x}))(t_2-\varphi_2(\mathbf{x}+i\mathbf{y}))\ |^2}\\
&=\left(\int_{-\infty}^{\infty}\frac{dt_1}{|\ t_1-\varphi_1(\mathbf{x}+i\mathbf{y})\ |^2}\right)\left(\int_{-\infty}^{\infty}\frac{dt_1}{|\ (t_2-\varphi_2(\mathbf{x}))(t_2-\varphi_2(\mathbf{x}+i\mathbf{y}))\ |^2}\right)
\end{aligned}$$

We have $\lim_{y\to 0}\varphi_j(\mathbf{x}+i\mathbf{y})=\varphi_j^*(\mathbf{x}),\ j=1,2$. Let

$$\varepsilon_0=\frac{\inf\{|\ t-\varphi_1^*(\mathbf{x}):t\in\mathbb{R}\}\cup\{|\ t-\varphi_2^*(\mathbf{x}):t\in\mathbb{R}\}}{2}$$

Choose $\varepsilon_0>\varepsilon>0$ such that $\forall\ 0<y<\delta$ we have

$$|\ \varphi_j(\mathbf{x}+i\mathbf{y})-\varphi_j(\mathbf{x})\ |<\frac{\varepsilon}{2}$$

Then one has by triangle inequality

$$|\ t_1-\varphi_1(\mathbf{x}+i\mathbf{y})\ |\geqslant|\ t_1-\varphi_1^*(\mathbf{x})\ |-\varepsilon_0>\varepsilon_0$$

and this implies that

$$\frac{1}{\mid t_1-\varphi_1(\mathbf{x}+i\mathbf{y})\mid} \leqslant \frac{1}{\mid t_1-\varphi_1^*(\mathbf{x})\mid -\varepsilon_0}$$

hence we have

$$\int_{-\infty}^{\infty}\frac{dt_1}{\mid t_1-\varphi_1(\mathbf{x}+i\mathbf{y})\mid^2} \leqslant \int_{-\infty}^{\infty}\frac{dt_1}{(\mid t_1-\varphi_1^*(\mathbf{x})\mid -\varepsilon_0)^2} = M_{\varepsilon_0,\mathbf{x}}$$

since the integral on the right hand side converges and its value only depends on ε_0 and $\mathbf{x}=(x_1,x_2)\in\mathbb{R}^2$. Similarly by the same arguments as in [2] we have

$$\begin{aligned}&\int_{-\infty}^{\infty}\frac{dt_2}{\mid (t_2-\varphi_2^*(\mathbf{x}))(t_2-\varphi_2(\mathbf{x}+i\mathbf{y}))\mid^2}\\ \leqslant\ &\int_{-\infty}^{\infty}\frac{dt_2}{(\mid t_2-\varphi_2^*(\mathbf{x})\mid -\varepsilon_0)^2\mid t_2-\varphi_2^*(\mathbf{x})\mid^2} = K_{\varepsilon_0,\mathbf{x}}\end{aligned}$$

As a result we have

$$\begin{aligned}&\left|C_\varphi(f)(\mathbf{x}+i\mathbf{y})-\frac{1}{(2\pi i)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f^*(t_1,t_2)dt_1dt_2}{(t_1-\varphi_1^*(\mathbf{x}))(t_2-\varphi_2^*(\mathbf{x}))}\right|\\ \leqslant\ &\frac{(M_{\varepsilon_0,\mathbf{x}}K_{\varepsilon_0,\mathbf{x}})^{\frac{1}{2}}}{4\pi^2}(\mid\varphi_1(\mathbf{x}+i\mathbf{y})-\varphi_1^*(\mathbf{x})\mid+\mid\varphi_2(\mathbf{x}+i\mathbf{y})-\varphi_2^*(\mathbf{x})\mid)\\ \leqslant\ &\frac{(M_{\varepsilon_0,\mathbf{x}}K_{\varepsilon_0,\mathbf{x}})^{\frac{1}{2}}}{4\pi^2}\varepsilon\end{aligned}$$

Therefore

$$\lim_{y\to 0}(C_\varphi f)(x_1+iy,x_2+iy) = \frac{1}{(2\pi i)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f^*(t_1,t_2)dt_1dt_2}{(t_1-\varphi_1^*(x_1,x_2))(t_2-\varphi_2^*(x_1,x_2))} \qquad \square$$

Throughout the rest of the paper we will identify a function f in $H^2(\mathbb{H}^2)$ or $H^\infty(\mathbb{H}^2)$ with its boundary function f^*. We formulate and prove our approximation scheme as the following proposition.

PROPOSITION 8. *Let $\varphi:\mathbb{H}^2\to\mathbb{H}^2$ be an analytic self-map of $\mathbb{H}^2$ such that*

$$\varphi(z_1,z_2) = (p_1z_1+\psi_1(z_1,z_2),p_2z_2+\psi_2(z_1,z_2))$$

$p_1,p_2>0$ and $\psi_j\in H^\infty$ is such that $\Im(\psi_j(z))>\varepsilon>0$, $j=1,2$ for all $(z_1,z_2)\in\mathbb{H}^2$. Then there is an $\alpha\in\mathbb{R}^+$ such that for $C_\varphi:H^2(\mathbb{H}^2)\to H^2(\mathbb{H}^2)$ we have

$$C_\varphi = V_{p_1,p_2}\sum_{n,m=0}^{\infty}T_{\tau_1^n}T_{\tau_2^m}D_{\vartheta_{1,n}}D_{\vartheta_{2,m}},$$

where the convergence of the series is in operator norm, $T_{\tau_1^n}$ and $T_{\tau_2^m}$ are the Toeplitz operators with symbols τ_1^n and τ_2^m respectively,

$$\tau_j(x_1,x_2) = i\alpha - \tilde{\psi}_j(x_1,x_2), \quad \tilde{\psi}(x_1,x_2) = \psi\left(\frac{x_1}{p_1},\frac{x_2}{p_2}\right),$$

V_{p_1,p_2} is the dilation operator defined as

$$(V_{p_1,p_2}f)(z_1,z_2) = f(p_1z_1,p_2z_2)$$

and $D_{\vartheta_{1,n}}$ and $D_{\vartheta_{2,m}}$ are the Fourier multipliers with $\vartheta_{1,n}(t_1,t_2) = \frac{(-it_1)^n e^{-\alpha t_1}}{n!}$ and $\vartheta_{2,m}(t_1,t_2) = \frac{(-it_2)^m e^{-\alpha t_2}}{m!}$ respectively.

Proof. Since for $\varphi(z_1,z_2) = (p_1z_1+\psi_1(z_1,z_2), p_2z_2+\psi_2(z_1,z_2))$ where $\psi_j \in H^\infty$ with $\Im(\psi_j(z_1,z_2)) > \varepsilon > 0$ for all $z \in \mathbb{H}$ and $p_1,p_2 > 0$, we have

$$\Im(\varphi_j^*(x_1,x_2)) \geqslant \varepsilon > 0 \quad \text{for almost every} \quad (x_1,x_2) \in \mathbb{R}^2.$$

We can use Proposition 7 for $C_\varphi : H^2 \to H^2$ to have

$$\begin{aligned}(C_\varphi f)(x_1,x_2) &= \frac{1}{(2\pi i)^2}\int_{\mathbb{R}^2} \frac{f(w_1,w_2)dw_1dw_2}{(w_1-\varphi_1(\mathbf{x}))(w_2-\varphi_2(\mathbf{x}))} \\ &= \frac{1}{(2\pi i)^2}\int_{\mathbb{R}^2} \frac{f(w_1,w_2)dw_1dw_2}{(w_1-px_1-\psi_1(\mathbf{x}))(w_2-px_2-\psi_2(\mathbf{x}))}.\end{aligned}$$

where $\mathbf{x} = (x_1,x_2) \in \mathbb{R}^2$. Without loss of generality, we take $p_1 = p_2 = 1$, since if $p_1 \neq 1$ or $p_2 \neq 1$ then we have

$$(V_{\frac{1}{p_1},\frac{1}{p_2}}C_\varphi)(f)(x_1,x_2) = \frac{1}{(2\pi i)^2}\int_{\mathbb{R}^2} \frac{f(w_1,w_2)dw_1dw_2}{\prod_{j=1}^2 (w_j - x_j - \tilde{\psi}_j(x_1,x_2))}, \tag{13}$$

where $\tilde{\psi}_j(x_1,x_2) = \psi_j(\frac{x_1}{p_1},\frac{x_2}{p_2})$. We observe that

$$\begin{aligned}&\frac{1}{(w_1-x_1-\psi_1(\mathbf{x}))(w_2-x_2-\psi_2(\mathbf{x}))} \qquad (14)\\ &= \frac{1}{(x_1-w_1+i\alpha-(i\alpha-\psi_1(\mathbf{x})))(x_2-w_2+i\alpha-(i\alpha-\psi_2(\mathbf{x})))} \\ &= \frac{1}{(x_1-w_1+i\alpha)(x_2-w_2+i\alpha)\left(1-\left(\frac{i\alpha-\psi_1(\mathbf{x})}{x_1-w_1+i\alpha}\right)\right)\left(1-\left(\frac{i\alpha-\psi_2(\mathbf{x})}{x_2-w_2+i\alpha}\right)\right)}.\end{aligned}$$

Since $\Im(\psi_j(z_1,z_2)) > \varepsilon > 0$ for all $(z_1,z_2) \in \mathbb{H}^2$ and $\psi_j \in H^\infty$, we have $\overline{\psi_j(\mathbb{H}^2)}$ is compact in $\mathbb{H}$, and then by Lemma 4 there is an $\alpha > 0$ such that

$$\left|\frac{i\alpha-\psi_j(\mathbf{x})}{x_j-w_j+i\alpha}\right| < \delta < 1$$

for all $(x_1,x_2),(w_1,w_2)\in\mathbb{R}^2$. So we have

$$\frac{1}{1-\left(\dfrac{i\alpha-\psi_j(\mathbf{x})}{x_j-w_j+i\alpha}\right)}=\sum_{n=0}^{\infty}\left(\frac{i\alpha-\psi_j(\mathbf{x})}{x_j-w_j+i\alpha}\right)^n.$$

Inserting this into equation (14) and then into equation (13), we have

$$(C_\varphi f)(x_1,x_2)=\sum_{n,m=0}^{(N_1-1),(N_2-1)}T_{\tau_1^n}T_{\tau_2^m}K_{n,m}f(\mathbf{x})+R_{1,N_1}f(\mathbf{x})+R_{2,N_2}f(\mathbf{x})+R_{N_1,N_2}f(\mathbf{x}),$$

where $T_{\tau_j^n}f(x_1,x_2)=\ \tau_j^n(x_1,x_2)f(x_1,x_2)$, $\tau_j(x_1,x_2)=\ i\alpha-\psi_j(x_1,x_2)$, $K_{n,m}$ is defined as

$$K_{n,m}f(x_1,x_2)=\frac{1}{(2\pi i)^2}\int_{\mathbb{R}^2}\frac{f(w_1,w_2)dw_1dw_2}{(x_1-w_1+i\alpha)^{n+1}(x_2-w_2+i\alpha)^{m+1}}$$

and

$$R_{1,N_1}f(x_1,x_2)=\sum_{m=0}^{N_2}T_{\tau_1^{N_1+1}}T_{\tau_2^m}K_{N_1+1,m}f(\mathbf{x}),$$

$$R_{2,N_2}f(x_1,x_2)=\sum_{n=0}^{N_1}T_{\tau_1^n}T_{\tau_2^{N_2+1}}K_{n,N_2+1}f(\mathbf{x}),$$

$$R_{N_1,N_2}f(x)=\frac{1}{(2\pi i)^2}T_{\tau_1^{N_1+1}}T_{\tau_2^{N_2+1}}\int_{\mathbb{R}^2}\frac{f(w_1,w_2)dw_1dw_2}{\prod_{j=1}^2(x_j-w_j+i\alpha)^{N_j+1}(w_j-x_j-\psi_j(\mathbf{x}))}.$$

Let $\varphi_1(z_1,z_2)=(z_1+\psi_1(z_1,z_2),z_2+i\alpha)$ and $\varphi_2(z_1,z_2)=(z_1+i\alpha,z_2+\psi_2(z_1,z_2))$ then we have the following estimates for R_{1,N_1} and R_{2,N_2}:

$$\| R_{1,N_1}\|\leqslant\| C_{\varphi_1}\|\,\delta^{N_1+1}(1-\delta)^{-1},$$

$$\| R_{2,N_2}\|\leqslant\| C_{\varphi_2}\|\,\delta^{N_2+1}(1-\delta)^{-1}.$$

By proposition 6, C_{φ_1} and C_{φ_2} are bounded so we have $\| R_{1,N_1}\|\to 0$ and $\| R_{2,N_2}\|\to 0$ as $N_1,N_2\to\infty$. We have the following estimate for R_{N_1,N_2}:

$$\| R_{N_1,N_2}\|\leqslant\| T_{\tau_1}\|\|\, T_{\tau_2}\|\|\, C_\varphi\|\,\delta^{N_1+N_2}$$

Hence $\| R_{N_1,N_2}\|\to 0$ as $N_1,N_2\to\infty$. We observe that

$$K_{n,m}=D_{\vartheta_{n,m}}$$

where $\vartheta_{n,m}(t_1,t_2)=\frac{(-it_1)^ne^{-\alpha t_1}}{n!}\frac{(-it_2)^me^{-\alpha t_2}}{m!}$. Hence we have

$$C_\varphi=\sum_{n,m=0}^{\infty}T_{\tau_1^n}T_{\tau_2^m}D_{\vartheta_{1,n}}D_{\vartheta_{2,m}},$$

where the convergence is in operator norm. $\square$

3. A Ψ-c*-algebra of operators on Hardy spaces of $\mathbb{H}^2$

In the preceding section we have shown that "quasi-parabolic" composition operators on the $\mathbb{H}^2$ lie in the C*-algebra generated by certain Toeplitz operators and Fourier multipliers. In this section we will identify the character space of the C*-algebra generated by Toeplitz operators with a class of symbols and Fourier multipliers. We identify this C*-algebra with the tensor product of its one variable version which is treated in [2] with itself. We will consider the C*-algebra of operators acting on $H^2(\mathbb{H}^2)$

$$\Psi(QC(\mathbb{R}^2), C_0((\mathbb{R}^+)^2)) = C^*(\mathscr{T}(QC(\mathbb{R}^2)) \cup F_{C_0((\mathbb{R}^+)^2)})$$

where

$$QC(\mathbb{R}^2) = QC(\mathbb{R}) \otimes QC(\mathbb{R}) = \overline{\{\sum_{j=1}^{n} f_j(x).g_j(y) : f_j, g_j \in QC(\mathbb{R})\}}$$

and

$$\mathscr{T}(QC(\mathbb{R}^2)) = C^*(\{T_\phi : \phi \in QC(\mathbb{R}^2)\}$$

is the Toeplitz C*-algebra with $QC(\mathbb{R}^2)$ symbols. Recall that

$$QC(\mathbb{R}) = \{f \in L^\infty(\mathbb{R}) : f \circ \mathfrak{C}^{-1} \in QC(\mathbb{T})\}$$

where

$$\mathfrak{C}(z) = \frac{z-i}{z+i}$$

is the Cayley transform and

$$QC(\mathbb{T}) = (H^\infty(\mathbb{D}) + C(\mathbb{T})) \cap \overline{H^\infty(\mathbb{D}) + C(\mathbb{T})}$$

is the class of quasi-continuous functions.

In ([2]) we showed that the following sequence

$$0 \to K(H^2(\mathbb{H})) \xrightarrow{j} \Psi(QC(\mathbb{R}), C_0(\mathbb{R}^+)) \xrightarrow{\pi} C(\mathbb{M}) \to 0 \tag{15}$$

is short exact where

$$\Psi(QC(\mathbb{R}), C_0(\mathbb{R}^+)) = C^*(\mathscr{T}(QC(\mathbb{R})) \cup F_{C_0(\mathbb{R}^+)})$$

is the C*-algebra generated by Toeplitz operators with QC symbols and continuous Fourier multipliers and

$$\mathbb{M} \cong (M_\infty(QC(\mathbb{R})) \times [0,\infty]) \cup (M(QC(\mathbb{R})) \times \{\infty\}) \tag{16}$$

is the maximal ideal space of $\Psi(QC(\mathbb{R}), C_0(\mathbb{R}^+))/K(H^2(\mathbb{H}))$. Here $M(QC(\mathbb{R}))$ is the maximal ideal space of $QC(\mathbb{R})$ and

$$M_\infty(QC(\mathbb{R})) = \{x \in M(QC(\mathbb{R})) : x|_{C(\dot{\mathbb{R}})} = \delta_\infty, \quad \delta_\infty(f) = \lim_{t\to\infty} f(t)\}$$

is the fiber of $M(QC(\mathbb{R})$ at ∞. Throughout the C*-algebra $\Psi(QC(\mathbb{R}),C_0(\mathbb{R}^+))$ will be denoted by Ψ and $\Psi(QC(\mathbb{R}^2),C_0((\mathbb{R}^+)^2))$ will be denoted by Ψ_2. Since $K(H^2)$ is nuclear and $C(\mathbb{M})$ is commutative and hence nuclear, by eqn. (15) Ψ is nuclear. Therefore all the C*-algebras that we will deal with in this paper will be nuclear and $A\otimes B$ will denote the closure of the algebraic tensor product of A and B with respect to this unique C* norm. Following the approach in [1] we identify Ψ_2 with $\Psi\otimes\Psi$ corresponding to the identification of $H^2(\mathbb{H}^2)$ with $H^2(\mathbb{H})\otimes H^2(\mathbb{H})$. Define the operators $W_f = T_f\hat{\otimes}I$ for any $f\in QC(\mathbb{R})$ as

$$(W_f a)(z_1,z_2) = P(f(z_1)a(z_1,z_2))$$

for $a\in H^2(\mathbb{H}^2)$ and $W_g = I\hat{\otimes}T_g$ as

$$(W_g a)(z_1,z_2) = P(g(z_2)a(z_1,z_2))$$

where P is the orthogonal projection of $L^2(\mathbb{R}^2)$ onto $H^2(\mathbb{H}^2)$. In the same way for the Fourier multipliers define $E_\vartheta = D_\vartheta\hat{\otimes}I$ as

$$(E_\vartheta a)(z_1,z_2) = (\mathscr{F}^{-1}M_\vartheta\mathscr{F})(a)(z_1,z_2)$$

where $\vartheta\in C_0(\mathbb{R}^+)$ and $E_\tau = I\hat{\otimes}D_\tau$ for $\tau\in C_0(\mathbb{R}^+)$ as

$$(E_\tau a)(z_1,z_2) = (\mathscr{F}^{-1}M_\tau\mathscr{F})(a)(z_1,z_2)$$

where M_ϑ is defined as

$$(M_\vartheta a)(t_1,t_2) = \vartheta(t_1)a(t_1,t_2)$$

M_τ is defined as

$$(M_\tau a)(t_1,t_2) = \tau(t_2)a(t_1,t_2)$$

and $\mathscr{F}$ is the Fourier transform defined as in equation (2). Since Ψ_2 is generated by $\{W_f,W_g,E_\vartheta,E_\tau : f,g\in QC(\mathbb{R}) \quad \vartheta,\tau\in C_0(\mathbb{R}^+)\}$ and Ψ is nuclear, $\Psi_2=\Psi\otimes\Psi$. By equation (6) we have $K(H^2(\mathbb{H}^2)) = K(H^2(\mathbb{H}))\otimes K(H^2(\mathbb{H}))$. Since $\Psi_2=\Psi\otimes\Psi$ we have

$$com(\Psi_2) = com(\Psi\otimes\Psi) = I^*(com(\Psi)\otimes\Psi\cup\Psi\otimes com(\Psi))$$

By equation (15) we have $com(\Psi) = K(H^2(\mathbb{H}))$ hence we have

$$K(H^2(\mathbb{H}^2)) = K(H^2(\mathbb{H}))\otimes K(H^2(\mathbb{H}))\subset com(\Psi\otimes\Psi) = com(\Psi_2)$$

Hence by equations (3) and (7) we have

$$M(\Psi_2/K(H^2(\mathbb{H}^2))) = M(\Psi_2) = M(\Psi\otimes\Psi)\cong\mathbb{M}\times\mathbb{M}$$

where $\mathbb{M}$ is as in equation (16). We summarize the result of this section as the following proposition:

PROPOSITION 9. *Let $\Psi_2 = C^*(\mathscr{T}(QC(\mathbb{R}^2)) \cup F_{C_0((\mathbb{R}^+)^2)})$ be the C*-algebra generated by Toeplitz operators with $QC(\mathbb{R}) \otimes QC(\mathbb{R})$ symbols and continuous Fourier multipliers acting on $H^2(\mathbb{H}^2)$. Then for the character space $M(\Psi_2/K(H^2(\mathbb{H}^2)))$ of $\Psi_2/K(H^2(\mathbb{H}^2))$ we have*

$$M(\Psi_2/K(H^2(\mathbb{H}^2))) \cong \mathbb{M} \times \mathbb{M}$$

where $\mathbb{M}$ is the maximal ideal space of the C-algebra $\Psi(QC(\mathbb{R}), C_0(\mathbb{R}^+))/K(H^2(\mathbb{H}))$ generated by Toeplitz operators with $QC(\mathbb{R})$ symbols and continuous Fourier multipliers modulo compact operators acting on $H^2(\mathbb{H})$.*

4. Main results

In this section we prove the main results of this paper which asserts that the essential spectra of quasi-parabolic composition operators on the Hardy space of the polydisc contain a non-trivial set which consists of spiral curves as in one variable case. In doing this we use multi-dimensional generalizations of the methods employed in [2]. We prove the following proposition which might be regarded as a weaker version of a multi-dimensional generalization of lemma 3:

PROPOSITION 10. *Let $\psi \in QC(\mathbb{R}) \otimes QC(\mathbb{R}) \cap H^\infty(\mathbb{H}^2)$ then we have*

$$\{(\phi_1 \hat{\otimes} \phi_2)(\psi) : \phi_1, \phi_2 \in M_\infty(QC(\mathbb{R}))\} \supseteq \mathscr{C}_{(\infty,\infty)}(\psi)$$

where $\phi_1 \hat{\otimes} \phi_2$ is as defined by equation (8) and $\mathscr{C}_{(\infty,\infty)}(\psi)$ is defined to be the set of points $w \in \mathbb{C}$ for which there is a sequence $\{z_n\} \subset \mathbb{H}^2$ so that $z_n \to (\infty,\infty)$ and $\psi(z_n) \to w$.

Proof. Let us first show the above inclusion for functions of the form

$$\psi(z_1, z_2) = \sum_{j=1}^{m} \varphi_j(z_1)\eta_j(z_2)$$

where $\varphi_j, \eta_j \in QC(\mathbb{R}) \cap H^\infty(\mathbb{H})$. Let $w \in \mathscr{C}_{(\infty,\infty)}(\psi)$ then there exists a sequence $\{z_n\} \in \mathbb{H}^2$, $z_n = (z_{1,n}, z_{2,n})$ such that $z_n = (z_{1,n}, z_{2,n}) \to (\infty,\infty)$ and $\psi(z_{1,n}, z_{2,n}) \to w$. Since $\varphi_j, \eta_j \in QC(\mathbb{R}) \cap H^\infty(\mathbb{H}) \subset H^\infty(\mathbb{H})$, the sequences $\{\varphi_j(z_{1,n})\}$ and $\{\eta_j(z_{2,n})\}$ have convergent subsequences, hence without loss of generality (by passing to a subsequence if needed) there are $w_{1,j}, w_{2,j} \in \mathbb{H}$ such that

$$\varphi_j(z_{1,n}) \to w_{1,j} \qquad \text{and} \qquad \eta_j(z_{2,n}) \to w_{2,j} \tag{17}$$

as $n \to \infty$, where $j \in \{1,2,...,m\}$. Since the index j takes finite number of values we observe that one may find a single sequence $z_n = (z_{1,n}, z_{2,n}) \in \mathbb{H}^2$ such that equation (17) holds for all $j \in \{1,2,...,m\}$. By proposition 2 and lemma 3 there are $\phi_1, \phi_2 \in M_\infty(QC(\mathbb{R}))$ such that $\phi_1(\varphi_j) = w_{1,j}$ and $\phi_2(\eta_j) = w_{2,j}$ for all $j \in \{1,2,...,m\}$. Since $\sum_{j=1}^m w_{1,j}w_{2,j} = w$ we have

$$(\phi_1 \hat{\otimes} \phi_2)(\psi) = w$$

Therefore we have

$$\mathscr{C}_{(\infty,\infty)}(\psi) \subseteq \{(\phi_1 \hat{\otimes} \phi_2)(\psi) : \phi_1, \phi_2 \in M_\infty(QC(\mathbb{R}))\}$$

For ψ having an infinite sum of the following form

$$\psi(z_1,z_2) = \sum_{j=1}^{\infty} \varphi_j(z_1)\eta_j(z_2)$$

one may choose subsequences of $\{z_{1,n}\}$ and $\{z_{2,n}\}$ through a Cantor diagonalization argument so that equation (17) holds for all $j \in \mathbb{N}$. The rest follows in the same way as above. $\square$

We are now ready to state and prove our first main result for quasi-parabolic composition operators acting on $H^2(\mathbb{H}^2)$:

MAIN THEOREM 1. *Let $\varphi : \mathbb{H}^2 \to \mathbb{H}^2$ be an analytic self-map of $\mathbb{H}^2$ such that*

$$\varphi(z_1,z_2) = (z_1 + \psi_1(z_1,z_2), z_2 + \psi_2(z_1,z_2))$$

where $\psi_j \in H^\infty(\mathbb{H}^2)$ with $\Im(\psi_j(z_1,z_2)) > \varepsilon > 0$ for all $(z_1,z_2) \in \mathbb{H}^2$, $j = 1,2$. Then $C_\varphi : H^2(\mathbb{H}^2) \to H^2(\mathbb{H}^2)$ is bounded. Moreover if $\psi_j \in (QC(\mathbb{R}) \otimes QC(\mathbb{R})) \cap H^\infty(\mathbb{H}^2)$ then we have

$$\sigma_e(C_\varphi) \supseteq \{e^{i(z_1t_1+z_2t_2)} : t_1,t_2 \in [0,\infty], z_1 \in \mathscr{C}_{(\infty,\infty)}(\psi_1) \text{and} \quad z_2 \in \mathscr{C}_{(\infty,\infty)}(\psi_2)\} \cup \{0\},$$

where $\mathscr{C}_{(\infty,\infty)}(\psi)$ is the set of cluster points of ψ at (∞,∞).

Proof. The boundedness of C_φ is a consequence of proposition 6. By Proposition 8 we have the following series expansion for C_φ:

$$C_\varphi = \sum_{n,m=0}^{\infty} T_{\tau_1^n} T_{\tau_2^m} D_{\frac{(-it_1)^n e^{-\alpha t_1}}{n!}} D_{\frac{(-it_2)^m e^{-\alpha t_2}}{m!}},$$

where $\tau_1(z_1,z_2) = i\alpha - \psi_1(z_1,z_2)$ and $\tau_2(z_1,z_2) = i\alpha - \psi_2(z_1,z_2)$. So we conclude that if $\psi_1, \psi_2 \in QC(\mathbb{R}) \otimes QC(\mathbb{R}) \cap H^\infty(\mathbb{H}^2)$ with $\Im(\psi_j(z_1,z_2)) > \varepsilon > 0$, $j = 1,2$, then

$$C_\varphi \in \Psi(QC(\mathbb{R}^2), C_0((\mathbb{R}^+)^2)) = \Psi_2$$

where $\varphi(z_1,z_2) = (z_1 + \psi_1(z_1,z_2), z_2 + \psi_2(z_1,z_2))$. We look at the values $\phi(C_\varphi)$ of ϕ where $\phi \in M(\Psi_2/K(H^2(\mathbb{H}^2)))$. By Proposition 9 we have

$$M(\Psi_2/K(H^2(\mathbb{H}^2))) = \mathbb{M} \times \mathbb{M}$$

where $\mathbb{M}$ is the maximal ideal space of the C*-algebra $\Psi(QC(\mathbb{R}), C_0(\mathbb{R}^+))/K(H^2(\mathbb{H}))$ generated by Toeplitz operators with $QC(\mathbb{R})$ symbols and continuous Fourier multipliers modulo compact operators acting on $H^2(\mathbb{H})$. Let

$$\phi = \phi_1 \hat{\otimes} \phi_2 \in M(\Psi_2/K(H^2(\mathbb{H})))$$

where $\phi_1, \phi_2 \in \mathbb{M}$ as in the identification done in equation (8). If $\phi_1 = (x_1, \infty)$ or $\phi_2 = (x_2, \infty)$ where $x_1, x_2 \in M(QC(\mathbb{R}))$ then we have

$$\phi(C_\varphi) = \sum_{n,m=0}^{\infty} \frac{1}{n!m!} \hat{\tau}_1(x_1,x_2)^n \hat{\tau}_2(x_1,x_2)^m \vartheta_{1,n}(\infty,t_2)\vartheta_{2,m}(\infty,t_2) = 0$$

$\forall x_1, x_2 \in M(QC(\mathbb{R}))$ and $t_2 \in [0,\infty)$ since $\vartheta_{1,n}(\infty,t_2) = 0$ for all $n \in \mathbb{N}$ where $\vartheta_{1,n}(t_1,t_2) = (-it_1)^n e^{-\alpha t_1}$ and $\vartheta_{2,m}(t_1,t_2) = (-it_2)^m e^{-\alpha t_2}$. If $\phi_1 = (x_1,t_1)$ and $\phi_2 = (x_2,t_2)$ where $x_1, x_2 \in M_\infty(QC(\mathbb{R}))$ and $t_1 \neq \infty$, $t_2 \neq \infty$, then we have

$$\begin{aligned}(\phi_1 \hat{\otimes} \phi_2)(C_\varphi) &= \sum_{n,m=0}^{\infty} \frac{1}{n!m!} \hat{\tau}_1(x_1,x_2)^n \hat{\tau}_2(x_1,x_2)^m \vartheta_{1,n}(t_1,t_2)\vartheta_{2,m}(t_1,t_2) \\ &= (e^{-\alpha t_1} \sum_{n=0}^{\infty} \frac{1}{n!} \hat{\tau}_1(x_1,x_2)^n (-it_1)^n)(e^{-\alpha t_2} \sum_{m=0}^{\infty} \frac{1}{m!} \hat{\tau}_2(x_1,x_2)^m (-it_2)^m) \\ &= e^{i(\hat{\psi}_1(x_1,x_2)t_1 + \hat{\psi}_2(x_1,x_2)t_2)} \end{aligned} \tag{18}$$

Since $\Psi_2/K(H^2(\mathbb{H}^2))$ is a closed *-subalgebra of $B(H^2(\mathbb{H}^2))/K(H^2(\mathbb{H}^2))$ we have, by equation (4),

$$\sigma_{\Psi_2/K(H^2(\mathbb{H}^2))}(C_\varphi) = \sigma_{B(H^2(\mathbb{H}^2))/K(H^2(\mathbb{H}^2))}(C_\varphi) = \sigma_e(C_\varphi)$$

and by equation (5) we have

$$\sigma_{\Psi_2/K(H^2(\mathbb{H}^2))}(C_\varphi) = \sigma_e(C_\varphi) \supseteq \{(\phi_1 \hat{\otimes} \phi_2)(C_\varphi) : \phi_1, \phi_2 \in \mathbb{M}\} \tag{19}$$

By proposition 10 we have

$$\begin{aligned}&\{\hat{\psi}_1(x_1,x_2) : x_1, x_2 \in M_\infty(QC(\mathbb{R}))\} \supseteq \mathscr{C}_{(\infty,\infty)}(\psi_1), \\ &\{\hat{\psi}_2(x_1,x_2) : x_1, x_2 \in M_\infty(QC(\mathbb{R}))\} \supseteq \mathscr{C}_{(\infty,\infty)}(\psi_2)\end{aligned} \tag{20}$$

Therefore by equations (18), (19) and (20) we have

$$\begin{aligned}&\sigma_e(C_\varphi) \supseteq \{(\phi_1 \hat{\otimes} \phi_2)(C_\varphi) : \phi_1, \phi_2 \in \mathbb{M}\} \supseteq \\ &\{e^{i(z_1t_1+z_2t_2)} : t_1, t_2 \in [0,\infty], z_1 \in \mathscr{C}_{(\infty,\infty)}(\psi_1) \quad \text{and} \quad z_2 \in \mathscr{C}_{(\infty,\infty)}(\psi_2)\} \cup \{0\} \quad \square\end{aligned}$$

MAIN THEOREM 2. *Let $\varphi : \mathbb{D}^2 \to \mathbb{D}^2$ be an analytic self-map of $\mathbb{D}^2$ such that*

$$\varphi(z_1,z_2) = \left(\frac{2iz_1 + \psi_1(z_1,z_2)(1-z_1)}{2i + \psi_1(z_1,z_2)(1-z_1)}, \frac{2iz_2 + \psi_2(z_1,z_2)(1-z_2)}{2i + \psi_2(z_1,z_2)(1-z_2)} \right)$$

where $\psi_j \in H^\infty(\mathbb{D}^2)$ with $\Im(\psi_j(z_1,z_2)) > \varepsilon > 0$ for all $(z_1,z_2) \in \mathbb{D}^2$, $j = 1,2$. Then $C_\varphi : H^2(\mathbb{D}^2) \to H^2(\mathbb{D}^2)$ is bounded. Moreover if $\psi_j \in (QC \otimes QC) \cap H^\infty(\mathbb{D}^2)$ then we have

$$\sigma_e(C_\varphi) \supseteq \{e^{i(z_1t_1+z_2t_2)} : t_1, t_2 \in [0,\infty], z_1 \in \mathscr{C}_{(1,1)}(\psi_1) \mathit{and} \quad z_2 \in \mathscr{C}_{(1,1)}(\psi_2)\} \cup \{0\},$$

where $\mathscr{C}_{(1,1)}(\psi)$ *is the set of cluster points of* ψ *at* $(1,1)\in\mathbb{T}^2$.

Proof. Using the isometric isomorphism $\Phi : H^2(\mathbb{D}^2) \longrightarrow H^2(\mathbb{H}^2)$ introduced in section 2, if $\varphi : \mathbb{D}^2 \to \mathbb{D}^2$ is of the form

$$\varphi(z_1,z_2)=\left(\frac{2iz_1+\psi_1(z_1,z_2)(1-z_1)}{2i+\psi_1(z_1,z_2)(1-z_1)},\frac{2iz_2+\psi_2(z_1,z_2)(1-z_2)}{2i+\psi_2(z_1,z_2)(1-z_2)}\right)$$

where $\psi_j \in H^\infty(\mathbb{D})^2$ satisfies $\Im(\psi_j(z_1,z_2)) > \delta > 0$ then, by equation (1), for $\tilde{\varphi} = \mathfrak{C}_2^{-1}\circ\varphi\circ\mathfrak{C}_2$ we have $\tilde{\varphi}(z_1,z_2) = (z_1+\psi_1\circ\mathfrak{C}_2(z_1,z_2), z_2+\psi_2\circ\mathfrak{C}_2(z_1,z_2))$ and

$$\Phi C_\varphi \Phi^{-1} = T_{(1+\frac{\psi_1\circ\mathfrak{C}_2(z_1,z_2)}{z_1+i})(1+\frac{\psi_2\circ\mathfrak{C}_2(z_1,z_2)}{z_2+i})} C_{\tilde{\varphi}} \tag{21}$$

Since both $T_{(1+\frac{\psi_1\circ\mathfrak{C}_2(z_1,z_2)}{z_1+i})(1+\frac{\psi_2\circ\mathfrak{C}_2(z_1,z_2)}{z_2+i})}$ and $C_{\tilde{\varphi}}$ are bounded and Φ is an isometric isomorphism, it follows that C_φ is also bounded. For $\psi_j \in QC\otimes QC,\ j=1,2$ we have both

$$C_{\tilde{\varphi}} \in \Psi_2 \quad\text{and}\quad T_{(1+\frac{\psi_1\circ\mathfrak{C}_2(z_1,z_2)}{z_1+i})(1+\frac{\psi_2\circ\mathfrak{C}_2(z_1,z_2)}{z_2+i})} \in \Psi_2$$

and hence

$$\Phi\circ C_\varphi\circ\Phi^{-1} \in \Psi_2.$$

For any $\phi_1\hat{\otimes}\phi_2 \in M(\Psi_2/K(H^2(\mathbb{H}))) = \mathbb{M}\times\mathbb{M}$ we observe that

$$(\phi_1\hat{\otimes}\phi_2)(T_{\frac{\psi_1\circ\mathfrak{C}_2(z_1,z_2)}{z_1+i}}) = (\phi_1\hat{\otimes}\phi_2)(T_{\frac{\psi_2\circ\mathfrak{C}_2(z_1,z_2)}{z_2+i}}) = 0$$

Hence we have

$$(\phi_1\hat{\otimes}\phi_2)(\Phi\circ C_\varphi\circ\Phi^{-1}) = (\phi_1\hat{\otimes}\phi_2)(C_{\tilde{\varphi}}) \tag{22}$$

By equation (9) we have

$$\sigma_e(C_\varphi) = \sigma_e(\Phi\circ C_\varphi\circ\Phi^{-1})$$

By equations (19) and (22), we have thus

$$\sigma_e(C_\varphi) \supseteq \{(\phi_1\hat{\otimes}\phi_2)(C_{\tilde{\varphi}}) : \phi_1,\phi_2\in\mathbb{M}\} \tag{23}$$

By equations (20) and (23) we have

$$\sigma_e(C_\varphi) \supseteq \{e^{i(z_1t_1+z_2t_2)} : t_1,t_2\in[0,\infty], z_1\in\mathscr{C}_{(\infty,\infty)}(\psi_1\circ\mathfrak{C}_2), z_2\in\mathscr{C}_{(\infty,\infty)}(\psi_2\circ\mathfrak{C}_2)\}\cup\{0\}$$

Since for any $\psi\in H^\infty(\mathbb{D}^2)$,

$$\mathscr{C}_{(\infty,\infty)}(\psi\circ\mathfrak{C}_2) = \mathscr{C}_{(1,1)}(\psi)$$

we conclude that

$$\sigma_e(C_\varphi) \supseteq \{e^{i(z_1t_1+z_2t_2)} : t_1,t_2\in[0,\infty], z_1\in\mathscr{C}_{(1,1)}(\psi_1)\text{and}\quad z_2\in\mathscr{C}_{(1,1)}(\psi_2)\}\cup\{0\} \qquad \square$$

REFERENCES

[1] R. G. DOUGLAS, R. HOWE, *On the C*-algebra of Toeplitz operators on the quarterplane*, Trans. Amer. Math. Soc., **158** (1971), pp. 203–217.

[2] U. GÜL, *Essential Spectra of Quasi-parabolic Composition Operators on Hardy Spaces of Analytic Functions*, J. Math. Anal. Appl., **377** (2011), pp. 771–791.

[3] K. HOFFMAN, *Banach Spaces of Analytic Functions*, Prentice-Hall Inc., Englewood Cliffs, N.J., 1962.

[4] F. JAFARI, *On Bounded and Compact Composition Operators in Poly-discs*, Canadian J. Math. 42 (1990), pp. 869–889.

[5] G. MURPHY, *C*-algebras and Operator Theory*, Academic Press Inc., 1990.

[6] W. RUDIN, *Functional Analysis*, McGraw Hill Inc., 1973.

[7] J. H. SHAPIRO, *Cluster set, essential range, and distance estimates in BMO*, Michigan Math. J. 34 (1987), no. 3, 323–336.

[8] H. UPMEIER, *Toeplitz Operators and Index Theory in Several Complex Variables*, Operator Theory Advances and Applications vol. 81, Birkhäuser, 1996.

(Received March 21, 2012)

Uğur Gül
Hacettepe University, Department of Mathematics
06800, Beytepe, Ankara
Turkey
e-mail: gulugur@gmail.com